Problems

Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
January 15, 2009

- 5032: *Proposed by Kenneth Korbin, New York, NY.*

  Given positive acute angles A, B, C such that
  $$\tan A \cdot \tan B + \tan B \cdot \tan C + \tan C \cdot \tan A = 1.$$
  Find the value of
  $$\frac{\sin A}{\cos B \cdot \cos C} + \frac{\sin B}{\cos A \cdot \cos C} + \frac{\sin C}{\cos A \cdot \cos B}.$$

- 5033: *Proposed by Kenneth Korbin, New York, NY.*

  Given quadrilateral $ABCD$ with coordinates $A(-3,0), B(12,0), C(4,15)$, and $D(0,4)$. Point P is on side $\overline{AB}$ and point Q is on side $\overline{CD}$. Find the coordinates of P and Q if area $\triangle PCD =$ area $\triangle QAB = \frac{1}{2}$area quadrilateral $ABCD$.

- 5034: *Proposed by Roger Izard, Dallas, TX.*

  In rectangle $MDCB$, $MB \perp MD$. F is the midpoint of BC, and points N, E and G lie on line segments DC, DM, and MB respectively, such that $NC = GB$. Let the area of quadrilateral $MGFC$ be A_1 and let the area of quadrilateral $MGFE$ be A_2. Determine the area of quadrilateral $EDNF$ in terms of A_1 and A_2.

- 5035: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be positive numbers. Prove that
  $$(a^a b^b c^c)^2 (a^{-(b+c)} + b^{-(c+a)} + c^{-(a+b)})^3 \geq 27.$$

- 5036: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Find all triples (x, y, z) of nonnegative numbers such that

$$\begin{cases} x^2+y^2+z^2=1 \\ 3^x+3^y+3^z=5 \end{cases}$$

- 5037: *Ovidiu Furdui, Toledo, OH.*

  Let k, p be natural numbers. Prove that

$$1^k+3^k+5^k+\cdots+(2n+1)^k=(1+3+\cdots+(2n+1))^p$$

  for all $n \geq 1$ if and only if $k=p=1$.

Solutions

- 5014: *Proposed by Kenneth Korbin, New York, NY.*

  Given triangle ABC with $a=100, b=105$, and with equal cevians $\overline{AD}$ and $\overline{BE}$. Find the perimeter of the triangle if $\overline{AE}\cdot\overline{BD}=\overline{CE}\cdot\overline{CD}$.

  Solution by David Stone and John Hawkins, Statesboro, GA.

  The solution to this problem is more complex than expected. There are infinitely many triangles satisfying the given conditions, governed in a sense by two types of degeneracy. The nicest of these solutions is a right triangle with integer sides, dictated by the given data: $100=5(20)$ and $105=5(21)$ and $(20, 21, 29)$ is a Pythagorean triple.

  One type of degeneracy is the usual: if $AB=5$ or $AB=205$, we have a degenerate triangle which can be shown to satisfy the conditions of the problem.

  The other type of degeneracy is problem specific: when neither cevian intersects the interior of its targeted side, but lies along a side of the triangle. In these two situations, the problem's condition are also met.

  Let $x=$ length of CE so $0\leq x\leq 105$. The following table summarizes our results.

$x=CE$	$\cos(C)$	C	AB	BD	*Perimeter*	$AD=BE$	*note*
0	$\frac{21}{40}$	$\cos^{-1}\left(\frac{21}{40}\right)$	100	0	305	100	1
21	$\frac{194}{350}$	$\cos^{-1}\left(\frac{194}{350}\right)$	$\sqrt{9385}$	20	301.88	$\sqrt{8113}$	2
$\frac{1985}{41}$	1	0	5	$\frac{1900}{41}$	210	$\frac{2105}{41}$	3
excluded	values	$*****$	$*****$	$*****$	$*****$	$*****$	
$\frac{2205}{41}$	-1	180^o	205	$\frac{2100}{41}$	410	$\frac{6305}{41}$	3
$\frac{105}{41}(\sqrt{178081}-400)$	0	90^o	145	≈ 53.65	350	≈ 114.775	4
105	$\frac{10}{21}$	$\cos^{-1}\left(\frac{10}{21}\right)$	105	100	310	105	5

Notes:

1. Cevian BE is side BC; Cevian AD is side AB. **2.** A "nice" value for x. **3.** Degenerate triangle. **4.** Right triangle. **5.** Cevian BE is side AB; Cevian AD is side AC.

In short, the perimeter assumes all values in $[210, 305] \cup [310, 410]$.

