Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before January 15, 2008

- 4984: *Proposed by Kenneth Korbin, New York, NY.*

  Prove that
  $$\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{5}+\sqrt{7}}+\cdots+\frac{1}{\sqrt{2009}+\sqrt{2011}} > \sqrt{120}.$$

- 4985: *Proposed by Kenneth Korbin, New York, NY.*

  A Heron triangle is one that has both integer length sides and integer area. Assume Heron triangle ABC is such that $\angle B = 2\angle A$ and with (a,b,c)=1.

  PartI : Find the dimensions of the triangle if side a = 25.
  PartII : Find the dimensions of the triangle if 100 < a < 200.

- 4986: *Michael Brozinsky, Central Islip, NY.*

  Show that if $0 < a < b$ and $c > 0$, that
  $$\sqrt{(a+c)^2+d^2}+\sqrt{(b-c)^2+d^2} \le \sqrt{(a-c)^2+d^2}+\sqrt{(b+c)^2+d^2}.$$

- 4987: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be the sides of a triangle ABC with area S. Prove that
  $$(a^2+b^2)(b^2+c^2)(c^2+a^2) \le 64S^3 \csc 2A \csc 2B \csc 2C.$$

- 4988: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Find all real solutions of the equation

$$3^{x^2-x-z}+3^{y^2-y-x}+3^{z^2-z-y}=1.$$

- 4989: *Proposed by Tom Leong, Scotrun, PA.*

  The numbers $1,2,3,\cdots,2n$ are randomly arranged onto $2n$ distinct points on a circle. For a chord joining two of these points, define its *value* to be the absolute value of the difference of the numbers on its endpoints. Show that we can connect the $2n$ points in disjoint pairs with n chords such that no two chords intersect inside the circle and the sum of the values of the chords is exactly n^2.

Solutions

- 4960: *Proposed by Kenneth Korbin, New York, NY.*

  Equilateral triangle ABC has an interior point P such that

$$\overline{AP}=\sqrt{5},\ \overline{BP}=\sqrt{12},\ \text{and}\ \overline{\text{CP}}=\sqrt{17}.$$

  Find the area of $\triangle APB$.

  Solution by Scott H. Brown, Montgomery, AL.

  First rotate $\triangle ABC$ about point C through a counter clockwise angle of 60^o. This will create equilateral triangle CBB' and interior point P'. Since triangle ABC is equialteral and $m\angle ACB=60^o$, $\overline{AC}$ falls on $\overline{BC}$, and $\overline{CP'}=\sqrt{17}$, $\overline{B'P'}=\sqrt{12}$, $\overline{BP'}=\sqrt{5}$. Now $\triangle CPA\cong\triangle CP'B$ and $m\angle ACP=m\angle BCP'$, so $m\angle PCP'=60^o$.

  Second, draw $\overline{PP'}$, forming isosceles triangle PCP'. Since $m\angle PCP'=60^o$, triangle PCP' is equilateral. We find $\overline{PP'}=\sqrt{17}$, $\overline{PA}=\overline{P'B}=\sqrt{5}$ and $\overline{PB}=\sqrt{12}$. So triangle PBP' is a right triangle.

  Third, $m\angle APB'=120^o$ and $m\angle PBP'=90^o$. We find $m\angle PBA+m\angle P'BB'=30^o$. Since $m\angle P'BB'=m\angle PAB$, then by substitution, $m\angle PBA+m\angle PAB=30^o$. Thus $m\angle APB=150^o$.

  Finally, we find the area of triangle APB$=\dfrac{1}{2}(\sqrt{5})(\sqrt{12})\sin(150^o)=\dfrac{\sqrt{15}}{2}$ square units.

  (Reference: Challenging Problems in Geometry 2, Posamentier & Salkind, p. 39.)

  Also solved by Mark Cassell (student, Saint George's School), Spokane, WA; Matt DeLong, Upland, IN; Grant Evans (student, Saint George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; Boris Rays and Jahangeer Kholdi (jointly), Chesapeake and Portsmouth, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4961: *Proposed by Kenneth Korbin, New York, NY.*

  A convex hexagon is inscribed in a circle with diameter d. Find the area of the hexagon if its sides are $3,3,3,4,4$ and 4.

  Solution 1 by John Nord, Spokane, WA.

For cyclic quadrilateral ABCD with sides a, b, c, and d, two different formulations of the area are given, Brahmagupta's formula and Bretschneider's formula.

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)} \text{ where } s = \frac{a+b+c+d}{2} \tag{1}$$

$$A = \frac{\sqrt{(ac+bd)(ad+bc)(ab+cd)}}{4R} \text{ where R is the circumradius} \tag{2}$$

In order to employ the cyclic quadrilateral theorems, place a diagonal into the hexagon to obtain two inscribed quadrilaterals. The first has side lengths of 3,3,3, and x and the second has side lengths of 4,4,4 and x.

Equating (1) and (2) and solving for R yields

$$R = \frac{1}{4}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{(s-a)(s-b)(s-c)(s-d)}} \tag{3}$$

Both quadrilaterals are inscribed in the same circle so (3) can be used for both quadrilaterals and they can be set equal to each other. Solving for x is surprisingly simple and the area computations can be calculated using (1) directly. The area of the inscribed hexagon with sides 3,3,3,4,4, and 4 is $\dfrac{73\sqrt{3}}{4}$.

Solution 2 by Jonathan Schrock, Seth Bird, and Jim Moore (jointly, students at Taylor University), Upland, IN.

