Problems

Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before December 15, 2013

- **5265:** *Proposed by Kenneth Korbin, New York, NY*

  Find positive integers x and y such that

  $$2x - y - \sqrt{3x^2 - 3xy + y^2} = 2014,$$

  with $(x, y) = 1$.

- **5266:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  The pentagonal numbers begin $1, 5, 12, 22, \cdots$ and in general satisfy $P_n = \dfrac{n(3n-1)}{2}$, $\forall n \geq 1$. The positive Jacobsthal numbers, which have applications in tiling and graph matching problems, begin $1, 1, 3, 5, 11, 21, \cdots$ with general term $J_n = \dfrac{2^n - (-1)^n}{3}$, $\forall n \geq 1$. Prove that there exists infinitely many pentagonal numbers that are the sum of three Jacobsthal numbers.

- **5267:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "Geroge Emil Palade" General School, Buzău, Romania*

  Let n be a positive integer. Prove that

  $$\frac{F_n L_{n+2}^2}{F_{n+3}} + \frac{F_{n+1} L_{n+3}^2}{F_n + F_{n+2}} + (L_n + L_{n+2})^2 \geq 2\sqrt{6}\left(\sqrt{L_n L_{n+1}}\right) L_{n+2},$$

  where F_n and L_n represents the n^{th} Fibonacci and Lucas Numbers defined by $F_0 = 0, F_1 = 1$, and for all $n \geq 0$, $F_{n+2} = F_{n+1} + F_n$; and $L_0 = 2, L_1 = 1$, and for all $n \geq 0$, $L_{n+2} = L_{n+1} + L_n$, respectively.

- **5268:** *Proposed by Pedro H.O. Pantoja, IMPA, Rio de Janeiro, Brazil*

  Let $N = 121^a + a^3 + 24$. Determine all positive integers a for which

  a) N is a perfect square.

  b) N is a perfect cube.

- **5269:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $\{a_n\}_{n\geq 1}$ be the sequence defined by

  $$a_1 = 1,\ a_2 = 5,\ a_{n-1}^2 - a_n a_{n-2} + 4 = 0.$$

  Show that all of the terms of the sequence are integers.

- **5270:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $k \geq 1$ be an integer. Calculate

  $$\int_0^1 \int_0^1 (x+y)^k \, (-1)^{\left\lfloor \frac{1}{x} - \frac{1}{y} \right\rfloor} \, dxdy,$$

  where$\lfloor x \rfloor$ denotes the integer part of x.

Solutions

- **5248:** *Proposed by Kenneth Korbin, New York, NY*

  A triangle with sides (a, a, b) has the same area and the same perimeter as a triangle with sides (c, c, d) where a, b, c and d are positive integers and with

  $$\frac{b^2 + bd + d^2}{b + d} = 7^6.$$

  Find the sides of the triangles.

  Solution 1 by Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

  First, note that the condition

  $$\frac{b^2 + bd + d^2}{b + d} = 7^6 \qquad (1)$$

  implies that $b \neq d$ and thus, we may assume that $b > d$. Further, the required equality of the perimeters and areas of the triangles yields

  $$\begin{aligned} 2a + b &= 2c + d, \text{and} && (2) \\ b^2(4a^2 - b^2) &= d^2(4c^2 - d^2) && (3) \end{aligned}$$

  By (1), (2), and (3),

  $$\begin{aligned} b^2(2a - b) &= d^2(2c - d) \\ \Rightarrow b^2(2a + b) - 2b^3 &= d^2(2c + d) - 2d^3 \\ \Rightarrow (b^2 - d^2)(2a + b) &= 2(b^3 - d^3) \end{aligned}$$

$$\begin{aligned} \Rightarrow (b+d)(2a+b) &= 2(b^2+bd+d^2) \\ \Rightarrow 2a+b &= 2c+d = 2(7^6). \qquad (4) \end{aligned}$$

It follows that b and d must be even.