Now we support these assertions. Consider $\triangle ABC$ with cevians BE (from $\angle B$ to side AC) and AD (from $\angle A$ to side BC). Let $CE = x, AE = 105 - x$ and $CD = 100 - BD$. To find the perimeter, we only need to compute AB.

We have $AE = 105 - x$, so

$$\begin{aligned} AE \cdot BD &= CE \cdot CD \\ (105 - x)BD &= x(100 - BD) \\ 100x &= 105BD \\ \text{so } BD &= \frac{20}{21}x \text{ and } CD = 100 - \frac{20}{21}x. \end{aligned}$$

Applying the Law of Cosines three times, we have

$$\begin{array}{ll} (1) & BE^2 = x^2 + 100^2 - 2(100)x\cos C \\ (2) & AD^2 = CD^2 + 105^2 - 2(105)CD\cos(C) \text{ and} \\ (3) & AB^2 = 100^2 + 105^2 - 2 \cdot 100 \cdot 105\cos(C). \end{array}$$

Because we must have $AD = BE$, we combine (1) and (2) to get

$$\begin{aligned} CD^2 + 105^2 - 2(105)CD\cos(C) &= x^2 + 100^2 - 2(100)x\cos(C) \text{ or} \\ \left(100 - \frac{20}{21}x\right)^2 + 105^5 - 210\left(100 - \frac{20}{21}x\right)\cos(C) &= x^2 + 100^2 - 2(100)x\cos(C). \end{aligned}$$

Solving for $\cos(C)$, we obtain a rational expression in x:

$$(4) \quad \cos(C) = \frac{41x^2 + 84000x - 2205^2}{21^2 \cdot 200(2x - 105)}.$$

Substituting this value into (3) we have

$$\begin{aligned} AB^2 &= 21025 - 2100 \cdot \frac{41x^2 + 84000x - 2205^2}{21^2 \cdot 200(200x - 105)}, \text{ so} \\ (5)\ AB^2 &= \frac{5}{21}\frac{41x^2 + 92610x + 4410000}{105 - 2x}. \end{aligned}$$

Thus we can then calculate AB and the perimeter

$$P = 205 + \sqrt{\frac{5}{21}\frac{41x^2 + 92610x + 4410000}{105 - 2x}}.$$

The graphs of $\cos(C)$ and of AB have vertical asymptotes at $x = \dfrac{105}{2}$, in the center of our interval $[0, 105]$. Other than an interval bracketing this singularity, each value of x produces a solution to the problem.

We explore the endpoints and the "degenerate" solutions, obtaining the values exhibited in the table above.

I. $x = 0$: That is $CE = 0$, so $E = C$ and the cevian from vertex B is actually the side BC. Therefore, $BE = BC = 100$. Hence, the condition

$$\begin{aligned} AE \cdot BD &= CE \cdot CD \text{ becomes} \\ AC \cdot BD &= 0 \cdot CD \text{ or} \\ 100 \cdot BD &= 0. \end{aligned}$$

Thus $BD = 0$, so $D = B$ and the cevian from vertex A is actually the side AB; $AD = AB$.
Computing by (4) and (5): $\cos(C) = \dfrac{21}{40}$ and $AB = 100$. Thus
$AD = AB = 100 = BC = BE$, so this triangle satisfies the required conditions. Its perimeter is 305.

II. $x = 105$**:** gives a similar result, a $(105, 105, 100)$ triangle with cevians lying along the sides and $P = 310$.

III. The degenerate case $C = 0$ occurs when $\cos(C) = 1$. By (4), this happens when $x = \dfrac{1985}{41}$. Also, $C = 0$ if and only if $AB = 5$, which is the smallest possible value (by the Triangle Inequality).

IV. The degenerate case $C = \pi$ occurs when $\cos(C) = -1$. By (4) this happens when $x = \dfrac{2205}{41}$. Also $C = \pi$ if and only if $AB = 205$, which is the largest possible value (by the Triangle Inequality).

The values of x appearing in III and IV are the endpoints of the interval of excluded values bracketing $\dfrac{105}{2}$.

V. The degenerate case $C = \pi/2$ occurs when $\cos(C) = 0$. By (4), this happens when x takes on the ugly irrational $\dfrac{105}{41}\left(\sqrt{178081} - 400\right)$. In this case, $AB = 145$ and our triangle is the $(20, 21, 29)$ Pythagorean triangle scaled up by a factor of 5. The common value of the cevians is $AD = BE = \dfrac{5}{41}\sqrt{49788121 - 352800\sqrt{178081}} \approx 114.775$.