Since the hexagon is convex and cyclic, a radius of the circumscribing circle can be drawn to each vertex producing six isosceles triangles. The formula for the height of one of these triangles is $\frac{1}{2}\sqrt{4r^2 - c^2}$ where c is the length of the base of the triangle and r is the radius of the circle. Since $2r = d$ (the diameter of the circle), the area of any one of these triangles will therefore be $\frac{c}{4}\sqrt{d^2 - c^2}$. The total area of the hexagon is the sum of the areas of the triangles. There are three triangles for which $c = 3$ and three for which $c = 4$. So the total area of the hexagon in terms of d is $3\sqrt{d^2 - 16} + \frac{9}{4}\sqrt{d^2 - 9}$.

We can determine d by rearranging the hexagon so that the side lengths alternate as 3,4,3,4,3,4. This creates three congruent quadrilaterals. Consider just one of these quadrilaterals and label it ABCO, where A, B, and C lie on the circle and O is the center of the circle. Since the interior angle for a circle is 360^o and there are three quadrilaterals, $\angle AOC = 120^o$. By constructing a line from A to C we can see by the symmetry of the rearranged hexagon, that $\angle ABC = 120^o$. Using the law of cosines,

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2\left(\overline{AB}\right)\left(\overline{BC}\right)\cos(120^o),$$

which can be written as $\overline{AC}^2 = 3^2 + 4^2 - 2(3)(4)\cos(120^o)$. That is, $\overline{AC} = \sqrt{37}$.
To determine d we use the law of cosines again. Here,

$$\overline{AC}^2 = \overline{AO}^2 + \overline{CO}^2 - 2\left(\overline{AO}\right)\left(\overline{CO}\right)\cos(120^o),$$

which can be written as $37 = \dfrac{d^2}{2} - \dfrac{d^2}{2}\cos(120^o)$. Solving for d gives $d = 2\sqrt{\dfrac{37}{3}}$.
Substituting this value of d into the formula $3\sqrt{d^2 - 16} + \frac{9}{4}\sqrt{d^2 - 9}$ gives the area of the hexagon as $\dfrac{73\sqrt{3}}{4}$.

Comment by editor: David Stone and John Hawkins of Statesboro GA generalized the problem for any convex, cyclic hexagon with side lengths a, a, a, b, b, b (with $0 < a \leq b$) and with d as the diameter of the circumscribing circle. They showed that d is uniquely determined by the values of a and b, $d = \sqrt{\frac{4}{3}(a^2 + ab + b^2)}$. Then they asked the question: What fraction of the circle's area is covered by the hexagon? They found that in general, the fraction of the circle's area covered by the hexagon is:

$$\frac{\frac{\sqrt{3}}{4}(a^2 + 4ab + b^2)}{\frac{\pi}{3}(a^2 + ab + b^2)} = \frac{3\sqrt{3}(a^2 + 4ab + b^2)}{4\pi(a^2 + ab + b^2)} = \frac{3\sqrt{3}}{4\pi}\frac{(a+b)^2 + 2ab}{(a+b)^2 - ab} = \left(\frac{3\sqrt{3}}{4\pi}\right)\frac{1+2c}{1-c}$$

where $c = \dfrac{ab}{(a+b)^2}$.

They continued on by stating that in fact, c takes on the values $0 < c \leq 1/4$, thus forcing $1 < \dfrac{1+2c}{1-c} \leq 2$. So by appropriate choices of a and b, the hexagon can cover from $\dfrac{3\sqrt{3}}{4\pi} \approx 0.4135$ of the circle up to $\dfrac{3\sqrt{3}}{4\pi} \cdot 2 \approx 0.827$ of the circle. A regular hexagon, where $a = b$ and $c = 1/4$, would achieve the upper bound and cover the largest possible fraction of the circle.

For instance, we can force the hexagon to cover exactly one half the circle by making $\left(\dfrac{3\sqrt{3}}{4\pi}\right)\dfrac{1+2c}{1-c} = \dfrac{1}{2}$. This would require $c = \dfrac{2\pi - 3\sqrt{3}}{2\left(3\sqrt{3} + \pi\right)} \approx 0.0651875$. Setting this equal to $\dfrac{ab}{(a+b)^2}$, we find that $\dfrac{a}{b} = \dfrac{\left(6\sqrt{3} - \pi\right) \pm \sqrt{3(27 - \pi^2)}}{2\pi - 3\sqrt{3}}$.
That is, if $b = 13.2649868a$, the hexagon will cover half of the circle.

Also solved by Matt DeLong, Upland, IN; Peter E. Liley, Lafayette, IN; Mandy Isaacson, Julia Temple, and Adrienne Ramsay (jointly, students at Taylor University), Upland, IN; Paul M. Harms, North Newton, KS; Boris Rays and Jahangeer Kholdi (jointly), Chesapeake and Portsmouth, VA , and the proposer.

- 4962: *Proposed by Kenneth Korbin, New York, NY.*

Find the area of quadrilateral $ABCD$ if the midpoints of the sides are the vertices of a square and if $AB = \sqrt{29}$ and $CD = \sqrt{65}$.

Solution by proposer.