Condition (1) can be re-written in the form

$$b^2 + (d-7^6)b + d^2 - 7^6 d = 0$$

and hence,

$$\begin{aligned} b &= \frac{7^6 - d \pm \sqrt{(d-7^6)^2 - 4(d^2 - 7^6 d)}}{2} \\ &= \frac{7^6 - d \pm \sqrt{(d+7^6)^2 - 4d^2}}{2}. \qquad (5) \end{aligned}$$

Since b and d are even integers, there must exist an odd positive integer k such that

$$\begin{aligned} (d+7^6)^2 - 4d^2 &= k^2, \text{ or} \\ (d+7^6)^2 &= 4d^2 + k^2. \end{aligned}$$

Using known properties of Pythagorean Triples, there are positive integers s, m, and n such that $m > n$, $(m,n) = 1$, $m - n \equiv 1 \pmod 2$, and

$$d + 7^6 = s(m^2+n^2), \quad k = s(m^2-n^2), \quad \text{and} \quad 2d = s(2mn). \qquad (6)$$

Note that since k and $m^2 - n^2$ are odd, s must also be odd. Then (6) implies that s divides d and s divides $d + 7^6$ and hence, s divides 7^6. Therefore, $s = 7^r$ for some $r \in \{0, 1, 2, \ldots 6\}$.

Next it follows from (6) that

$$\begin{aligned} 7^r(m^2+n^2) &= d + 7^6 = 7^r(mn) + 7^6, \text{ or} \\ m^2 - mn + n^2 &= 7^{6-r}. \qquad (7) \end{aligned}$$

Using

$$m^2 + n^2 = \frac{1}{2}\left[(m+n)^2 + (m-n)^2\right] \quad \text{and } mn = \frac{1}{4}\left[(m+n)^2 - (m-n)^2\right],$$

(7) can be re-written

$$(m+n)^2 + 3(m-n)^2 = 4 \cdot 7^{6-r}. \qquad (8)$$

Also, (5) and (6) imply that

$$\begin{aligned} b &= \frac{7^r(m-n)^2 \pm 7^r(m^2-n^2)}{2} \\ &= 7^r m(m-n) \quad \text{or} \quad 7^r n(n-m). \end{aligned}$$

Since $m > n$, we have $b = 7^r m(m-n)$ to go with $d = 7^r mn$ (from (6)). The fact that b is even now forces m to be even and n to be odd.

We can now solve (8) for m and n and thereby solve for b and d. Our work is reduced by the facts that $m+n$ and $m-n$ are odd, $(m+n, m-n) = (m,n) = 1$, and

$$\begin{aligned} 4 \cdot 7^{6-r} &= (m+n)^2 + 3(m-n)^2 > 4(m-n)^2, \text{i.e.,} \\ m-n &< \sqrt{7^{6-r}}. \end{aligned}$$

Using these and some help from MuPAD, there are only two feasible solutions for (8), namely

r	m	n	b	d
0	360	37	116,280	13,320
4	8	3	96,040	57,624.

Then, (4) may be employed to get the final solutions

a	b	c	d
59,509	116,280	110,989	13,320
69,629	96,040	88,837	57,624.

Solution 2 by Brian D. Beasley, Presbyterian College, Clinton, SC

Without loss of generality, we assume $b < d$. Since $2a + b = 2c + d$, we have $d - b = 2(a - c)$, so b and d have the same parity. But if both b and d are odd, then $b^2 + bd + d^2$ is odd and $7^6(b+d)$ is even, a contradiction. Thus both b and d are even. Letting $b = 2x$ and $d = 2y$ for positive integers x and y, we obtain $y - x = a - c$ and $2(x^2 + xy + y^2) = 7^6(x+y)$. Then

$$y = \frac{7^6 - 2x \pm \sqrt{(6x + 7^6)(7^6 - 2x)}}{4}.$$

A quick search via computer program yields two possible solutions with $x < y$:

$$(x, y) = (6660, 58140) \text{ or } \ (x, y) = (28812, 48020).$$

Next, since the areas of the two isosceles triangles must be equal, we have

$$\begin{aligned} \frac{1}{2}b\sqrt{a^2 - \frac{1}{4}b^2} &= \frac{1}{2}d\sqrt{c^2 - \frac{1}{4}d^2} \ \text{ and thus} \\ b^2(4a^2 - b^2) &= d^2(4c^2 - d^2). \end{aligned}$$

Since $2a+b=2c+d$, we obtain $b^2(2a-b)=d^2(2c-d)$, or $x^2(a-x)=y^2(c-y)$. Then $cy^2-ax^2=y^3-x^3$, so

$$c(y+x)(y-x)+x^2(c-a)=(y-x)(x^2+xy+y^2).$$

Applying $y-x=a-c$ and $x^2+xy+y^2=7^6(x+y)/2$, we have

$$\begin{aligned} c(y+x)-x^2 &= 7^6(x+y)/2 \text{ and hence} \\ c &= x^2/(x+y)+7^6/2. \end{aligned}$$

Similarly,

$$a=y^2/(x+y)+7^6/2.$$

In particular, we note that this implies

$$2a+b=\frac{2(x^2+xy+y^2)}{x+y}+7^6=2\cdot 7^6,$$

the perimeter of all such triangles.