VI. Because $BD = \dfrac{20}{21}x$, some nice results occur when x is a multiple of 21. The table shows the values for $x = 21$.

Excel has produced many values of these triangles, letting x range from 0 to 105, except for the excluded interval $\left(\dfrac{1985}{41}, \dfrac{2205}{41}\right)$, but in summary,

- the perimeter assumes all values in $[210, 305] \cup [310, 410]$.
- side AB assumes all values in $[5, 100] \cup [105, 205]$.
- $\angle C$ assumes all values in

$$\left[0, \cos^{-1}\left(\frac{21}{40}\right)\right] \cup \left[\cos^{-1}\left(\frac{10}{21}, 180^o\right)\right] = [0, 58.33^o] \cup [61.56^o, 180^o].$$

– The common cevians achieve the values

$$\left[\frac{2105}{41}, 100\right] \cup \left[105, \frac{6305}{41}\right] \approx [51.34, 100] \cup [105, 153.78].$$

Our final comment: AB assumes all integer values in $[5, 100] \cup [105, 205]$, so the right triangle described above is not the only solution with all sides integral. For any integer AB in $[5, 100] \cup [105, 205]$, we can use (5) to determine the appropriate value of x, C, etc. Of course, this raises another question: are any of these triangles Heronian?

Also solved by the proposer.

5015: *Proposed by Kenneth Korbin, New York, NY.*

Part I: Find the value of

$$\sum_{x=1}^{10} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right).$$

Part II: Find the value of

$$\sum_{x=1}^{\infty} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right).$$

Solution by David C. Wilson, Winston-Salem, N.C.

First, let's look for a pattern.

$\mathbf{x = 1}$: $\text{Arcsin}(\frac{4}{5})$.

$\mathbf{x = 2}$: $\text{Arcsin}(\frac{4}{5}) + \text{Arcsin}(\frac{16}{65}) = \text{Arcsin}(\frac{12}{13})$.

Let $\theta = \text{Arcsin}(\frac{4}{5})$ and $\phi = \text{Arcsin}(\frac{16}{65})$.

$$\sin\theta = \frac{4}{5} \qquad \sin\phi = \frac{16}{65}$$

$$\cos\theta = \frac{3}{5} \qquad \cos\phi = \frac{63}{65}$$

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi = (\frac{4}{5})(\frac{63}{65}) + (\frac{3}{5})(\frac{16}{65}) = \frac{300}{325} = \frac{12}{13} = \text{Arcsin}(\frac{12}{13}).$$

$\mathbf{x = 3}$: $\text{Arcsin}(\frac{12}{13}) + \text{Arcsin}(\frac{36}{325}) = \text{Arcsin}(\frac{24}{25})$.

Let $\theta = \text{Arcsin}(\frac{12}{13})$ and $\phi = \text{Arcsin}(\frac{36}{325})$.

$$\sin\theta = \frac{12}{13} \qquad \sin\phi = \frac{36}{325}$$

$$\cos\theta = \frac{5}{13} \qquad \cos\phi = \frac{323}{325}$$

$$\sin(\theta+\phi) = (\frac{12}{13})(\frac{323}{325}) + (\frac{5}{13})(\frac{36}{325}) = \frac{4056}{4225} = \frac{24}{25} = \text{Arcsin}(\frac{24}{25}).$$

$\mathbf{x = 4}$: $\text{Arcsin}(\frac{24}{25}) + \text{Arcsin}(\frac{64}{1025}) = \text{Arcsin}(\frac{40}{41})$.

Let $\theta = \text{Arcsin}(\frac{24}{25})$ and $\phi = \text{Arcsin}(\frac{64}{1025})$.

$$\sin\theta = \frac{24}{25} \qquad \sin\phi = \frac{64}{1025}$$

$$\cos\theta = \frac{7}{25} \qquad \cos\phi = \frac{1023}{1025}$$

$$\sin(\theta+\phi) = (\frac{24}{25})(\frac{1023}{1025}) + (\frac{7}{25})(\frac{64}{1025}) = \frac{25000}{25625} = \frac{40}{41} = \text{Arcsin}(\frac{40}{41}).$$

Therefore, the conjecture is

$$\sum_{x=1}^{n} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) = \text{Arcsin}\left(\frac{2n^2+2n}{2n^2+2n+1}\right).$$

Proof is by induction.

