Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before December 15, 2012

- **5218:** *Proposed by Kenneth Korbin, New York, NY*

  Find positive integers x and y such that,

  $$2x - y - \sqrt{3x^2 - 3xy + y^2} = 2013$$

  with $(x, y) = 1$.

- **5219:** *Proposed by David Manes and Albert Stadler, SUNY College at Oneonta, Oneonta, NY and Herrliberg, Switzerland (respectively)*

  Let k and n be natural numbers. Prove that:

  $$\sum_{j=1}^{n} \cos^k\left(\frac{(2j-1)\pi}{2n+1}\right) = \begin{cases} \dfrac{2n+1}{2^{k+1}}\dbinom{k}{k/2} - \dfrac{1}{2}, & k \text{ even} \\[2ex] \dfrac{1}{2}, & k \text{ odd.} \end{cases}$$

- **5220:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  The pentagonal numbers begin $1, 5, 12, 22\ldots$ and are generally defined by $P_n = \dfrac{n(3n-1)}{2}$, $\forall n \geq 1$. The triangular numbers begin $1, 3, 6, 10, \ldots$ and are generally defined by $T_n = \dfrac{n(n+1)}{2}, \forall n \geq 1$. Find the greatest common divisor, $\gcd(T_n, P_n)$.

- **5221:** *Proposed by Michael Brozinsky, Central Islip, NY*

  If x, y and z are positive numbers find the maximum of

  $$\frac{\sqrt{(x+y+z)\cdot xyz}}{(x+y)^2 + (y+z)^2 + (x+z)^2}.$$

- **5222:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

Calculate without the aid of a computer the following sum

$$\sum_{n=0}^{\infty}(-1)^n\,(n+1)(n+3)\left(\frac{1}{1+2\sqrt{2}i}\right)^n, \quad \text{where } i=\sqrt{-1}.$$

- **5223:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  a) Find the value of

$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\cdots\right).$$

  b) More generally, if $x\in(-1,1]$ is a real number, calculate

$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}+\frac{x^{n+3}}{n+3}-\cdots\right).$$

Solutions

- **5200:** *Proposed by Kenneth Korbin, New York, NY*

  Given positive integers (a,b,c,d) such that,

$$(a+b+c+d)^2=2\left(a^2+b^2+c^2+d^2\right)$$

  with $a<b<c<d$. Find positive integers x,y and z such that

$$\begin{aligned} x &= \sqrt{ab+ad+bd}-\sqrt{ab+ac+bc}, \\ y &= \sqrt{bc+bd+cd}-\sqrt{bc+ab+ac}, \\ z &= \sqrt{bc+bd+cd}-\sqrt{ac+ad+cd}. \end{aligned}$$

Solution by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

The first equation can be treated as a quadratic in d and solved:

$$d=(a+b+c)\pm 2\sqrt{ab+ac+bc}.$$

The simplest way to force d to be an integer is to find a,b and c such that the discriminate $ab+ac+bc$ is a square. (Note that we must use the + sign, because the negative choice would make $d<c$.) (Note also that we could cast a slightly wider net and look for a,b and c such that $ab+ac+bc$ has the form $n^2/4$.)

Suppose $ab+ac+bc=N^2$, so that $d=(a+b+c)+2N$. Then we need $x=\sqrt{ab+ad+bd}-N$ to be an integer, so $ab+ad+bd$ must be a square. Successively, $bc+bd+cd$ and $ac+ad+cd$ must also be squares.

Thus we look for values of a, b, c and d such that $ab + ac + bc, ab + ad + bd, bc + bd + cd$ and $ac + ad + cd$ are all squares.

Surprisingly, there are many such. We used MATLAB to find a sampling and conjecture that there are infinitely many of them. The smallest set is $a = 1, b = 4, c = 9, d = 28$ which give $x = 5, y = 13, z = 3$.

Editor's note: David and John then listed about 145 different 4-tuplets (a, b, c, d) which produce positive integer values for x, y, z. Listed below is a sampling of the values they listed.

$$\begin{pmatrix} a & b & c & d & x & y & z \\ 1 & 4 & 9 & 28 & 5 & 13 & 3 \\ 1 & 4 & 12 & 33 & 5 & 16 & 3 \\ 1 & 4 & 28 & 57 & 5 & 32 & 3 \\ 1 & 4 & 33 & 64 & 5 & 37 & 3 \\ 1 & 4 & 57 & 96 & 5 & 61 & 3 \\ 1 & 4 & 64 & 105 & 5 & 68 & 3 \\ 1 & 4 & 96 & 145 & 5 & 100 & 3 \\ 1 & 9 & 16 & 52 & 10 & 25 & 8 \\ 1 & 9 & 28 & 72 & 10 & 37 & 8 \\ 1 & 9 & 52 & 108 & 10 & 61 & 8 \\ 1 & 9 & 72 & 136 & 10 & 81 & 8 \\ 1 & 12 & 24 & 73 & 13 & 36 & 11 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

David and John also asked if there were an infinite number of such integers and **Paul M. Harms of North Newton, KS** answered this affirmatively in his solution by showing that for any positive integer a, $(a, b, c, d) = (a, 4a, 9a, 28a)$ satisfies the conditions of the problem and yields positive integers for x, y, z. Note that Paul's parameterization of the simplest solution $(1, 4, 9, 28)$ produces an infinite number of solutions to the problem, but not all solutions. E.g., there is no integer value of a for which $(a, 4a, 9a, 28a)$ will give $(1, 4, 12, 33,)$, the second tuple in the above listing.

Most solvers showed that if the conditions of the problem are satisfied then two cases exist:

$$\begin{cases} x = d - c,\ y = d - a,\ z = b - a & \text{if } a+b+c-d > 0, \\ x = a + b,\ y = b + c,\ z = b - a & \text{if } a+b+c-d < 0. \end{cases}$$

So the main question becomes: when is $(a + b + c + d)^2 = 2(a^2 + b^2 + c^2 + d^2)$ solvable?

Albert Stadler of Herrliberg, Switzerland stated that by labeling the equation $(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2)$ as (1), we see that (1) is equivalent to $(a-b)^2+(c-d)^2 = 2(a+b)(c+d)$. So if we choose odd integers u and v such that

$$u^2+v^2 = 2rs \text{ with } r \geq \ u \text{ and } s \geq v$$

then r and s are both odd and $(a,b,c,d) = ((r-u)/2, (r+u)/2, (s-v)/2, (s+v)/2)$ satisfies (1).