Finally, we verify that the two possible solutions for (x,y) yield the required triangles:

$$(x,y)=(6660,58140) \quad\Rightarrow\quad (a,b,c,d)=(110989,13320,59509,116280).$$
$$(x,y)=(28812,48020) \quad\Rightarrow\quad (a,b,c,d)=(88837,57624,69629,96040).$$

In the first solution, both triangles have area 737,854,740. In the second solution, both triangles have area 2,421,216,420. Also, the second solution may be written in the form $(a,b,c,d)=7^4(37,24,29,40)$.

Editor's comments: **David Stone and John Hawkins** stated in their solution that it would be nice if an analytical solution for b and d in the following could be found.

$$\begin{aligned} \frac{b^2+bd+d^2}{b+d} &= 7^6, \Longrightarrow \\ b+\frac{d^2}{b+d} &= 7^6, \text{ and } \frac{b^2}{b+d}+d=7^6, \text{ and } b+d-\frac{bd}{b+d}=7^6. \end{aligned}$$

This allowed them to put some conditions onto b and d. But then they stated: "we see no path towards a complete solution. Finding integers b and d whose sum divides their product seems to be a difficult problem."

Ed Gray of Highland Beach, FL also reached the equation $u^2+uv+v^2=7^6$ and found that the general solution to

$$x^2+xy+y^2=z^2$$

has been characterized parametrically by J. Neuburg and G.B. Mathews (See L. E. Dickson's, History of History of The Theory of Numbers, vol.II, 2005, Dover Books on Mathematics, p.406). Specifically,

$$\begin{cases} x=p^2-q^2 \\ y=2pq+q^2 \\ z=x^2+pq+q^2. \end{cases}$$

He then applied this generic solution to the problem by solving

$$x^2 + xy + y^2 = (7^3)^2 = 343^2.$$

There are two positive integer solutions to this equation: $(x, y) = (18, 1)$ and $(x, y) = (14, 7)$. With these solutions it was possible for him, by retracing his steps, to obtain two sets, wherein each set contains two isosceles triangles with sides (a, a, b) and (c, c, d), and for which the triangles have the same perimeter, the same area, and for which $\dfrac{b^2 + bd + d^2}{b + d} = 7^6$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; David Stone and John Hawkins (jointly), Georgia Southern University, Statesboro GA, and the proposer.

- **5249:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

(a) Let n be an odd positive integer. Prove that $a^n + b^n$ is the square of an integer for infinitely many integers a and b.

(b) Prove that $a^2 + b^3$ is the square of an integer for infinitely many integers a and b.

Solution 1 by Arkady Alt, San Jose, CA

(a) Let $a = x\left(x^n + y^n\right), b = y\left(x^n + y^n\right)$ where $x, y \in \mathcal{N}$ then

$$a^n + b^n = x^n\left(x^n + y^n\right)^n + y^n\left(x^n + y^n\right)^n = \left(x^n + y^n\right)^{n+1}$$

and, since $n = 2m - 1, m \in \mathcal{N}$ then

$$a^n + b^n = \left(\left(x^n + y^n\right)^m\right)^2.$$

(b) We will show that equation $a^2 + b^3 = c^2$ have infinitely many solutions in integers. Assuming that $c = 2a$ we obtain $b^3 = 3a^2$. Let $a = 3t^3, t \in \mathcal{Z}$ then

$$b^3 = 3 \cdot 9t^6 \iff b = 3t^2.$$

Thus, for $(a, b) = \left(3t^3, 3t^2\right)$, where t is any integer we have

$$a^2 + b^3 = 9t^6 + 27t^6 = 36t^6 = \left(6t^3\right)^2.$$

Solution 2 by Pat Costello, Eastern Kentucky University, Richmond, KY

(a) Let n be an odd positive integer. Let $a = 2 \cdot 2^{2j}$ and $b = 2 \cdot 2^{2j}$ for an arbitrary positive integer j. Then

$$\begin{aligned} a^n + b^n &= (2 \cdot 2^{2j})^n + (2 \cdot 2^{2j})^n \\ &= 2^n \cdot 2^{2nj} + 2^n \cdot 2^{2nj} \\ &= 2 \cdot \left(2^n \cdot 2^{2nj}\right) \\ &= 2^{n+1} \cdot 2^{2nj} \end{aligned}$$

$$= \left(2^{(n+1)/2} \cdot 2^{nj}\right)^2,$$

the square of an integer since n is odd.