Conclude that $AC \perp BD$ and that $AC = BD$. Then, there are positive numbers (w, x, y, z) such that

$$\begin{aligned} w + x &= AC, \\ y + z &= BD, \\ w^2 + y^2 &= 29, \text{ and} \\ x^2 + z^2 &= 65. \end{aligned}$$

Then, $(w, x, y, z) = (\dfrac{11}{\sqrt{10}}, \dfrac{19}{\sqrt{10}}, \dfrac{13}{\sqrt{10}}, \dfrac{17}{\sqrt{10}})$ and $AC = BD = \dfrac{30}{\sqrt{10}}$. The area of the quadrilateral then equals $\frac{1}{2}(AC)(BD) = \frac{1}{2}\left(\frac{30}{\sqrt{10}}\right)\left(\frac{30}{\sqrt{10}}\right) = 45.$

Also solved by Peter E. Liley, Lafayette, IN, and by Boris Rays and Jahangeer Kholdi (jointly), Chesapeake and Portsmouth, VA.

- 4963: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Calculate

$$\lim_{n\to\infty} \sum_{1\le i<j\le n} \frac{1}{3^{i+j}}.$$

Solution 1 by Ken Korbin, New York, NY.

$$\begin{aligned}
\sum_{1\le i<j\le n} \frac{1}{3^{i+j}} &= \left(\frac{1}{3^3}+\frac{1}{3^4}\right)+\left(\frac{2}{3^5}+\frac{2}{3^6}\right)+\left(\frac{3}{3^7}+\frac{3}{3^8}\right)+\left(\frac{4}{3^9}+\frac{4}{3^{10}}\right)+\cdots\\
&= \frac{4}{3^4}+\frac{8}{3^6}+\frac{12}{3^8}+\frac{16}{3^{10}}+\cdots\\
&= \frac{4}{3^4}\left[1+\frac{2}{3^2}+\frac{3}{3^4}+\frac{4}{3^6}+\cdots\right]\\
&= \frac{4}{3^4}\left[1+\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+\cdots\right]^2\\
&= \frac{4}{3^4}\left[\frac{1}{1-\dfrac{1}{3^2}}\right]^2\\
&= \frac{4}{3^4}\left[\frac{9}{8}\right]^2 = \frac{1}{16}.
\end{aligned}$$

Solutions 2 and 3 by Pat Costello, Richmond, KY.

2) When $n=2$ we have $\dfrac{1}{3^{1+2}}$.

When $n=3$ we have $\dfrac{1}{3^{1+3}}+\dfrac{1}{3^{2+3}}$.

When $n=4$ we have $\dfrac{1}{3^{1+4}}+\dfrac{1}{3^{2+4}}+\dfrac{1}{3^{3+4}}$.

Adding down the columns we obtain:

$$\begin{aligned}
&\sum_{k=3}^{\infty}\frac{1}{3^k}+\sum_{k=5}^{\infty}\frac{1}{3^k}+\sum_{k=7}^{\infty}\frac{1}{3^k}+\cdots\\
&\qquad= \frac{(1/3)^3}{1-1/3}+\frac{(1/3)^5}{1-1/3}+\frac{(1/3)^7}{1-1/3}+\cdots\\
&\qquad= \frac{3}{2}\left(\frac{1}{3}\right)^3\left(1+(1/3)^2+(1/3)^4+\cdots\right)\\
&\qquad= \frac{3}{2}\left(\frac{1}{3}\right)^3\left(1+(1/9)+(1/9)^2+\cdots\right)\\
&\qquad= \frac{3}{2}\left(\frac{1}{3}\right)^3\left(\frac{1}{1-1/9}\right)=\frac{1}{16}.
\end{aligned}$$

3) Another way to see that the value is 1/16 is to write the limit as the double sum

$$\begin{aligned}
\sum_{n=2}^{\infty}\sum_{i=2}^{n-1}\frac{1}{3^{n+i}} &= \sum_{n=2}^{\infty}\frac{1}{3^n}\sum_{i=2}^{n-1}\frac{1}{3^i} = \sum_{n=2}^{\infty}\frac{1}{3^n}\left(\frac{(1/3)-(1/3)^n}{1-(1/3)}\right) \\
&= \frac{3}{2}\sum_{n=2}^{\infty}\frac{1}{3^n}\Big((1/3)-(1/3)^n\Big) \\
&= \frac{3}{2}\left((1/3)\sum_{n=2}^{\infty}\frac{1}{3^n}-\sum_{n=2}^{\infty}\frac{1}{9^n}\right) \\
&= \frac{3}{2}\left((\frac{1}{3})\frac{1/9}{1-1/3}-\frac{1/(81)}{1-1/9}\right) \\
&= \frac{3}{2}\left(\frac{1}{18}-\frac{1}{72}\right)=\frac{1}{16}.
\end{aligned}$$

Also solved by Bethany Ballard, Nicole Gottier, Jessica Heil (jointly, students at Taylor University), Upland, IN; Matt DeLong, Upland, IN; Paul M. Harms, North Newton, KS; Carl Libis, Kingston, RI; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4964: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let x, y be real numbers and we define the law of composition

$$x \perp y = x\sqrt{1+y^2} + y\sqrt{1+x^2}.$$

Prove that $(R,+)$ and $(R,\perp)$ are isomorphic and solve the equation $x \perp a = b$.