Also solved by Brian D. Beasley, Clinton, SC; Samuel Judge, Justin Wydra and Karen Wydra (jointly, students at Taylor University), Upland, IN; Adrian Naco, Polytechnic University, Tirana, Albania; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5201:** *Proposed by Kenneth Korbin, New York, NY*

Given convex cyclic quadrilateral ABCD with integer length sides where $\left(\overline{AB}, \overline{BC}, \overline{CD}\right) = 1$ and with $\overline{AB} < \overline{BC} < \overline{CD}$.

The inradius, the circumradius, and both diagonals have rational lengths. Find the possible dimensions of the quadrilateral.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

A **Brahmagupta quadrilateral** [1] is a cyclic quadrilateral with integer sides, integer diagonals, and integer area. All Brahmagupta quadrilaterals with sides a, b, c, d, diagonals e, f, area K, and circumradius R can be obtained by clearing denominators from the following expressions involving rational parameters t, u, and v:

$$\begin{aligned}
a &= \Big(t(u+v)+1-uv\Big)\Big(u+v-t(1-uv)\Big),\\
b &= (1+u^2)(v-t)(1+tv),\\
c &= t(1+u^2)(1+v^2),\\
d &= (1+v^2)(u-t)(1+tv),\\
e &= u(1+t^2)(1+v^2),\\
f &= v(1+t^2)(1+u^2),\\
K &= \Big|uv\Big(2t(1-uv)-(u+v)(1-t^2)\Big)\Big(2(u+v)t+(1-uv)(1-t^2)\Big)\Big|,\\
4R &= (1+u^2)(1+v^2)(1+t^2).
\end{aligned}$$

(Source: http://en.wikipedia.org/wiki/Cyclic quadrilateral; we have corrected a minor slip in the formula for K as we must take the absolute value of the defining expression of K.)

The condition $\max\left(0, \dfrac{uv-1}{u+v}\right) < t < \min(u,v)$ ensures that $a>0, b>0, c>0$, and $d>0$.

If the cyclic quadrilateral is in addition tangential (as in the problem statement) then $a + c = b + d$. So ,

$$t + tu^2 + 2tuv - u^2v + t^2u^2v + tv^2 - uv^2 + t^2uv^2 - tu^2v^2 = 0, \text{or,}$$

$$t = \frac{(uv-1)(uv+1) - (u+v)^2 + \sqrt{(1+u^2)(1+v^2)\Big((1+uv)^2 + (u+v)^2\Big)}}{2uv(u+v)}$$

There are many tuples (u, v) of rational numbers such that

$$\sqrt{(1+u^2)(1+v^2)\Big((1+uv)^2 + (u+v)^2\Big)}$$

is rational. Here are a few examples:

$$\begin{pmatrix} t & u & v \\ 31/384 & 4/3 & 124/957 \\ 31/384 & 4/3 & 496/3828 \\ 1443/1276 & 4/3 & 1914/248 \\ 1443/1276 & 4/3 & 7656/992 \\ 93/1924 & 6/8 & 124/957 \\ 93/1924 & 6/8 & 496/3828 \\ 216/319 & 6/8 & 1914/248 \\ 44/273 & 14/48 & 156/133 \\ 171/1372 & 14/48 & 266/312 \\ 31/384 & 16/12 & 124/957 \\ 1443/1276 & 16/12 & 1914/248 \\ 2816/3705 & 20/21 & 912/215 \\ 93/1924 & 24/32 & 124/957 \\ 896/1053 & 24/7 & 156/133 \\ 152/231 & 24/7 & 266/312 \\ 896/1053 & 24/7 & 624/532 \\ 896/1053 & 96/28 & 156/133 \\ 2625/1664 & 140/51 & 260/69 \end{pmatrix}$$

In what follows we consider only the first entry in this table.

The triple $(t, u, v) = (31/384, 4/3, 124/957)$ yields the quadruple

$$(a, b, c, d) = \left(\frac{23280625}{17639424}, \frac{13885495975}{101285572608}, \frac{721699375}{3165174144}, \frac{447919225}{317509632}\right).$$

Clearing denominators yields

$$(a, b, c, d) = (143550, 14911, 24800, 153439)$$

which is equivalent to

$$(a^{'}, b^{'}, c^{'}, d^{'}) = (14911, 24800, 153439, 143550).$$

Obviously $a^{'} < b^{'} < c^{'}$ and these numbers are coprime.

We see that $a^{'} + c^{'} = b^{'} + d^{'}$, so the quadrilateral is tangential. We have

$$\begin{aligned}
s &= \frac{(a+b+c+d)}{2} = 168350, \\
K &= \sqrt{(s-a)(s-b)(s-c)(s-d)} = 2853965400 = \sqrt{abcd}, \\
r &= \frac{2K}{a+b+c+d} = \frac{K}{s} = \frac{118668}{7}, \\
R &= \frac{1}{4}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{(s-a)(s-b)(s-c)(s-d)}} = \frac{3710425}{48}, \\
e &= \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}} = 148417, \\
f &= \sqrt{\frac{(ab+cd)(ac+bd)}{ad+bc}} = \frac{7604641}{193}.
\end{aligned}$$

References: [1] Sastry, K.R.S., "Brahmagupta quadrilaterals" Forum Geometricorum, 2, 2002, 167-173.

Solution 2 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

We let $b = \overline{AB}, c = \overline{BC}, d = \overline{CD}, a = \overline{DA}$.

From Wolfram Math World at http://mathworld.wolfram.com/CyclicQuadrilateral.html and http://mathworld.wolfram.com/Bicentri Quadrilateral.html, we find for a bicentric quadrilateral with sides a, b, c, and d (in order around the quadrilateral), having inradius r circumradius R and area A, semiperimeter s, the following conditions must be fulfilled:

$$\begin{aligned}
a+c &= b+d, \\
A &= \sqrt{abcd} = \sqrt{(s-a)(s-b)(s-c)(s-d)}, \\
r &= \frac{\sqrt{abcd}}{s},
\end{aligned}$$

$$R = \frac{1}{4}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{abcd}} = \frac{1}{4}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{(s-a)(s-b)(s-c)(s-d)}}.$$

Diagonal lengths are given by $\sqrt{\dfrac{(ab+cd)(ac+bd)}{ad+bc}}$ and $\sqrt{\dfrac{(ad+bc)(ac+bd)}{ab+cd}}$.