(b) Let $a = 2^{3n}$ and $b = 2 \cdot 2^{2n}$ for an arbitrary positive integer n. Then

$$\begin{aligned} a^2 + b^3 &= \left(2^{3n}\right)^2 + \left(2 \cdot 2^{2n}\right)^3 \\ &= 2^{6n} + 8 \cdot 2^{6n} \\ &= 9 \cdot 2^{6n} \\ &= \left(3 \cdot 2^{3n}\right)^2, \end{aligned}$$

the square of an integer.

Solution 3 by David Stone and John Hawkins (jointly), Georgia Southern University, Statesboro, Ga

(a) Let $n = 2k+1$ for $k \geq 0$. Then let $a = b = 2m^2$, for any $m \geq 1$. Then

$$a^n + b^n = 2a^n = 2\left(2m^2\right)^{2k+1} = 2^{2k+2}m^{2(2k+1)} = \left(2^{k+1}m^{2k+1}\right)^2, \text{ which is square.}$$

Of course, there is also a trivial solution; let a be any square and $b = 0$.

(b) Let $a = m^2\left(16m^2 - 1\right)$ and $b = 4m^2$, for any integer m. Then

$$\begin{aligned} a^2 + b^3 &= m^4\left(16m^2 - 1\right)^2 + \left(4m^2\right)^3 \\ &= m^4\left(256m^4 - 32m^2 + 1\right) + 64m^6 \\ &= 256m^8 + 32m^4 + 1 \\ &= \left(16m^4 + m^2\right)^2; \text{ a square.} \end{aligned}$$

In addition to the trivial solution, let a be any square and $b = 0$, there is also a "semi-trivial" solution: For any $c, m \geq 1$, let $a = c^{3m}, b = -c^{2m}$, so that

$$a^2 + b^3 = \left(c^{3m}\right)^2 + \left(-c^{2m}\right)^3 = c^{6m} - c^{6m} = 0; \text{ a square.}$$

Solution 4 by Ken Korbin, New York, NY

(a) Let $a = N^2, b = 2N^2$ where N is a positive integer. Then $a^3 + b^3 = (3N^3)^2$, and it follows that for n odd, $a^n + b^n$ is a perfect square.

(b) Let $a = 4N^3 + 6N^2 + 3N$, and $b = 2N + 1$, where N is a positive integer. Then $a^2 + b^3 = (4N^3 + 6N^2 + 3N + 1)^2$.

Also solved by Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Brian D. Beasley, Presbyterian

College, Clinton, SC; Roberto de la Cruz Moreno, Centre de Recerca Matematica, Campus de Bellaterra, Barcelona, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Charles McCracken, Dayton, OH, and the proposer.

- **5250:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

Let $a \in \left(0, \frac{\pi}{2}\right)$ and $b, c \in (1, \infty)$. Calculate:

$$\int_{-a}^{a} \ln\left(b^{\sin^3 x} + c^{\sin^3 x}\right) \cdot \sin x \cdot dx.$$

Solution 1 by Anastasios Kotronis, Athens, Greece

We have

$$\begin{aligned} I = \int_{-a}^{a} \ln\left(b^{\sin^3 x} + c^{\sin^3 x}\right) \sin x\, dx \quad &\overset{y=-x}{=} \quad -\int_{-a}^{a} \ln\left(b^{-\sin^3 x} + c^{-\sin^3 x}\right) \sin x\, dx \\ &= \int_{-a}^{a} \ln\left(\frac{(bc)^{\sin^3 x}}{b^{\sin^3 x} + c^{\sin^3 x}}\right) \sin x\, dx \\ &= \ln(bc) \int_{-a}^{a} \sin^4 x\, dx - I. \end{aligned}$$

So

$$I = \frac{\ln(bc)}{2} \int_{-a}^{a} \sin^4 x\, dx = \frac{\ln(bc)}{2} \left(\frac{3a}{4} - \frac{\sin(2a)}{2} + \frac{\sin(4a)}{16}\right).$$