Solution by R. P. Sealy, Sackville, New Brunswick, Canada

Define $f : (R,+) \to (R,\perp)$ by $f(x) = \sinh x$.
Then f is one-to-one and onto, and

$$\begin{aligned}
f(a+b) &= \sinh(a+b) \\
&= \sinh a \cosh b + \cosh a \sinh b \\
&= \sinh a\sqrt{1+\sinh^2 b} + \sinh b\sqrt{1+\sinh^2 a} \\
&= f(a) \perp f(b)
\end{aligned}$$

Therefore $(R,+)$ and $(R,\perp)$ are isomorphic abelian groups.
Note that:

$$\begin{cases}
\textbf{i)}\ \ \mathrm{f}(0) = 0 \text{ and that } \mathrm{f}(-\mathrm{a}) = -\mathrm{f}(\mathrm{a}). \\
\textbf{ii)}\ \ \text{In } (\mathrm{R},\perp) \\
\qquad 0 \perp a = 0\sqrt{1+a^2} + a\sqrt{1+0^2} = a \text{ and} \\
\qquad a \perp (-a) = a\sqrt{1+a^2} - a\sqrt{1+a^2} = 0.
\end{cases}$$

If $x \perp a = b$, then $x = b \perp (-a) = b\sqrt{1+a^2} - a\sqrt{1+b^2}$.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY, and the proposer.

- 4965: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let h_a, h_b, h_c be the heights of triangle ABC. Let P be any point inside $\triangle ABC$. Prove that

$$(a)\quad \frac{h_a}{d_a}+\frac{h_b}{d_b}+\frac{h_c}{d_c} \geq 9, \quad (b)\quad \frac{d_a^2}{h_a^2}+\frac{d_b^2}{h_b^2}+\frac{d_c^2}{h_c^2} \geq \frac{1}{3},$$

where d_a, d_b, d_c are the distances from P to the sides BC, CA and AB respectively.

Solution to part (a) by Scott H. Brown, Montgomery, AL.

Suppose P is any point inside triangle ABC. Let AP, BP, and CP be the line segments whose distances from the vertices are x, y, and z respectively. Let AP, BP, and CP intersect the sides BC, CA, and AB, at points L, M, and N respectively. Denote PL, PM, and PN by u, v, and w respectively.
In reference [1] it is shown that

$$\frac{x}{u}+\frac{y}{v}+\frac{z}{w} \geq 6, \qquad (1)$$

with equality holding only if P in the centroid of triangle ABC.

Considering the heights h_a, h_b, and h_c, and the distances respectively to the sides from P as d_a, d_b, and d_c in terms of u, v, w, x, y, and z gives:

$$\frac{h_a}{d_a}=\frac{x+u}{u}, \qquad \frac{h_b}{d_b}=\frac{y+v}{v}, \qquad \frac{h_c}{d_c}=\frac{z+w}{w}. \qquad (2)$$

Applying inequality (1) gives:

$$\frac{h_a}{d_a}+\frac{h_b}{d_b}+\frac{h_c}{d_c} \geq 9,$$

with equality holding only if P is the centroid of triangle ABC.

Reference [1]. Some Inequalities For A Triangle, L. Carlitz,
American Mathematical Monthly, 1964, pp. 881-885.

Solution to part (b) by the proposers.

For the triangles BPC, APC, APB we have,

$$\begin{aligned}
[BPC] &= d_a \times \frac{BC}{2} = \frac{d_a}{h_a} \times \frac{h_a\,BC}{2} = \frac{d_a}{h_a} \times [ABC] \\
[APC] &= d_b \times \frac{AC}{2} = \frac{d_b}{h_b} \times \frac{h_b\,AC}{2} = \frac{d_b}{h_b} \times [ABC] \\
[APB] &= d_c \times \frac{AB}{2} = \frac{d_c}{h_c} \times \frac{h_c\,ABC}{2} = \frac{d_c}{h_c} \times [ABC]
\end{aligned}$$

Adding up the preceding expressions yields,

$$\left(\frac{d_a}{h_a}+\frac{d_b}{h_b}+\frac{d_c}{h_c}\right)[ABC] = [ABC]$$

and

$$\frac{d_a}{h_a} + \frac{d_b}{h_b} + \frac{d_c}{h_c} = 1$$

Applying AM-QM inequality, we get

$$\sqrt{\frac{\frac{d_a^2}{h_a^2} + \frac{d_b^2}{h_b^2} + \frac{d_c^2}{h_c^2}}{3}} \geq \frac{1}{3}\left(\frac{d_a}{h_a} + \frac{d_b}{h_b} + \frac{d_c}{h_c}\right) = \frac{1}{3}$$

from which the inequality claimed immediately follows. Finally, notice that equality holds when $d_a/h_a = d_b/h_b = d_c/h_c = 1/3$. That is, when $\triangle ABC$ is equilateral and P is its centroid.

- 4966: *Proposed by Kenneth Korbin, New York, NY.*

Solve:

$$16x + 30\sqrt{1-x^2} = 17\sqrt{1+x} + 17\sqrt{1-x}$$

with $0 < x < 1$.

Solution 1 by Elsie Campbell, Dionne Bailey, & Charles Diminnie, San Angelo, TX.