We are requiring $b < c < d$ (which also forces $b < a < d$), and $(b, c, d) = 1$, which forces any three sides to be coprime.

Rationalizing the denominator in the expressions for the diagonals, we see that $\sqrt{(ad+bc)(ab+cd)(ac+bd)}$ must be an integer if the diagonals are to have rational length.

Since the circumradius must also be rational, we deduce that the area must also be rational. Since it is the square root of a product of integers, it must be an integer.

Using the two formulas for the area $A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ and $A = \sqrt{abcd}$ were s is its semiperimeter, we see that $8abcd + 2(a^4+b^4+c^4+d^4) = (a^2+b^2+c^2+d^2)^2$. Thus the side lengths of the quadrilateral must satisfy the following:

- $8abcd + 2(a^4+b^4+c^4+d^4) = (a^2+b^2+c^2+d^2)^2$,
- $a + c = b + d$,
- the product $abcd$ must be a perfect square,
- $\sqrt{(ad+bc)(ab+cd)(ac+bd)}$ must be an integer.

We wrote a MATLAB program to search through integers b, c, and d where $b < c < d$ from 1 to 4000 where these conditions were satisfied. The results give us the possible dimensions of the cyclic quadrilaterals satisfying the requirements of the problem. We found 7 solutions.

Note that the position of the side can be rearranged as long as opposing pairs have the same sum. In the following table we have re-lettered to let a be the smallest entry.

Results are shown below with rational numbers in lowest terms:

a	b	c	d	$2s$	$Area$	$Inradius$	$circumradius$	$diag1$	$diag2$
21	85	140	204	450	7140	476/15	221/2	195	104
91	36	260	315	702	16380	140/3	325/2	280	125
190	231	399	440	1260	87780	418/3	1885/8	13650/29	377
2397	483	1316	1564	5760	1543668	128639/240	2405/2	22015/13	11544/5
4756	123	1428	3451	9758	1697892	348	7565/2	7743/5	414715/89
3256	629	1080	2805	7770	2490840	71224/111	1628	1653	15973/5
4828	1060	2125	3763	11776	6397100	3400	2414	3025	23551/5
2849	1480	2145	2184	8658	4444440	4070	3145/2	2975	15703/5

Comment: It is helpful to look at the prime-power decomposition of a, b, c and d. For instance,

$$21 = 3 \cdot 7, \quad 85 = 5 \cdot 17, \quad 140 = 2^2 \cdot 5 \cdot 7, \quad \text{and} \quad 204 = 2^2 \cdot 3 \cdot 17.$$

Thus the product $abcd = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 17^2$ is clearly a square. But recognizing such patterns does not help us in generating solutions. In fact, it would seem so difficult to satisfy the required conditions that no solutions could exist.

Conjectures: Each of our solutions consists of two even integers and two odd ones, so that would be a reasonable conjecture. We suspect there are infinitely many solutions.

Also solved by the proposer.

- **5202:** *Proposed by Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

Solve in $\Re^2$,

$$\begin{cases} \ln\left(x+\sqrt{x^2+1}\right) = \ln \dfrac{1}{y+\sqrt{y^2+1}} \\ 2^{y-x}\left(1-3^{x-y+1}\right) = 2^{x-y+1}-1. \end{cases}$$

Solution 1 by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany.

From the first equation, considering the fact that functions $f(x) = \sqrt{x^2+1}+x$ and $g(x) = \sqrt{x^2+1}-x$ are symmetric with respect to the y-axis, one can easily observe that this is satisfied for $x = -y$.

Replacing $x = -y$ in the second equation we have

$$2^{2y}\left(1-3^{-2y+1}\right) = 2^{-2y+1}-1.$$

Let's consider the function $f(y) = 2^{2y}\left(1 - 3^{-2y+1}\right) - 2^{-2y+1} + 1$ and find the roots of $f(y) = 0$. One can easily observe that $f(0.5) = 0$.

If $y > 0.5$ then

$$f(y) = 2^{2y}\left(1 - 3^{-2y+1}\right) - 2^{-2y+1} + 1 > 2^1\left(1 - 3^0\right) - 2^0 + 1 = 0.$$

And if $y < 0.5$ then

$$f(y) = 2^{2y}\left(1 - 3^{-2y+1}\right) - 2^{-2y+1} + 1 < 2^1\left(1 - 3^0\right) - 2^0 + 1 = 0.$$

So the only solution of the system is $(x, y) = (-0.5, 0.5)$ and this is end of the proof.

Solution 2 by Kee-Wai Lau, Hong Kong, China

The simultaneous equations have the unique solution $(x, y) = \left(-\frac{1}{2}, \frac{1}{2}\right)$.

For $s \in \Re$ let $f(s) = 2\left(3^s\right) + 4^s - 2^s - 2$, so that

$$\frac{df(s)}{ds} = 2\ln 3\left(3^s\right) + \ln 4\left(4^s\right) - \ln 2\left(2^s\right).$$

It is easy to check that the second equation of the system is equivalent to $f(1 + x - y) = 0$. We need to show that $f(s) = 0$ if and only if $s = 0$.

Since $f(s) < 2\left(3^{-1}\right) + 4^{-1} - 2 < 0$ for $s < -1$ and $f(0) = 0$, it suffices to show that $f(s)$ is strictly increasing for $s > -1$.