1) For $n = 1$, we obtain $\text{Arcsin}(\frac{4}{5}) = \text{Arcsin}(\frac{4}{5})$.

2) Assume true for n; i.e.,

$$\sum_{x=1}^{n} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) = \text{Arcsin}\left(\frac{2n^2+2n}{2n^2+2n+1}\right).$$

3) For $n+1$, we have

$$\begin{aligned}
\sum_{x=1}^{n+1} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) &= \sum_{x=1}^{n} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) + \text{Arcsin}\left(\frac{4(n+1)^2}{4(n+1)^4+1}\right) \\
&= \sum_{x=1}^{n} \text{Arcsin}\left(\frac{2n^2+2n}{2n^2+2n+1}\right) + \text{Arcsin}\left(\frac{4(n+1)^2}{4(n+1)^4+1}\right).
\end{aligned}$$

Let $\theta = \text{Arcsin}\left(\frac{\text{2n}^2+\text{2n}}{\text{2n}^2+\text{2n}+1}\right)$ and $\phi = \text{Arcsin}\left(\frac{4(\text{n}+1)^2}{4(\text{n}+1)^4+1}\right)$.

$$\sin\theta = \frac{2n^2+2n}{2n^2+2n+1} \qquad \sin\phi = \frac{4(n+1)^2}{4(n+1)^4+1}$$

$$\cos\theta = \frac{2n+1}{2n^2+2n+1} \qquad \cos\phi = \frac{4(n+1)^4-1}{4(n+1)^4+1}$$

$$\begin{aligned}
\sin(\theta+\phi) &= \sin\theta\cos\phi + \cos\theta\sin\phi \\
&= \left(\frac{2n^2+2n}{2n^2+2n+1}\right)\left[\frac{4(n+1)^4-1}{4(n+1)^4+1}\right] + \left(\frac{2n+1}{2n^2+2n+1}\right)\left[\frac{4(n+1)^2}{4(n+1)^4+1}\right] \\
&= \frac{8n^6+40n^5+80n^4+88n^3+58n^2+22n+4}{(2n^2+2n+1)(2n^2+6n+5)(2n^2+2n+1)} \\
&= \frac{(2n^2+2n+1)^2(2n^2+6n+4)}{(2n^2+2n+1)^2(2n^2+6n+5)} = \frac{2n^2+6n+4}{2n^2+6n+5} = \frac{2(n+1)^2+2(n+1)}{2(n+1)^2+2(n+1)+1}.
\end{aligned}$$

Thus $\sum_{x=1}^{n+1} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) = \text{Arcsin}\left[\frac{2(n+1)^2+2(n+1)}{2(n+1)^2+2(n+1)+1}\right]$ and this proves the conjecture.

Part I: $\sum_{x=1}^{10} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) = \text{Arcsin}\left[\frac{220}{221}\right].$

Part II:

$$\begin{aligned}
\sum_{x=1}^{\infty} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) &= \lim_{n\to\infty} \sum_{x=1}^{n} \text{Arcsin}\left[\frac{4x^2}{4x^4+1}\right] \\
&= \lim_{n\to\infty} \text{Arcsin}\left(\frac{2n^2+2n}{2n^2+2n+1}\right) \\
&= \text{Arcsin}\left[\lim_{n\to\infty} \frac{2n^2+2n}{2n^2+2n+1}\right] = \text{Arcsin}(1) = \frac{\pi}{2}.
\end{aligned}$$

Also solved by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Roger Zarnowski (jointly), San Angelo, TX; Brian D. Beasley, Clinton, SC; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy; Boris Rays, Chesapeake, VA; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

5016: *Proposed by John Nord, Spokane, WA.*

Locate a point (p, q) in the Cartesian plane with integral values, such that for any line through (p, q) expressed in the general form $ax + by = c$, the coefficients a, b, c form an arithmetic progression.

Solution 1 by Nate Wynn (student at Saint George's School), Spokane, WA.

As $\{a, b, c\}$ is an arithmetic progression, b can be written as $a + n$ and c can be written as $a + 2n$. Then using a series of two equations:

$$\begin{cases} ap + (a+n)q & = a+2n \\ tp + (t+u)q & = t+2u \end{cases}$$

Solving this system gives

$$(tn - au)q = 2tn - 2au, \text{thus } q = 2.$$

Placing this value into the first equation and solving gives

$$\begin{aligned}
ap + 2a + 2n &= a + 2n \\
a(p+1) &= 0 \\
p &= -1.
\end{aligned}$$

Therefore the point is $(-1, 2)$.