Let $x = \cos\theta$ where $\theta \in (0, \frac{\pi}{2})$. Then,

$$16x + 30\sqrt{1-x^2} = 17\sqrt{1+x} + 17\sqrt{1-x}$$

becomes

$$\begin{aligned}
16\cos\theta + 30\sqrt{1-\cos^2\theta} &= 17\sqrt{1+\cos\theta} + 17\sqrt{1-\cos\theta} \\
&= 17\sqrt{2}\left(\sqrt{\frac{1+\cos\theta}{2}} + \sqrt{\frac{1-\cos\theta}{2}}\right) \\
&= 34\left(\frac{1}{\sqrt{2}}\cos\frac{\theta}{2} + \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\right) \\
&= 34\left(\cos\frac{\pi}{4}\cos\frac{\theta}{2} + \sin\frac{\pi}{4}\sin\frac{\theta}{2}\right) \\
&= 34\cos(\frac{\pi}{4} - \frac{\theta}{2}). \qquad (1)
\end{aligned}$$

Let $\cos\theta_0 = \frac{8}{17}$. Then by (1),

$$\begin{aligned}
\cos(\frac{\pi}{4} - \frac{\theta}{2}) &= \frac{8}{17}\cos\theta + \frac{15}{17}\sin\theta \\
&= \cos\theta_0\cos\theta + \sin\theta_0\sin\theta \\
&= \cos(\theta_0 - \theta).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\theta_0 - \theta &= \tfrac{\pi}{4} - \tfrac{\theta}{2} & \qquad & & \theta_0 - \theta &= -(\tfrac{\pi}{4} - \tfrac{\theta}{2}) \\
\Rightarrow \quad \theta &= 2\theta_0 - \tfrac{\pi}{2} & \text{or} & \quad \Rightarrow & \theta &= \tfrac{2}{3}\theta_0 + \tfrac{\pi}{6} \\
\Rightarrow \quad x &= \tfrac{240}{289} & & \quad \Rightarrow & x &= \cos(\tfrac{2}{3}\cos^{-1}\tfrac{8}{17} + \tfrac{\pi}{6}).
\end{aligned}$$

Remark: This solution is an adaptation of the solution on pp.13-14 from *Mathematical Miniatures* by Savchev and Andreescu.

Solution 2 by Brian D. Beasley, Clinton, SC.

Since $0 < x < 1$, each side of the given equation will be positive, so we may square both sides without introducing any extraneous solutions. After simplifying, this yields

$$(480x - 289)\sqrt{1-x^2} = 161(2x^2 - 1).$$

For each side of this equation to have the same sign (or zero), we require $x \in (0, 289/480] \cup [\sqrt{2}/2, 1)$. We now square again, checking for actual as well as extraneous solutions. This produces

$$(1156x^3 - 867x + 240)(289x - 240) = 0,$$

so one potential solution is $x = 240/289$. The cubic formula yields three more, namely

$$x \in \{-\cos(\frac{1}{3}\cos^{-1}(\frac{240}{289})), \sin(\frac{1}{3}\sin^{-1}(\frac{240}{289})), \cos(\frac{1}{3}\cos^{-1}(-\frac{240}{289}))\}.$$

Of these four values, only two are in $x \in (0, 289/480] \cup [\sqrt{2}/2, 1)$:

$$x = \frac{240}{289} \quad \text{and} \quad x = \sin(\frac{1}{3}\sin^{-1}(\frac{240}{289})).$$

Addendum. The given equation generalizes nicely to

$$2ax + 2b\sqrt{1-x^2} = c\sqrt{1+x} + c\sqrt{1-x},$$

where $a^2 + b^2 = c^2$ with $a < b$. The technique outlined above produces

$$(4c^2x^3 - 3c^2x + 2ab)(c^2x - 2ab) = 0,$$

so one solution (which checks in the original equation) is $x = 2ab/c^2$. Another solution (does it always check in the original equation?) is $x = \sin(\frac{1}{3}\sin^{-1}(\frac{2ab}{c^2}))$, which is connected to the right triangle with side lengths $(b^2 - a^2, 2ab, c^2)$ in the following way:

If we let 3θ be the angle opposite the side of length $2ab$ in this triangle, then we have $2ab/c^2 = \sin(3\theta) = -4\sin^3\theta + 3\sin\theta$, which brings us right back to $4c^2x^3 - 3c^2x + 2ab = 0$ for $x = \sin\theta$.

Similarly, we may show that the other two solutions are $x = -\cos(\frac{1}{3}\cos^{-1}(\frac{2ab}{c^2}))$ and $x = \cos(\frac{1}{3}\cos^{-1}(-\frac{2ab}{c^2}))$; the first of these is never in $(0, 1)$, but will the second ever be a solution of the original equation?

Also solved by John Boncek, Montgomery, AL; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4967: *Proposed by Kenneth Korbin, New York, NY.*

Given equilateral triangle ABC with an interior point P such that $\overline{AP}^2 + \overline{BP}^2 = \overline{CP}^2$, and with an exterior point Q such that $\overline{AQ}^2 + \overline{BQ}^2 = \overline{CQ}^2$, where points C, P, and Q are in a line. Find the lengths of $\overline{AQ}$ and $\overline{BQ}$ if $\overline{AP} = \sqrt{21}$ and $\overline{BP} = \sqrt{28}$.

Solution by Paul M. Harms, North Newton, KS.