But this follows immediately from the facts that

$$\frac{df(s)}{ds} > 2\ln 3\left(3^{-1}\right) + \ln 4\left(4^{-1}\right) - \ln 2 > 0 \text{ for } -1 < s \le 0, \text{ and } \frac{df(s)}{ds} > 2\ln 3 > 0 \text{ for } s > 0.$$

Hence $1 + x - y = 0$ and the first equation of the system can now be written as

$$x + \sqrt{x^2+1} = \frac{1}{y + \sqrt{y^2+1}} = \sqrt{y^2+1} - y = \sqrt{x^2+2x+2} - x - 1, \text{ or}$$

$$\left(2x + 1 + \sqrt{x^2+1}\right)^2 = x^2 + 2x + 2.$$

Expanding and simplifying the last equation, we obtain $2(2x+1)\left(x + \sqrt{x^2+1}\right) = 0$, so that $x = -\frac{1}{2}$ and $y = \frac{1}{2}$ as claimed.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

$$\text{Arcsh}x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\begin{aligned}
&= \ln \frac{1}{y+\sqrt{y^2+1}} \\
&= \ln 1 - \ln\left(y+\sqrt{y^2+1}\right) \\
&= -\text{Arsh}y = \text{Arsh(−y)} \iff \\
x &= -y.
\end{aligned}$$

If $x-y+1<0$, then $1-3^{x-y+1} > 1-3^0 = 0$ and $2^{x-y+1}-1 < 2^0-1 = 0$. So, since $2^{y-x} > 0$, we have that $0 < 2^{y-x}\left(1-3^{x-y+1}\right) = 2^{x-y+1}-1 < 0$, which is a contradiction.

And if $x-y+1>0$, then $1-3^{x-y+1} < 1-3^0 = 0$, and $2^{x-y+1}-1 > 2^0-1 = 0$, so, since $2^{y-x} > 0$, we have $0 > 2^{y-x}\left(1-3^{x-y+1}\right) = 2^{x-y+1}-1 > 0$, which is a contradiction, so $x-y+1=0$.

Hence the given system is equivalent to

$$\begin{aligned}
x+y &= 0 \\
x-y &= -1,
\end{aligned}$$

whose only solution in $\Re^2$ is $(x,y) = (-1/2, 1/2)$.

Solution 4 by David Manes, SUNY College at Oneonta, Oneonta, NY

The unique solution for the system of equations is $x = -\dfrac{1}{2}, y = \dfrac{1}{2}$.

Note that $\ln \dfrac{1}{y+\sqrt{y^2+1}} = \ln\left(\sqrt{y^2+1}-y\right)$ and the natural logarithm function is one-to-one.

Therefore, $x+\sqrt{x^2+1} = \sqrt{y^2+1}-y$. Squaring both sides of the equation yields

$$x\sqrt{x^2+1} + y\sqrt{y^2+1} = y^2 - x^2.$$

Squaring this equation one obtains

$$x^2 + y^2 + 2x^2y^2 = -2xy\sqrt{x^2+1}\sqrt{y^2+1},$$

an equation that also implies that x and y have opposite signs. Finally, squaring this equation, we get

$$\left(x^2 - y^2\right)^2 = 0 \iff |x| = |y|.$$

Therefore, $y=-x$, since $y=x$ is impossible. With $y=-x$, the second equations reduces to

$$\frac{1}{2x}\left(1-3^{2x+1}\right) = 2^{2x+1}-1.$$

If $t = 2x+1$, then this equation can be written as $4^t + 2\cdot 3^t - 2^t = 2$, whose only solution is $t = 0$; hence, $x = -\dfrac{1}{2}$ and $y = \dfrac{1}{2}$ as claimed.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro Spain (two solutions); Paul M. Harms, North Newton, KS; Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, "Tor Vergata Roma," Italy; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5203: *Proposed by Pedro Pantoja, Natal-RN, Brazil*

Evaluate,

$$\int_0^{\pi/4} \ln\left(\frac{1+\sin^2 2x}{\sin^4 x + \cos^4 x}\right) dx.$$

Solution 1 by Marius Damian, Brăila, "Nicolae Balcescu" College, Brailia, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzau, Romania

First, we have:

$$1 = \left(\sin^2 x + \cos^2 x\right)^2 = \sin^4 x + \cos^4 x + 2\sin^2 x\cos^2 x = \sin^4 x + \cos^4 x + \frac{1}{2}\sin^2 2x,$$

so

$$\sin^4 x + \cos^4 x = 1 - \frac{1}{2}\sin^2 2x.$$

Then the integral becomes:

$$\begin{aligned} I = \int_0^{\pi/4} \ln\left(\frac{1+\sin^2 2x}{1-\frac{1}{2}\sin^2 2x}\right)dx &= \int_0^{\pi/4} \ln\left[2\left(\frac{1+\sin^2 2x}{2-\sin^2 2x}\right)\right]dx \\ &= \int_0^{\pi/4}\left[\ln 2 + \ln\left(\frac{1+\sin^2 2x}{2-\sin^2 2x}\right)\right]dx \\ &= \frac{\pi\ln 2}{4} + \int_0^{\pi/4} \ln\left(\frac{1+\sin^2 2x}{2-\sin^2 2x}\right)dx. \end{aligned}$$

We denote:

$$J = \int_0^{\pi/4} \ln\left(\frac{1+\sin^2 2x}{2-\sin^2 2x}\right)dx,$$

and we substitute $t = \dfrac{\pi}{4} - x$, therefore we deduce that $J = -J$, so $J = 0$.

Hence $I = \dfrac{\pi \ln 2}{4}$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We have

$$\sin^2 x = \frac{1-\cos(2x)}{2},\ \cos^2 x = \frac{1+\cos(2x)}{2}.$$

So

$$\sin^4 x + \cos^4 x = \left(\frac{1-\cos(2x)}{2}\right)^2 + \left(\frac{1+\cos(2x)}{2}\right)^2 = \frac{1+\cos^2(2x)}{2}$$

and

$$\begin{aligned}\int_0^{\pi/4} \ln\left(\frac{1+\sin^2 2x}{\sin^4 x + \cos^4 x}\right) dx &= \int_0^{\pi/4} \ln 2 dx + \int_0^{\pi/4} \ln\left(\frac{1+\sin^2(2x)}{1+\cos^2(2x)}\right) dx \\ &= \frac{\pi}{4}\ln 2 + \frac{1}{2}\int_0^{\pi/2} \ln\left(\frac{1+\sin^2 y}{1+\cos^2 y}\right) dy \\ &= \frac{\pi}{4}\ln 2,\end{aligned}$$

since $\displaystyle\int_0^{\pi/2} \ln\left(\frac{1+\sin^2 y}{1+\cos^2 y}\right) dy = \int_0^{\pi/2} \ln\left(1+\sin^2 y\right) dy - \int_0^{\pi/2} \ln\left(1+\cos^2 y\right) dy = 0.$

Also solved by Arkady Alt, San Jose, CA; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, Sydney Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Anastasios Kotronis, Athens, Greece; Kee-Wai Lau, Hong Kong, China; David E, Manes, Oneonta, NY; Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, University "Tor Vergata Roma," Italy; Luke Sly, Joseph Kasper, and Daniel Crane (jointly, students at Taylor University), Upland, IN, and the proposer.