Solution 2 by Paolo Perfetti, Department of Mathematics, University of Tor Vergata Roma, Rome, Italy

Answer: $\frac{1}{2}\left(\frac{3}{4}a - \frac{1}{2}\sin(2a) + \frac{1}{16}\sin(4a)\right)\ln(bc)$

Proof: We observe

$$\begin{aligned} \ln\left(b^{\sin^3(-x)} + c^{\sin^3(-x)}\right)\sin(-x) &= -\ln\left(b^{-\sin^3 x} + c^{-\sin^3 x}\right)\sin x \\ &= -\ln\left(b^{\sin^3 x} + c^{\sin^3 x}\right)\sin x + \ln\left((bc)^{\sin^3 x}\right)\sin x \end{aligned}$$

thus

$$\begin{aligned} 2\int_{-a}^{a} \ln\left(b^{\sin^3 x} + c^{\sin^3 x}\right)\sin x\, dx &= \int_{-a}^{a} \sin^4 x dx \ln(bc) \\ &= \int_{-a}^{a} \sin^2 x(1 - \cos^2 x) dx \end{aligned}$$

$$
\begin{aligned}
&= \left.\frac{x - \sin x \cos x}{2}\right|_{-a}^{a} - \int_{-a}^{a} \frac{1}{4}(\sin^2(2x))dx \\
&= a - \frac{1}{2}\sin(2a) - \int_{-2a}^{2a} \frac{1}{8}(\sin^2 x)dx \\
&= a - \frac{1}{2}\sin(2a) - \frac{a}{4} + \frac{1}{16}\sin(4a)
\end{aligned}
$$

Also solved by Arkady Alt, San Jose, CA; Roberto de la Cruz Moreno, Centre de Recerca Matematica, Campus de Bellaterra, Barcelona, Spain; Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Adrian Naco, Polytechnic University, Tirana, Albania; Boris Rays, Brooklyn, NY, and the proposers.

- **5251:** *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany*

Compute the following sum:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+n}\frac{\cos(m+n)}{(m+n)^2}.$$

Solution 1 by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania

The sum equals

$$\frac{\pi^2}{12} - \frac{1}{4} - \ln\left(2\cos\frac{1}{2}\right).$$

The problem is a particular case of the following theorem (see the first citation below [Theorem 1, p. 2]).

Theorem 1. *Suppose that both series*

$$\sum_{k=1}^{\infty} a_k \qquad \textit{and} \qquad \sum_{k=1}^{\infty} k a_k$$

converge and let σ and $\widetilde{\sigma}$ denote their sums, respectively. Then, the iterated series

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{n+m},$$

converges and its sum s equals $\widetilde{\sigma} - \sigma$.

The following two formulae are well-known (see citation 2, [Formula 1.441(4), p. 44])

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos kx}{k} = \ln\left(2\cos\frac{x}{2}\right), \quad -\pi < x < \pi$$

and (citation 2 [Formula 1.443(4), p. 45])

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos kx}{k^2}=\frac{\pi^2}{12}-\frac{x^2}{4},\quad -\pi\le x\le\pi.$$

Now, we apply the Theorem in citation 1 with $a_k=(-1)^k\cdot\dfrac{\cos k}{k^2}$, and we have that

$$\sigma=\sum_{k=1}^{\infty}a_k=\sum_{k=1}^{\infty}(-1)^k\frac{\cos k}{k^2}=\frac{1}{4}-\frac{\pi^2}{12}$$

and

$$\widetilde{\sigma}=\sum_{k=1}^{\infty}ka_k=\sum_{k=1}^{\infty}(-1)^k\frac{\cos k}{k}=-\ln\left(2\cos\frac{1}{2}\right).$$

It follows, based on the Theorem in citation 1, that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_{n+m}=\widetilde{\sigma}-\sigma=\frac{\pi^2}{12}-\frac{1}{4}-\ln\left(2\cos\frac{1}{2}\right).$$

Citations:

1) Ovidiu Furdui and Tiberiu Trif, *On the Summation of Certain Iterated Series*, Journal of Integer Sequences, Vol. 14, 2011, Issue 6, article 11.6.1, article available online at https://cs.uwaterloo.ca/journals/JIS/VOL14/Furdui/furdui3.pdf

2) I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* Sixth Edition, Academic Press, 2000