Put the equilateral triangle on a coordinate system with A at $(0, 0)$, B at $(a, \sqrt{3}a)$ and C at $(2a, 0)$ where $a > 0$. The point P is at the intersection of the circles

$$x^2 + y^2 \quad = \quad 21$$

$$\begin{aligned}(x-a)^2+(y-\sqrt{3}a)^2 &= 28 \text{ and}\\ (x-2a)^2+y^2 &= 28+21=49.\end{aligned}$$

Using $x^2+y^2=21$ in the last two circles we obtain

$$\begin{aligned}-2ax-2\sqrt{3}ay+4a^2=28-21 &= 7 \text{ and}\\ -4ax+4a^2 &= 49-21=28.\end{aligned}$$

From the last equation $x=\dfrac{a^2-7}{a}$ and, using the linear equation, we get $y=\dfrac{2a^2+7}{2\sqrt{3}a}$. Putting these x, y values into $x^2+y^2=21$ yields the quadratic in a^2, $16a^4-392a^2+637=0$. From this equation $a^2=22.75$ or $a^2=1.75$. From the distances given in the problem, a^2 must be 22.75. The coordinates of P are $x=3.3021$ and $y=3.1774$. The line through C and P is $y=-0.5094x+4.85965$.

Let Q have coordinates (x_1, y_1). An equation for $\overline{AQ}^2+\overline{BQ}^2=\overline{CQ}^2$ can be found using the coordinates $Q(x_1,y_1), A(0,0), B(4.7697, 8.2614)$, and $C(9.5394, 0)$. An equation is

$$x_1^2+y_1^2+(x_1-4.7697)^2+(y_1-8.2614)^2=(x_1-9.5394)^2+y_1^2.$$

Simplifying and replacing y_1 by $-0.5094x_1+4.85965$ yields the quadratic equation $1.2595x_1^2+13.0052x_1-56.6783=0$. In order that Q is exterior to the triangle we need the solution $x_1=-13.6277$. Then $y_1=-0.5094x_1+4.85965=11.8020$. The distance from A to Q is $\sqrt{325}=18.0278$ and the distance from B to Q is $\sqrt{351}=18.7350$.

Also solved by Zhonghong Jiang, New York, NY, and the proposer.

- **4968:** *Proposed by Kenneth Korbin, New York, NY.*

Find two quadruples of positive integers (a, b, c, d) such that

$$\frac{a+i}{a-i}\cdot\frac{b+i}{b-i}\cdot\frac{c+i}{c-i}\cdot\frac{d+i}{d-i}=\frac{a-i}{a+i}\cdot\frac{b-i}{b+i}\cdot\frac{c-i}{c+i}\cdot\frac{d-i}{d+i}$$

with $a<b<c<d$ and $i=\sqrt{-1}$.

Solution 1 by Brian D. Beasley, Clinton, SC.

We need $((a+i)(b+i)(c+i)(d+i))^2=((a-i)(b-i)(c-i)(d-i))^2$, so

$$(a+i)(b+i)(c+i)(d+i)=\pm(a-i)(b-i)(c-i)(d-i).$$

Then either

$$(ab-1)(c+d)+(a+b)(cd-1)=0 \quad \text{or} \quad (ab-1)(cd-1)-(a+b)(c+d)=0.$$

But $(ab-1)(c+d)>0$ and $(a+b)(cd-1)>0$, so the first case cannot occur. In the second case, since $d=(ab+ac+bc-1)/(abc-a-b-c)>0$, we have $abc>a+b+c$. Then $d\ge 4$ implies

$$3\le c\le \frac{ab+4a+4b-1}{4ab-a-b-4},$$

where we note that $1\le a<b$ implies $4ab>a+b+4$. Thus $2\le b\le(7a+11)/(11a-7)$, so $a\le 5/3$. Thus $a=1$, which yields $b\in\{2,3,4\}$.

If $(a,b)=(1,2)$, then $d=(3c+1)/(c-3)$, so $c<d$ forces $c\in\{4,5,6\}$. Only $c\in\{4,5\}$ will yield integral values for d, producing the two solutions $(1,2,4,13)$ and $(1,2,5,8)$ for (a,b,c,d).

If $(a,b)=(1,3)$, then $d=(2c+1)/(c-2)$, so $3<c<d$ forces $c=4$. But this yields $d=9/2$.

If $(a,b)=(1,4)$, then $d=(5c+3)/(3c-5)$, but $4<c<d$ forces the contradiction $c\leq 3$.

Hence the only two solutions for (a,b,c,d) are $(1,2,4,13)$ and $(1,2,5,8)$.

Solution 2 by Dionne Bailey, Elsie Campbell, & Charles Diminnie, San Angelo, TX.

By using the following properties of complex numbers,

$$(\overline{z_1 z_2})=\bar{z}_1\bar{z}_2,\quad \overline{\left(\tfrac{z_1}{z_2}\right)}=\tfrac{\bar{z}_1}{\bar{z}_2},\quad \bar{\bar{z}}=z,$$

we see that the left and right sides of the equation are conjugates and hence, the equation reduces to

$$Im\left(\frac{a+i}{a-i}\cdot\frac{b+i}{b-i}\cdot\frac{c+i}{c-i}\cdot\frac{d+i}{d-i}\right)=0. \qquad (1)$$

If $z=(a+i)(b+i)(c+i)(d+i)=A+Bi$, then (1) becomes

$$Im\left(\frac{z}{\bar{z}}\right)=0,$$

which reduces to $AB=0$ or equivalently, $A=0$ or $B=0$. With some labor, we get

$$\begin{aligned} A &= 1-(ab+ac+ad+bc+bd+cd)+abcd \\ &= (ab-1)(cd-1)-(a+b)(c+d) \text{ and} \\ B &= (abc+abd+acd+bcd)-(a+b+c+d) \\ &= (a+d)(bc-1)+(b+c)(ad-1). \end{aligned}$$

Therefore, a,b,c,d must satisfy

$$(ab-1)(cd-1)=(a+b)(c+d) \qquad (2)$$

or

$$(a+d)(bc-1)+(b+c)(ad-1)=0. \qquad (3)$$

Immediately, the condition $1\leq a<b<c<d$ rules out equation (3) and we may restrict our attention to equation (2).