5204: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let $f : \Re \to \Re$ be a non-constant function such that,

$$f(x+y) = \frac{f(x)+f(y)}{1+f(x)f(y)}$$

for all $x, y \in \Re$. Show that $-1 < f(x) < 1$ for all $x \in \Re$.

Solution 1 by Michael Brozinsky, Central Islip, NY

The functional equation implies $f(0) + \Big(f(0)\Big)^3 = 2f(0)$ and so $f(0) = 0, 1$ or -1. The two latter possibilities lead to similar contradictions. For example if $f(0) = 1$ then

$$f(x) = f(x+0) = \frac{f(x)+1}{1+f(x)\cdot 1} = 1, \text{ a constant.}$$

Thus we must have $f(0) = 0$.

Now since $(u+1)^2 \geq 0$ and $(u-1)^2 \geq 0$ we have

$$-1 \leq \frac{2u}{1+u^2} \leq 1 \quad (*)$$

with equalities (on the side) occurring only if $u = 1$ or $u = -1$.

If there exits an x_0 such that $f(x_0) = 1$ then

$$f(x) = f\left((x - x_0) + x_0\right) = \frac{f(x-x_0)+1}{1+f(x-x_0)} = 1$$

contrary to the stated condition that $f(x)$ is not constant. A similar contradiction follows if there exits an x_0 such that $f(x_0) = -1$.

Finally, since $f(x) = \dfrac{2f\left(\dfrac{x}{2}\right)}{1+\left(f\left(\dfrac{x}{2}\right)\right)^2}$ we have the given inequality follows upon setting $u = f\left(\dfrac{x}{2}\right)$, and using $(*)$ and the last two results.

Solution 2 by Brian D. Beasley, Presbyterian College, Clinton, SC

Since $f(0) = 2f(0)/(1+(f(0))^2)$, we have $f(0) \in \{0, \pm 1\}$. But $f(0) = 1$ would imply $f(x) = (f(x)+1)/(1+f(x)) = 1$ for each real x, contradicting the non-constant condition of the hypothesis. Similarly, $f(0) = -1$ would imply $f(x) = (f(x)-1)/(1-f(x)) = -1$ for each real x, another contradiction. Thus $f(0) = 0$. This yields

$$0 = \frac{f(x)+f(-x)}{1+f(x)f(-x)}$$

and hence $f(-x) = -f(x)$ for each real x. Also, given any real x, we have $f(x) = 2f(x/2)/(1+(f(x/2))^2)$.

If $f(x) \geq 1$ for some real x, then $2f(x/2) \geq 1 + (f(x/2))^2$, so $0 \geq (f(x/2)-1)^2$ and thus $f(x/2) = 1$. Then $f(x) = 1$ and $f(2x) = 1$, but $f(-x) = -1$, which would mean that

$$f(x) = \frac{f(2x)+f(-x)}{1+f(2x)f(-x)}$$

is undefined.

Similarly, if $f(x) \leq -1$ for some real x, then $2f(x/2) \leq -1 - (f(x/2))^2$, so $(f(x/2)+1)^2 \leq 0$ and thus $f(x/2) = -1$. Then $f(x) = -1$ and $f(2x) = -1$, but $f(-x) = 1$, which would again mean that $f(x)$ is undefined.

Hence $-1 < f(x) < 1$ for each real x.

Solution 3 by Arkady Alt, San Jose, CA

First note that $f(x) \cdot\ f(y) \neq -1$ for any $x, y \in R$.

Since $f(x) = f\left(\dfrac{x}{2} + \dfrac{x}{2}\right) = \dfrac{2f\left(\dfrac{x}{2}\right)}{1 + \ f^2\left(\dfrac{x}{2}\right)} \Rightarrow |f(x)| = \dfrac{2\left|f\left(\dfrac{x}{2}\right)\right|}{1 + \left|\ f\left(\dfrac{x}{2}\right)\right|^2}$

and then we have $\left(\left|f\left(\dfrac{x}{2}\right)\right| - 1\right)^2 \geq 0 \iff \dfrac{2\left|f\left(\dfrac{x}{2}\right)\right|}{1 + \left|\ f\left(\dfrac{x}{2}\right)\right|^2} \leq 1 \iff |f(x)| \leq 1.$

If we suppose $|f(x_0)| = 1,$ for some $x_0,$ then $\left|f\left(\dfrac{x_0}{2}\right)\right| = 1$ and $f(x)$ becomes a constant function. Indeed, if $f(x_0) = 1,$ then for any $x \in R$ we have $f(x + x_0) = \dfrac{f(x) +\ 1}{1 +\ f(x)} = 1,$ because $f(x) = f(x) \cdot f(x_0) \neq -1.$

If $f(x_0) = -1,$ then for any $x \in R$ we have $f(x + x_0) = \dfrac{f(x) -\ 1}{1 -\ f(x)} = -1,$ because $-f(x) = f(x) \cdot f(x_0) \neq -1.$ Thus, $|f(x)| < 1 \iff -1 < f(x) < 1$ for any x.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo TX; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, Sydney Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; David Manes, Oneonta, NY; Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, University "Tor Vergata Roma," Italy; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Statesboro, GA; Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, Buzău, Romania, and the proposer.