Solution 2 by Kee-Wai Lau, Hong Kong, China

Denote the double sum by S. We show that

$$S=\frac{\pi^2-3-6\ln\left(2(1+\cos 1)\right)}{12}=0.00990\ldots.$$

Let m and M be positive integers with $m\le M$. We have

$$\sum_{n=1}^{\infty}(-1)^{m+n}\frac{\cos(m+n)}{(m+n)^2}=\sum_{k=m+1}^{\infty}(-1)^k\frac{\cos k}{k^2}=\sum_{k=m+1}^{M^2+1}(-1)^k\frac{\cos k}{k^2}+r,$$

where $|r|\le\displaystyle\sum_{k=M^2+2}^{\infty}\frac{1}{k^2}<\frac{1}{M^2}$. Hence,

$$\sum_{m=1}^{M}\sum_{n=1}^{\infty}(-1)^{m+n}\frac{\cos(m+n)}{(m+n)^2}=\sum_{k=2}^{M^2+1}(-1)^k\frac{(k-1)\cos k}{k^2}+R,$$

where $|R|<\dfrac{1}{M}$. By taking the limit as M tends to infinity, we have

$$S=\sum_{k=2}^{\infty}(-1)^k\frac{(k-1)\cos k}{k^2}=\sum_{k=2}^{\infty}(-1)^k\frac{\cos k}{k}-\sum_{k=2}^{\infty}(-1)^k\frac{\cos k}{k^2}.$$

For $-\pi < x < \pi$, is known ([1], formula 17.2.6, p.239) that

$$\sum_{k=1}^{\infty} \frac{(-1)^k \cos kx}{k} = -\frac{\ln\left(2\left(1+\cos x\right)\right)}{2},$$

and ([1] formula 17.2.9, p.239) that

$$\sum_{k=1}^{\infty} \frac{(-1)^k \cos kx}{k^2} = \frac{3x^2 - \pi^2}{12}.$$

By putting $x = 1$, we obtain our result for S.

Reference: 1. E.R. Hansen: *A Table of Series and Products,* Prentice-Hall, Inc. (1975).

Also solved by Ed Gray, Highland Beach FL; Anastasios Kotronis, Athens, Greece; Paolo Perfetti, Department of Mathematics, University of Tor Vergata Roma, Rome, Italy, and the proposers.

- **5252:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

Let $\{a_n\}_{n\geq 1}$ be the sequence of real numbers defined by $a_1 = 3, a_2 = 5$ and for all $n \geq 2, a_{n+1} = \dfrac{1}{2}\left(a_n^2 + 1\right)$. Prove that

$$1 + 2\left(\sum_{k=1}^{n} \sqrt{\frac{F_k}{1+a_k}}\right)^2 < F_{n+2},$$

where F_n represents the n^{th} Fibonacci number defined by $F_1 = F_2 = 1$ and for $n \geq 3, F_n = F_{n-1} + F_{n-2}$.

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

By the Cauchy-Schwarz inequality

$$\left(\sum_{k=1}^{n} \sqrt{\frac{F_k}{1+a_k}}\right)^2 \leq \left(\sum_{k=1}^{n} F_k\right)\left(\sum_{k=1}^{n} \frac{1}{1+a_k}\right),$$

and since $\displaystyle\sum_{k=1}^{n} F_k = F_{n+2} - 1$, it is enough to prove that $\displaystyle\sum_{k=1}^{n} \frac{1}{1+a_k} < \frac{1}{2}$.

We will prove by induction that $\displaystyle\sum_{k=1}^{n} \frac{1}{1+a_k} = \frac{\frac{a_{n+1}-1}{2} - 1}{a_{n+1} - 1}$, which is less than $\frac{1}{2}$.

Clearly it is true for $n = 1$. Let us suppose it holds for n. Then, for $n+1$ we have

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{1}{1+a_k} &= \sum_{k=1}^{n} \frac{1}{1+a_k} + \frac{1}{1+a_{n+1}} \\
&= \frac{\frac{a_{n+1}-1}{2} - 1}{a_{n+1} - 1} + \frac{1}{1+a_{n+1}} \quad \text{by hypotesis of induction} \\
&= \frac{\frac{a_{n+1}^2-1}{2} - a_{n+1} - 1 + a_{n+1} - 1}{a_{n+1}^2 - 1} = \frac{\frac{a_{n+1}^2-1}{2} - 2}{a_{n+1}^2 - 1}
\end{aligned}$$

$$= \frac{\frac{2a_{n+2}-2}{2} - 2}{2a_{n+2} - 2} \quad \text{by the definition of sequence } \{a_n\}$$

$$= \frac{\frac{a_{n+2}-1}{2} - 1}{a_{n+2} - 1}.$$

And, therefore, the conclusion follows.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