Since $c\geq 3$ and $d\geq 4$, we obtain

$$(cd-1)-(c+d)=(c-1)(d-1)-2>0$$

and hence,

$$c+d<cd-1.$$

Using this and the fact that $(ab-1)>0$, equation (2) implies that

$$(ab-1)(c+d)<(ab-1)(cd-1)=(a+b)(c+d),$$

or

$$ab-1<a+b.$$

This in turn implies that

$$0\leq(a-1)(b-1)<2.$$

Then, since $1 \le a < b$, we must have $a = 1$ and equation (2) becomes

$$(b-1)(cd-1) = (b+1)(c+d). \qquad (4)$$

Finally, $b \ge 2$ implies that

$$cd - 1 = \frac{b+1}{b-1}(c+d) = \left(1 + \frac{2}{b-1}\right)(c+d) \le 3(c+d)$$

or

$$0 \le (c-3)(d-3) \le 10. \qquad (5)$$

To complete the solution, we solve each of the 11 possibilities presented by (5) and then substitute back into (4) to solve for the remaining variable. It turns out that the only situation which yields feasible answers for b, c, d is the case where $(c-3)(d-3) = 10$. We show this case and two others to indicate the reasoning applied.

Case 1. If

$$(c-3)(d-3) = 0,$$

then since $1 = a < b < c < d$, we must have $c = 3$ and $b = 2$. When these are substituted into (4), we get

$$3d - 1 = 3(3+d)$$

which is impossible.

Case 2. If

$$(c-3)(d-3) = 6,$$

then since $c < d$, we must have $c - 3 = 1$, $d - 3 = 6$ or $c - 3 = 2$, $d - 3 = 3$. These yield $c = 4$, $d = 9$ or $c = 5$, $d = 6$. However, neither pair gives an integral answer for b when these are substituted into (4).

Case 3. If

$$(c-3)(d-3) = 10,$$

then since $c < d$, we must have $c - 3 = 1$, $d - 3 = 10$ or $c - 3 = 2$, $d - 3 = 5$. These yield $c = 4$, $d = 13$ or $c = 5$, $d = 8$. When substituted into (4), both pairs give the answer $b = 2$.

Therefore, the only solutions for which a, b, c, d are integers, with $1 \le a < b < c < d$, are $(a, b, c, d) = (1, 2, 4, 13)$ or $(1, 2, 5, 8)$.

Also solved by Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Raul A. Simon, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

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- 4969: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{1}{a^2\left(\frac{1}{a} + \frac{1}{c}\right)} + \frac{1}{b^2\left(\frac{1}{b} + \frac{1}{a}\right)} + \frac{1}{c^2\left(\frac{1}{c} + \frac{1}{b}\right)} \ge \frac{3}{2}$$

Solution by Kenneth Korbin, New York, NY.

Let $x = \dfrac{1}{a}, y = \dfrac{1}{b}, z = \dfrac{1}{c}$. Then, $K = \dfrac{x^2}{x+z} + \dfrac{y^2}{y+x} + \dfrac{z^2}{z+y}$.
Let $U_1 = \dfrac{x}{\sqrt{x+z}}, U_2 = \dfrac{y}{\sqrt{y+x}}, U_3 = \dfrac{z}{\sqrt{z+y}}$. Then, $K = (U_1)^2 + (U_2)^2 + (U_3)^2$.
Let $V_1 = \sqrt{x+z}, V_2 = \sqrt{y+x}, V_3 = \sqrt{z+y}$. Then, by the Cauchy inequality,

$$\begin{aligned} K &= (U_1)^2 + (U_2)^2 + (U_3)^2 \\ &\geq \frac{(U_1V_1 + U_2V_2 + U_3V_3)^2}{(V_1)^2 + (V_2)^2 + (V_3)^2} \\ &= \frac{(x+y+z+)^2}{2(x+y+z)} = \frac{x+y+z}{2} \end{aligned}$$

Then, by the AM-GM inequality,

$$\begin{aligned} K &\geq \frac{x+y+z}{2} \\ &\geq \frac{1}{2}(3)(\sqrt[3]{xyz}) \\ &= \frac{3}{2}(1) = \frac{3}{2}. \end{aligned}$$

Note: $abc = 1$ implies $xyz = 1$.

Comment by editor: John Boncek of Montgomery, AL noted that this problem is a variant of an exercise given in Andreescu and Enescu's *Mathematical Olympiad Treasures,* (Birkhauser, 2004, problem 6, page 108.)

Also solved by John Boncek; David E. Manes, Oneonta, NY, and the proposer.

- 4970: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Let $f : [0,1] \longrightarrow \mathbf{R}$ be a contintuous convex function. Prove that

$$\frac{3}{4}\int_0^{1/5} f(t)dt + \frac{1}{8}\int_0^{2/5} f(t)dt \geq \frac{4}{5}\int_0^{1/4} f(t)dt.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China.