5205: *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

Find the sum,

$$\sum_{n=1}^{\infty}\left(1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{(-1)^{n-1}}{n} - \ln 2\right) \cdot \ln\frac{n+1}{n}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

For each integer $m > 1$, is easy to prove by induction that

$$\sum_{n=1}^{m}\left(1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{(-1)^{n-1}}{n}\right)\ln\frac{n+1}{n}$$

$$= \left(1-\frac{1}{2}+\frac{1}{3}+\cdots+\frac{(-1)^{m-1}}{m}\right)\ln(m+1)+\sum_{n=2}^{m}\frac{(-1)^n\ln n}{n}.$$

Since

$$\left|1-\frac{1}{2}+\frac{1}{3}+\cdots+\frac{(-1)^{m-1}}{m}-\ln 2\right|$$

$$= \frac{1}{m+1}\left(1-\frac{m+1}{m+2}+\frac{m+1}{m+3}-\frac{m+1}{m+4}+\cdots\right)<\frac{1}{m+1},$$

so

$$\lim_{m\to\infty}\left(1-\frac{1}{2}+\frac{1}{3}+\cdots+\frac{(-1)^{m-1}}{m}-\ln 2\right)\ln(m+1)=0.$$

It is known [E. R. Hansen: *A Table of Series and Products,* Prentice-Hall, Inc., 1975, p. 288 entry (44.1.8)] that $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n\ln n}{n}=\gamma\ln 2-\frac{(\ln 2)^2}{2}$, where γ is Euler's constant. Hence the sum of the problem equals $\gamma\ln 2-\dfrac{(\ln 2)^2}{2}=0.1598\ldots.$

Solution 2 by Paolo Perfetti, Department of Mathematics, University "Tor Vergata Roma," Italy

By writing $q_n=1-\frac{1}{2}+\frac{1}{3}+\ldots+\frac{(-1)^{n-1}}{n}-\ln 2$ the series is

$$\sum_{n=1}^{\infty}q_n\ln\frac{n+1}{n}=\sum_{n=1}^{\infty}\left((q_n\ln(n+1)-q_{n-1}\ln n)+\ln n(q_{n-1}-q_n)\right)$$

$$\sum_{n=1}^{\infty}(q_n\ln(n+1)-q_{n-1}\ln n)=\lim_{n\to\infty}q_n\ln(n+1).$$

The series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

is Leibniz and converges to $\ln 2$ thus it satisfies

$$\left|\ln 2-\sum_{n=1}^{r}\frac{(-1)^{n-1}}{n}\right|\leq\frac{1}{r+1}.$$

Since this is a well known property of all Leibniz series present in all books on the subject, we omit it. The immediate consequence is

$$\lim_{n\to\infty}q_n\ln(n+1)=0.$$

We remain with

$$\sum_{n=1}^{\infty} \ln n(q_{n-1}-q_n) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \ln n = \gamma \ln 2 - \frac{1}{2}\ln^2 2$$

where γ is the Euler–Mascheroni constant. Also $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \ln n = \gamma \ln 2 - \frac{1}{2}\ln^2 2$ is a well known result. Nevertheless we write it here. For $p \geq 4$,

$$\sum_{k=2}^{2p}(-1)^k \frac{\ln k}{k} = \sum_{k=1}^{p} \frac{\ln 2}{2k} + \sum_{k=1}^{p} \frac{\ln k}{2k} - \sum_{k=1}^{p-1} \frac{\ln(2k+1)}{2k+1}.$$

$$-\sum_{k=1}^{p-1} \frac{\ln(2k+1)}{2k+1} = -\sum_{k=2}^{2p-1} \frac{\ln k}{k} + \sum_{k=1}^{p-1} \frac{\ln(2k)}{2k} = -\sum_{k=2}^{2p-1} \frac{\ln k}{k} + \sum_{k=1}^{p-1} \frac{\ln 2}{2k} + \sum_{k=1}^{p-1} \frac{\ln k}{2k}.$$

By summing we get

$$\sum_{k=1}^{p-1} \frac{\ln 2}{k} + \frac{\ln 2}{2p} + \frac{\ln p}{2p} - \sum_{k=p}^{2p-1} \frac{\ln k}{k}.$$

Now we employ the well known

$\sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + o(1)$. Moreover we observe that

$$\int_p^{2p} \frac{\ln x}{x} dx \leq \sum_{k=p}^{2p-1} \frac{\ln k}{k} = \int_{p-1}^{2p-1} \frac{\ln x}{x} dx,$$

(*Editor's note:* We note that the function $\frac{\ln x}{x}$ is decreasing for $x \geq e$. So $\int_k^{k+1} \frac{\ln x}{x} dx \leq \frac{\ln k}{k} \leq \int_{k-1}^{k} \frac{\ln x}{x} dx$. The claimed inequalities follow by summing over k from $k = p$ to $k = 2p-1$.)

thus

$$\begin{aligned}
\sum_{k=p}^{2p-1} \frac{\ln k}{k} &= \int_p^{2p-1} \frac{\ln x}{x} dx + o(1) = \frac{1}{2}(\ln^2(2p-1) - \ln^2 p) + o(1) \\
&= \frac{\ln^2 2}{2} + \frac{\ln^2 p}{2} + \ln 2 \ln p + \ln(2p)\ln(1-\frac{1}{2p}) + \frac{1}{2}\ln^2(1-\frac{1}{2p}) - \frac{\ln^2 p}{2} + o(1) \\
&= \frac{\ln^2 2}{2} + \ln 2 \ln p + o(1).
\end{aligned}$$

We get

$$\sum_{k=1}^{p-1}\frac{\ln 2}{k}+\frac{\ln 2}{2p}+\frac{\ln p}{2p}-\sum_{k=p}^{2p-1}\frac{\ln k}{k}=\ln 2(\ln(p-1)+\gamma)-\frac{\ln^2 2}{2}-\ln 2\ln p+o(1)$$

$$=\quad \gamma\ln 2-\frac{\ln^2 2}{2},\ \text{ as } p\to\infty.$$

Solution 3 by Anastasios Kotronis, Athens, Greece

We set

$$f_m(x)=\sum_{n=1}^{m}\left(-\sum_{k=1}^{n}\frac{x^k}{k}-\ln(1-x)\right)\ln\left(\frac{n+1}{n}\right)\qquad x<1,$$

and we wish to find

$$\lim_{m\to+\infty} f_m(-1).$$

For $x<1$ we have

$$\begin{aligned}
f_m'(x) &= \left(\sum_{n=1}^{m}\left(-\sum_{k=1}^{n}\frac{x^k}{k}-\ln(1-x)\right)\ln\left(\frac{n+1}{n}\right)\right)'\\
&= \sum_{n=1}^{m}\left(-\sum_{k=0}^{n-1}x^k+\frac{1}{1-x}\right)\ln\left(\frac{n+1}{n}\right)\\
&= \sum_{n=1}^{m}\left(-\frac{1-x^n}{1-x}+\frac{1}{1-x}\right)\ln\left(\frac{n+1}{n}\right)\\
&= \frac{1}{1-x}\sum_{n=1}^{m}x^n\left(\ln(n+1)-\ln n\right)\\
&= \frac{1}{1-x}\left(\sum_{n=2}^{m}\left(x^{n-1}-x^n\right)\ln n+x^m\ln(m+1)\right)\\
&= \sum_{n=2}^{m}x^{n-1}\ln n+\frac{x^m}{1-x}\ln(m+1).
\end{aligned}$$