Solution 2 by Eric Malm (graduate student at Stanford University, and an alumnus of Saint George's School in Spokane), Stanford, CA.

The only such point is $(-1, 2)$.

Suppose that each line through (p, q) is of the form $ax + by = c$ with (a, b, c) an arithmetic progression. Then $c = 2b - a$. Taking $a = 0$ yields the line $by = 2b$ or $y = 2$, so $q = 2$. Taking $a \neq 0$, p= must satisfy $ap + 2b = 2b - a$, so $p = -1$.

Conversely, any line through $(p, q) = (-1, 2)$ must be of the form
$ax + by = ap + bq = 2b - a$, in which case the coefficients $(a, b, 2b - a)$ form an arithmetic progression.

Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Matt DeLong, Upland, IN; Rachel Demeo, Matthew Hussey, Allison Reece, and Brian Tencher (jointly, students at Talyor University, Upland, IN); Michael N. Fried, Kibbutz Revivim, Israel; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; Raul A. Simon, Chile; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

5017: *Proposed by M.N. Deshpande, Nagpur, India.*

Let ABC be a triangle such that each angle is less than 90^0. Show that

$$\frac{a}{c \cdot \sin B} + \frac{1}{\tan A} = \frac{b}{a \cdot \sin C} + \frac{1}{\tan B} = \frac{c}{b \cdot \sin A} + \frac{1}{\tan C}$$

where $a = l(\overline{BC}), b = l(\overline{AC})$, and $c = l(\overline{AB})$.

Solution by John Boncek, Montgomery, AL.

From the Law of Sines:

$$\begin{aligned} a \sin B &= b \sin A \quad \to \quad \sin B = \frac{b \sin A}{a} \\ b \sin C &= c \sin B \quad \to \quad \sin C = \frac{c \sin B}{b} \\ c \sin A &= a \sin C \quad \to \quad \sin A = \frac{a \sin C}{c}, \end{aligned}$$

and from the Law of Cosines, we have

$$\begin{aligned} bc \cos A &= \frac{1}{2}(b^2 + c^2 - a^2) \\ ac \cos B &= \frac{1}{2}(a^2 + c^2 - b^2) \\ ab \cos C &= \frac{1}{2}(a^2 + b^2 - c^2). \end{aligned}$$

Thus,

$$\begin{aligned} \frac{a}{c \sin B} + \frac{1}{\tan A} &= \frac{a^2}{bc \sin A} + \frac{\cos A}{\sin A} \\ &= \frac{a^2 + bc \cos A}{bc \sin A} \\ &= \frac{a^2 + b^2 + c^2}{2bc \sin A}, \end{aligned}$$

$$\frac{b}{a \sin C} + \frac{1}{\tan B} = \frac{b^2}{ac \sin B} + \frac{\cos B}{\sin B}$$

$$
\begin{aligned}
&= \frac{b^2 + ac\cos B}{ac\sin B} \\
&= \frac{a^2+b^2+c^2}{2ac\sin B} \\
&= \frac{a^2+b^2+c^2}{2c(a\sin B)} \\
&= \frac{a^2+b^2+c^2}{2bc\sin A},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{c}{b\sin A} + \frac{1}{\tan C} &= \frac{c^2}{ab\sin C} + \frac{\cos C}{\sin C} \\
&= \frac{c^2 + ab\cos C}{ab\sin C} \\
&= \frac{a^2+b^2+c^2}{2ab\sin C} \\
&= \frac{a^2+b^2+c^2}{2b(a\sin C)} \\
&= \frac{a^2+b^2+c^2}{2bc\sin A}.
\end{aligned}
$$

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Michael C. Faleski, University Center, MI; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

5018: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Write the polynomial $x^{5020} + x^{1004} + 1$ as a product of two polynomials with integer coefficients.

Solution by Kee-Wai Lau, Hong Kong, China.

Clearly the polynomial $y^5 + y + 1$ has no linear factor with integer coefficients.
We suppose that for some integers a, b, c, d, e

$$
\begin{aligned}
y^5 + y + 1 &= (y^3 + ay^2 + by + c)(y^2 + dy + e) \\
&= y^5 + (a+d)y^4 + (b+e+ad)y^3 + (ae+bd+c)y^2 + (be+cd)y + ce.
\end{aligned}
$$

Hence

$$a + d = b + e + ad = ae + bd + c = 0, \quad be + cd = ce = 1.$$

It is easy to check that $a = -1, b = 0, c = 1, d = 1, e = 1$ so that

$$y^5 + y + 1 = (y^3 - y^2 + 1)(y^2 + y + 1)$$

and

$$x^{5020} + x^{1004} + 1 = \left(x^{3012} - x^{2008} + 1\right)\left(x^{2008} + x^{1004} + 1\right).$$

Comment by Kenneth Korbin, New York, NY. Note that $(y^2 + y + 1)$ is a factor of $(y^N + y + 1)$ for all N congruent to 2(mod3) with $N > 1$.