By the Cauchy-Schwarz inequality applied to the vectors $(\sqrt{F_1}, \cdots, \sqrt{F_n})$ and $\left(\frac{1}{\sqrt{1+a_1}}, \cdots, \frac{1}{\sqrt{1+a_n}}\right)$, we have for $n \geq 1$

$$\left(\sum_{k=1}^{n} \sqrt{\frac{F_k}{1+a_k}}\right)^2 = \left(\sum_{k=1}^{n} \sqrt{F_k}\frac{1}{\sqrt{1+a_k}}\right)^2 \leq \left(\sum_{k=1}^{n} F_k\right)\left(\sum_{k=1}^{n} \frac{1}{1+a_k}\right) \qquad (1)$$

The sequence $\{a_n\}_{n\geq 1}$ is related to the Sylvester sequence $\{b_n\}_{n\geq 1}$ defined by $b_1 = 2$ and for $n \geq 1, b_{n+1} = b_n^2 - b_n + 1$, by the equality $b_n = \frac{1}{2}(a_n + 1)$, and it is known that the sum of the reciprocals of the Sylvester sequence is 1. So for $n \geq 1$, we have that

$$\sum_{k=1}^{n} \frac{1}{1+a_k} < \sum_{k=1}^{n} \frac{1}{1+a_k} = \sum_{k=1}^{\infty} \frac{1}{2b_k} = \frac{1}{2}. \qquad (2)$$

From (1) and (2) and the property $1 + \sum_{k=1}^{n} F_k = F_{n+2}$, it follows that, for $n \geq 1$

$$1 + 2\left(\sum_{k=1}^{n} \sqrt{\frac{F_k}{1+a_k}}\right)^2 \leq 1 + 2\left(\sum_{k=1}^{n} F_k\right)\left(\sum_{k=1}^{n} \frac{1}{1+a_k}\right) < 1 + \sum_{k=1}^{n} F_k = F_{n+2}.$$

Solution 3 by Roberto de la Cruz Moreno, Centre de Recerca Matematica, Campus de Bellaterra, Barcelona, Spain

Lemma. Let $\{b_n\}_{n\geq 1}$ be the sequence of real numbers defined by $b_1 = 5$ and for all $n \geq 1$, $b_{n+1} = \frac{1}{2}(b_n^2 + 1)$. Then:

$$\sum_{k=1}^{m} \frac{1}{b_k + 1} = \frac{1}{4} - \frac{1}{b_{m+1} - 1}, \quad \forall m \in \mathcal{Z}^+$$

Proof. By induction:
$m = 1$:

$$\frac{1}{b_1 + 1} = \frac{1}{6} = \frac{1}{4} - \frac{1}{12} = \frac{1}{4} - \frac{1}{b_2 - 1}$$

$m \Rightarrow m + 1$:

$$\sum_{k=1}^{m+1} \frac{1}{b_k + 1} = \sum_{k=1}^{m} \frac{1}{b_k + 1} + \frac{1}{b_{m+1} + 1} = \frac{1}{4} - \frac{1}{b_{m+1} - 1} + \frac{1}{b_{m+1} + 1}$$

$$= \frac{1}{4} - \frac{2}{b_{m+1}^2 - 1} = \frac{1}{4} - \frac{1}{b_{m+2} - 1}$$

Corollary. $\displaystyle\sum_{k=1}^{m} \frac{1}{a_k+1} < \frac{1}{2}, \ \forall m \in \ \mathcal{Z}^{+}$

By Cauchy-Schwarz inequality:

$$\begin{aligned} 1+2\left(\sum_{k=1}^{n}\sqrt{\frac{F_k}{1+a_k}}\right)^2 &\leq 1+2\left(\sum_{i=1}^{n}F_i\right)\left(\sum_{j=1}^{n}\frac{1}{1+a_j}\right) \\ &= 1+2(F_{n+2}-1)\left(\sum_{j=1}^{n}\frac{1}{1+a_j}\right) < F_{n+2} \end{aligned}$$

Also solved by Ed Gray, Highland Beach, FL; Adrian Naco, Polytechnic University, Tirana, Albania, and the proposer.