So we integrate from 0 to y, where $y<1$, to get

$$f_m(y)=\sum_{n=2}^{m}\frac{y^n}{n}\ln n+\ln(m+1)\int_0^y\frac{x^m}{1-x}\,dx$$

and set $y=-1$ to get

$$\begin{aligned}
f_m(-1) &= \sum_{n=2}^{m} \frac{(-1)^n}{n} \ln n + \ln(m+1) \int_0^{-1} \frac{x^m}{1-x}\, dx \\
&\overset{x=-t}{===} \sum_{n=2}^{m} \frac{(-1)^n}{n} \ln n + (-1)^{m+1} \ln(m+1) \int_0^{1} \frac{t^m}{1+t} dt \\
&= A_m + (-1)^{m+1} \ln(m+1) B_m. \qquad (1)
\end{aligned}$$

Now integrating by parts,

$$\begin{aligned}
B_m &= \left.\frac{t^{m+1}}{(m+1)(1+t)}\right|_0^1 + \frac{1}{m+1} \int_0^1 \frac{t^{m+1}}{(1+t)^2}\, dt \\
&\leq \frac{1}{2(m+1)} + \frac{1}{m+1} \int_0^1 \frac{1}{(1+t)^2}\, dt \\
&= \frac{1}{m+1} < \frac{1}{m} \qquad (2)
\end{aligned}$$

and for A_m, since it converges from Leibniz Criterion, (see: http://mathworld.wolfram.com/Leibniz Criterion.html) we can write

$$\lim_{m\to+\infty} A_m = \lim_{m\to+\infty} A_{2m}$$

and

$$\begin{aligned}
A_{2m} &= \sum_{n=1}^{2m} \frac{(-1)^n}{n} \ln n \\
&= \sum_{n=1}^{m} \frac{\ln 2n}{2n} - \sum_{n=1}^{m} \frac{\ln(2n-1)}{2n-1} \\
&= \frac{\ln 2}{2} \sum_{n=1}^{m} \frac{1}{n} + \frac{1}{2} \sum_{n=1}^{m} \frac{\ln n}{n} - \left(\sum_{n=1}^{2m} \frac{\ln n}{n} - \sum_{n=1}^{m} \frac{\ln 2n}{2n} \right) \\
&= \ln 2 H_m + \sum_{n=1}^{m} \frac{\ln n}{n} - \sum_{n=1}^{2m} \frac{\ln n}{n} \\
&= \ln 2 H_m - \sum_{n=1}^{m} \frac{\ln(m+n)}{m+n} \\
&= \ln 2 H_m - \sum_{n=1}^{m} \frac{\ln m + \ln(1+n/m)}{m+n} \\
&= \ln 2 H_m - \ln m (H_{2m} - H_m) - \frac{1}{m} \sum_{n=1}^{m} \frac{\ln(1+n/m)}{1+n/m}
\end{aligned}$$

$$
\begin{aligned}
&= \quad H_m \ln(2m) - H_{2m} \ln m - \frac{1}{m} \sum_{n=1}^{m} \frac{\ln(1+n/m)}{1+n/m} \\
&\overset{H_m = \ln m + \gamma + O(1/m)}{=} \quad \gamma \ln 2 + O(1/m) - \frac{1}{m} \sum_{n=1}^{m} \frac{\ln(1+n/m)}{1+n/m} \qquad (3)
\end{aligned}
$$

Now with (2) and (3), (1) will give

$$
f_m(-1) \to \gamma \ln 2 - \int_0^1 \frac{\ln(1+x)}{1+x}\, dx = \gamma \ln 2 - \frac{\ln^2 2}{2}.
$$

Comment: In fact, one can easily show that

$$
\frac{1}{m} \sum_{n=1}^{m} \frac{\ln(1+n/m)}{1+n/m} = \frac{\ln^2 2}{2} + O(1/m), \quad \text{so}
$$

$$
\sum_{n=1}^{m} \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-1}}{n} - \ln 2\right) \cdot \ln\left(\frac{n+1}{n}\right) = \gamma \ln 2 - \frac{\ln^2 2}{2} + O\left(m^{-1} \ln m\right).
$$

Editor's comment: The sum in (3) is a Riemann sum whose limit as m tends to infinity equals the Riemann integral.

Solution 4 by Arkady Alt, San Jose, CA

Let $h_n = \sum\limits_{k=1}^{n} \frac{1}{k}, a_n = \sum\limits_{k=1}^{n} \frac{(-1)^{k-1}}{k} - \ln 2$, and $S = \sum\limits_{n=1}^{\infty} a_n \ln \frac{n+1}{n}$.

Note that

$$
\begin{aligned}
\sum_{k=1}^{n} a_k \ln \frac{k+1}{k} &= \sum_{k=1}^{n} a_k \left(\ln(k+1) - \ln k\right) \\
&= \sum_{k=1}^{n} a_k \ln(k+1) - \sum_{k=1}^{n} a_k \ln k \\
&= \sum_{k=2}^{n+1} a_{k-1} \ln k - \sum_{k=2}^{n} a_k \ln k \\
&= a_n \ln(n+1) - \sum_{k=2}^{n} (a_k - a_{k-1}) \ln k \\
&= a_n \ln(n+1) - \sum_{k=2}^{n} \frac{(-1)^{k-1} \ln k}{k} \\
&= a_n \ln(n+1) + \sum_{k=2}^{n} \frac{(-1)^{k} \ln k}{k}.
\end{aligned}
$$

First we will prove $\lim\limits_{n\to\infty} a_n \ln(n+1) = 0$.