Also solved by Landon Anspach, Nicki Reishus, Jessi Byl, and Laura Schindler (jointly, students at Taylor University), Upland, IN; Brian D. Beasley, Clinton, SC; John Boncek, Montgomery, AL; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX; Matt DeLong, Upland, IN; Paul M. Harms, North Newton, KS; Matthew Hussey Rachel DeMeo, Brian Tencher, and Allison Reece (jointly, students at Taylor University), Upland IN; Kenneth Korbin, New York, NY; N. J. Kuenzi, Oshkosh, WI; Carl Libis, Kingston, RI; Eric Malm, Stanford, CA; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

5019: *Michael Brozinsky, Central Islip, NY.*

In a horse race with 10 horses the horse with the number one on its saddle is referred to as the number one horse, and so on for the other numbers. The outcome of the race showed the number one horse did not finish first, the number two horse did not finish second, the number three horse did not finish third and the number four horse did not finish fourth. However, the number five horse did finish fifth. How many possible orders of finish are there for the ten horses assuming no ties?

Solution 1 by R. P. Sealy, Sackville, New Brunswick, Canada.

There are 229,080 possible orders of finish.

For $k = 0, 1, 2, 3, 4$ we perform the following calculations:

a) Choose the k horses numbered 1 through 4 which finish in places 1 through 4.

b) Arrange the k horses in places 1 through 4 and count the permutations with no "fixed points."

c) Arrange the remaining $(4 - k)$ horses numbered 1 through 4 in places 6 through 10.

d) Arrange the 5 horses numbered 6 through 10 in the remaining 5 places.

Case 1: K=0.

${}_4C_0 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 5! = 120 \cdot 5!$

Case 2: K=1.

${}_4C_1 \cdot 3 \cdot 5 \cdot 4 \cdot 3 \cdot 5! = 720 \cdot 5!$

Case 3: K=2.

${}_4C_2 \cdot 7 \cdot 5 \cdot 4 \cdot 5! = 840 \cdot 5!$

Case 4: K=3.

${}_4C_3 \cdot 11 \cdot 5 \cdot 5! = 220 \cdot 5!$

Case 5: K=4.

${}_4C_4 \cdot 9 \cdot 5! = 9 \cdot 5!$

Solution 2 by Matt DeLong, Upland, IN.

We must count the total number of ways that 10 horses can be put in order subject to the given conditions. Since the number five horse always finishes fifth, we are essentially only counting the total number of way that 9 horses can be put in order subject to the other given conditions. Thus there are at most 9! possibilities.

However, this over counts, since it doesn't exclude the orderings with the number one horse finishing first, etc. By considering the number of ways to order the other eight horses, we can see that there are 8! ways in which the number one horse does finish first. Likewise, there are 8! ways in which each of the horses numbered two through four finish in the position corresponding to its saddle number. By eliminating these from consideration, we see that there are at least 9!-4(8!) possibilities.

However, this under counts, since we twice removed orderings in which both horse one finished first and horse two finished second etc. There are 6(7!) such orderings, since there are 6 ways to choose 2 horses from among 4, and once those are chosen the other 7 horses must be ordered. We can add these back in, but then we will again be over counting. We would need to subtract out those orderings in which three of the first four horses finish according to their saddle numbers. There are 4(6!) of these, since there are 4 ways to choose 3 horses from among 4, and once those are chosen the other 6 horses must be ordered. Finally, we would then need to add back in the number of orderings in which all four horses numbered one through four finish according to their saddle numbers. There are 5! such orderings.

In sum, we are applying the inclusion-exclusion principle, and the total that we are interested in is $9! - 4(8!) + 6(7!) - 4(6!) + 5! = 229,080$.

Also solved by Michael C. Faleski, University Center, MI; Paul M. Harms, North Newton, KS; Nate Kirsch and Isaac Bryan (students at Taylor University), Upland, IN; N. J. Kuenzi, Oshkosh, WI; Kee-Wai Lau, Hong Kong, China; Carl Libis, Kingston, RI; David E. Manes, Oneonta, NY; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.