- **5253:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate

$$\int_0^1\int_0^1 \frac{\ln x\cdot\ln(xy)}{1-xy}dxdy.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

We show that the double integral equals $\dfrac{\pi^4}{30}$.

For $s,t\geq 0$ we have

$$\begin{aligned} \int_0^1\int_0^1 \frac{x^{s+t}y^s}{1-xy}dxdy &= \int_0^1\int_0^1 \sum_{k=0}^{\infty} x^{k+s+t}y^{k+s}dxdy \\ &= \sum_{k=0}^{\infty}\int_0^1\int_0^1 x^{k+s+t}y^{k+s}dxdy \\ &= \sum_{k=0}^{\infty}\frac{1}{(k+s+t+1)(k+s+1)}. \end{aligned}$$

Differentiating with respect to t, then setting $t=0$, we obtain

$$\int_0^1\int_0^1 \frac{\ln x.x^sy^s}{1-xy}dxdy = \sum_{k=0}^{\infty}\frac{-1}{(k+s+1)^3}$$

Differentiating with respect to s, then setting $s=0$, we obtain

$$\int_0^1\int_0^1 \frac{\ln x.\ln(xy)}{1-xy}dxdy = 3\sum_{k=0}^{\infty}\frac{1}{(k+1)^4}.$$

Now it is well known that the sum $\displaystyle\sum_{k=0}^{\infty}\frac{1}{(k+1)^4}$ equals $\dfrac{\pi^4}{90}$. Hence the result.

Solution 2 by Anastasios Kotronis, Athens, Greece

We have

$$
\begin{aligned}
\int_0^1 \int_0^1 \frac{\ln x \cdot \ln(xy)}{1-xy}\, dx\, dy \quad &\stackrel{1}{=} \quad \int_0^1 \int_0^1 \frac{\ln x \cdot \ln(xy)}{1-xy}\, dy\, dx = \int_0^1 \int_0^1 \sum_{k\geq 0} (xy)^k \ln x \ln(xy)\, dy\, dx \\
&\stackrel{xy=u}{=} \quad \int_0^1 \frac{\ln x}{x} \int_0^x \sum_{k\geq 0} u^k \ln u\, du\, dx \\
&\stackrel{2}{=} \quad \int_0^1 \frac{\ln x}{x} \sum_{k\geq 0} \int_0^x u^k \ln u\, du\, dx \\
&= \quad \int_0^1 \frac{\ln x}{x} \sum_{k\geq 0} \left(\frac{u^{k+1}}{k+1} \ln u \Big|_0^x - \frac{1}{k+1} \int_0^x u^k\, du \right) dx \\
&= \quad \int_0^1 \sum_{k\geq 0} \frac{x^k}{k+1} \ln^2 x\, dx - \int_0^1 \sum_{k\geq 0} \frac{x^k}{(k+1)^2} \ln x\, dx \\
&\stackrel{3}{=} \quad \sum_{k\geq 0} \int_0^1 \frac{x^k}{k+1} \ln^2 x\, dx - \sum_{k\geq 0} \int_0^1 \frac{x^k}{(k+1)^2} \ln x\, dx
\end{aligned}
$$

But integrating by parts twice and once respectively,

$$\int_0^1 x^k \ln^2 x\, dx = \frac{2}{(k+1)^3} \quad \text{and} \quad \int_0^1 x^k \ln x\, dx = -\frac{1}{(k+1)^2},$$

so

$$\int_0^1 \int_0^1 \frac{\ln x \cdot \ln(xy)}{1-xy}\, dx\, dy = 3 \sum_{k\geq 0} \frac{1}{(k+1)^4} = 3\zeta(4) = \frac{\pi^4}{30}.$$

Notes:
[1]From Fubini's theorem $< http://on.wikipedia.org/wiki/Fubini\#27 >$,
since the integrand doesn't change sign.
[2] Again from Fubini's theorem
[3] Again from Fubini's theorem

Also solved by Ed Gray, Highland Beach, FL; Paolo Perfetti, Department of Mathematics, University of Tor Vergata Roma, Rome, Italy, and the proposer.

Mea Culpa

Enkel Hysnelaj of the University of Technology in Sydney Australia and Elton Bojaxhiu of Kriftel, Germany were inadvertently omitted from the list of those having solved problem 5232 that appeared in the April issue this column. Once again, sorry.