Since $a_{2n+1} = a_{2n} + \dfrac{1}{2n+1}$ then it suffices to prove

$$\lim_{n\to\infty} a_{2n} \ln(2n+1) = 0.$$

We have $a_{2n} = h_{2n} - h_n - \ln 2$ and, since $\ln n + \gamma < h_n < \ln(n+1) + \gamma$, where $\gamma = \lim\limits_{n\to\infty} (h_n - \ln n)$ is Euler's constant, then

$$\begin{aligned}
& \ln 2n - \ln(n+1) - \ln 2 < a_{2n} < \ln(2n+2) - \ln n - \ln 2 \\
\iff \quad & -\ln\frac{n+1}{n} < a_{2n} < \ln\frac{n+1}{n} \\
\iff \quad & |a_{2n}| < \ln\frac{n+1}{n} < \frac{1}{n} \\
& \left(1+\frac{1}{n}\right)^n < e \iff \ln\frac{n+1}{n} < \frac{1}{n}.
\end{aligned}$$

Hence, $0 < |a_{2n}| \ln(2n+1) < \frac{\ln(2n+1)}{n}$ yields $\lim\limits_{n\to\infty} \frac{\ln(2n+1)}{n} = 0$, and, therefore $\lim\limits_{n\to\infty} a_{2n} \ln(2n+1) = 0$.

Thus, $S = \lim\limits_{n\to\infty} \sum\limits_{k=2}^{n} s_n$, where $s_n := \sum\limits_{k=2}^{n} \frac{(-1)^k \ln k}{k}$.

Since $s_{2n+1} = s_{2n} - \frac{\ln(2n+1)}{2n+1}$ and $\lim\limits_{n\to\infty} \frac{\ln(2n+1)}{2n+1} = 0$ then $S = \lim\limits_{n\to\infty} s_{2n}$.

Let $b_n := \sum\limits_{k=1}^{n} \frac{\ln k}{k}$ then

$$\begin{aligned}
s_{2n} &= \sum_{k=1}^{2n} \frac{(-1)^k \ln k}{k} \\
&= \sum_{k=1}^{n} \frac{\ln 2k}{2k} - \sum_{k=1}^{n} \frac{\ln(2k-1)}{2k-1} \\
&= 2\sum_{k=1}^{n} \frac{\ln 2k}{2k} - \sum_{k=1}^{2n} \frac{\ln k}{k} \\
&= \sum_{k=1}^{n} \frac{\ln 2k}{k} - b_{2n} \\
&= \sum_{k=1}^{n} \frac{\ln 2}{k} + \sum_{k=1}^{n} \frac{\ln k}{k} - b_{2n} \\
&= \ln 2 \cdot h_n + b_n - b_{2n}.
\end{aligned}$$

Consider now two sequences $\left(b_n - \frac{\ln^2(n+1)}{2}\right)_{n\geq 1}$ and $\left(b_n - \frac{\ln^2 n}{2}\right)_{n\geq 1}$.

Since $b_n - \frac{\ln^2(n+1)}{2}$ is increasing and $b_n - \frac{\ln^2 n}{2}$ is decreasing in n then

$$b_1 - \frac{\ln^2 2}{2} \leq b_n - \frac{\ln^2(n+1)}{2} < b_n - \frac{\ln^2 n}{2} \leq b_1$$

and, therefore, both sequences converges to the same limit.

Let $\delta = \lim\limits_{n\to\infty}\left(b_n - \frac{\ln^2(n+1)}{2}\right) = \lim\limits_{n\to\infty}\left(b_n - \frac{\ln^2 n}{2}\right)$ then

$$b_n - \frac{\ln^2(n+1)}{2} < \delta < b_n - \frac{\ln^2 n}{2} \iff \frac{\ln^2 n}{2} + \delta < b_n < \frac{\ln^2(n+1)}{2} + \delta, n \in N.$$

Hence,

$$\begin{aligned} \frac{\ln^2 2n - \ln^2(n+1)}{2} &< b_{2n} - b_n < \frac{\ln^2(2n+2) - \ln^2 n}{2} \iff \\ \beta_n &< b_{2n} - b_n - \ln 2 \cdot \ln n < \alpha_n, \end{aligned}$$

where $\alpha_n = \dfrac{\ln^2(2n+2) - \ln^2 n}{2} - \ln 2 \cdot \ln n$ and $\beta_n = \dfrac{\ln^2 2n - \ln^2(n+1)}{2} - \ln 2 \cdot \ln n$.

Noting that

$\dfrac{\ln^2 2n - \ln^2 n}{2} - \ln 2 \cdot \ln n = \dfrac{\ln 2\,(\ln 2 + 2\ln n)}{2} - \ln 2 \cdot \ln n = \dfrac{\ln^2 2}{2}$, we obtain

$$\lim_{n\to\infty}\left(\alpha_n - \frac{\ln^2 2}{2}\right) = \lim_{n\to\infty}\left(\frac{\ln^2(2n+2) - \ln^2 2n}{2}\right) = \frac{1}{2}\lim_{n\to\infty}\ln\left(\frac{n+1}{n}\right)\ln(4n(n+1)) = 0, \text{ and}$$

$$\lim_{n\to\infty}\left(\beta_n - \frac{\ln^2 2}{2}\right) = \lim_{n\to\infty}\frac{\ln^2 n - \ln^2(n+1)}{2} = -\frac{1}{2}\lim_{n\to\infty}\ln\left(\frac{n+1}{n}\right)\ln(n(n+1)) = 0.$$

This gives us

$$\lim_{n\to\infty}(b_{2n} - b_n - \ln 2 \cdot \ln n) = \lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = \frac{\ln^2 2}{2}.$$

Since $\lim\limits_{n\to\infty}(h_n - \ln n) = \gamma$ then

$$\begin{aligned} S &= \lim_{n\to\infty}(b_n - b_{2n} + \ln 2 \cdot h_n) \\ &= \lim_{n\to\infty}(b_n - b_{2n} + \ln 2 \cdot \ln n + \ln 2 \cdot (h_n - \ln n)) \\ &= \lim_{n\to\infty}(b_n - b_{2n} + \ln 2 \cdot \ln n) + \lim_{n\to\infty}\ln 2 \cdot (h_n - \ln n) \end{aligned}$$

$$= \quad \ln 2 \cdot \left(\gamma - \frac{\ln 2}{2} \right).$$

Also solved by Adrian Naco, Polytechnic University, Tirana, Albania; Albert Stadler, Herrliberg, Switzerland, and the proposer.

Editor's comment: Mea Culpa once again. I inadvertently gave credit to David Stone and John Hawkins for having solved problem 5199 when they should have been credited for having solved 5198. And I inadvertently forgot to acknowledge **Achilleas Sinefakopoulos of Larissa, Greece** for having correctly solved 5184.