By the convexity of f we have

$$\frac{3}{4}f\left(\frac{s}{5}\right) + \frac{1}{4}f\left(\frac{2s}{5}\right) \geq f\left((\frac{3}{4})(\frac{s}{5}) + (\frac{1}{4})(\frac{2s}{5})\right) = f\left(\frac{s}{4}\right)$$

for $0 \leq s \leq 1$. Hence,

$$\frac{3}{4}\int_0^1 f\left(\frac{s}{5}\right)ds + \frac{1}{4}\int_0^1 f\left(\frac{2s}{5}\right)ds \ \geq \ \int_0^1 f\left(\frac{s}{4}\right)ds.$$

By substituting $s = 5t$ in the first integral, $s = \dfrac{5t}{2}$ in the second at the left and $s = 4t$ in the integral at the right, we obtain the inequality of the problem.

Solution 2 by David Stone and John Hawkins, Statesboro, GA.

Note 1. Consider the behavior in the extreme case: if f is a linear function, then equality holds:

$$\frac{3}{4}\int_0^{1/5}(mt+b)dt+\frac{1}{8}\int_0^{2/5}(mt+b)dt=\frac{3}{4}\left[\frac{m}{2}\left(\frac{1}{5}\right)^2+b\frac{1}{5}\right]+\frac{1}{8}\left[\frac{m}{2}\left(\frac{2}{5}\right)^2+b\frac{2}{5}\right]=\frac{1}{40}m+\frac{1}{5}b,$$

and

$$\frac{4}{5}\int_0^{1/4}(mt+b)dt=\frac{4}{5}\left[\frac{m}{2}\left(\frac{1}{4}\right)^2+b\frac{1}{4}\right]=\frac{1}{40}m+\frac{1}{5}b.$$

We rewrite the inequality in an equivalent form by clearing fractions and splitting the integrals so that they are taken over non-overlapping intervals:

$$\frac{3}{4}\int_0^{1/5}f(t)dt+\frac{1}{8}\int_0^{2/5}f(t)dt\geq\frac{4}{5}\int_0^{1/4}f(t)dt \iff$$

$$30\int_0^{1/5}f(t)dt+5\left[\int_0^{1/5}f(t)dt+\int_{1/5}^{1/4}f(t)dt+\int_{1/4}^{2/5}f(t)dt\right]\geq 32\left[\int_0^{1/5}f(t)dt+\int_{1/5}^{1/4}f(t)dt\right] \iff$$

$$3\int_0^{1/5}f(t)dt+5\int_{1/4}^{2/5}f(t)dt\geq 27\int_{1/5}^{1/4}f(t)dt. \qquad (1)$$

So we see that the interval of interest, $\left[0,\frac{2}{5}\right]$, has been partitioned into three subintervals $\left[0,\frac{1}{5}\right]$, $\left[\frac{1}{5},\frac{1}{4}\right]$ and $\left[\frac{1}{4},\frac{2}{5}\right]$.

Consider the secant line through the two points $\left(\frac{1}{5},f\left(\frac{1}{5}\right)\right)$ and $\left(\frac{1}{4},f\left(\frac{1}{4}\right)\right)$. The linear function giving this line is $s(t)=20\left[f\left(\frac{1}{4}\right)-f\left(\frac{1}{5}\right)\right]t+\left[5f\left(\frac{1}{5}\right)-4f\left(\frac{1}{4}\right)\right]$. It is straightforward to use the convexity condition to show that this line lies above $f(t)$ on the middle interval$\left[\frac{1}{5}\right]$, and lies below $f(t)$ on the outside intervals $\left[0,\frac{1}{5}\right]$ and $\left[\frac{1}{4},\frac{2}{5}\right]$. That is

$$s(t)\ \geq\ f(t) \text{ on } \left[\frac{1}{5},\frac{1}{4}\right] \text{ and} \qquad (2)$$

$$s(t)\ \leq\ f(t) \text{ on } \left[0,\frac{1}{5}\right],\text{ and } \left[\frac{1}{4},\frac{2}{5}\right] \qquad (3).$$

Considering the sides of (1),

$$3\int_0^{1/5}f(t)dt+5\int_{1/4}^{2/5}f(t)dt\geq 3\int_0^{1/5}s(t)dt+5\int_{1/4}^{2/5}s(t)dt \text{ by (3).}$$

and

$$3\int_0^{1/5}s(t)dt+5\int_{1/4}^{2/5}s(t)dt=27\int_{1/5}^{1/4}s(t)dt \text{ by (Note 1),}$$

and

$$27\int_{1/5}^{1/4}s(t)dt\geq 27\int_{1/5}^{1/4}f(t)dt \text{ by (2).}$$

Therefore (1) is true.

Also solved by John Boncek, Montgomery, AL and the proposers.

• 4971: *Proposed by Howard Sporn, Great Neck, NY and Michael Brozinsky, Central Islip, NY.*

Let $m \geq 2$ be a positive integer and let $1 \leq x < y$. Prove:

$$x^m - (x-1)^m < y^m - (y-1)^m.$$

Solution 1 by Brian D. Beasley, Clinton, SC.

We let $f(x) = x^m - (x-1)^m$ for $x \geq 1$ and show that f is strictly increasing on $[1, \infty)$. Since $f'(x) = mx^{m-1} - m(x-1)^{m-1}$, we have $f'(x) > 0$ if and only if $x^{m-1} > (x-1)^{m-1}$. Since $x \geq 1$ and $m \geq 2$, this latter inequality holds, so we are done.

Solution 2 by Matt DeLong, Upland, IN.

Let $X = x - 1$ and $Y = y - 1$. Then $0 \leq X < Y$, $x = X + 1$, and $y = Y + 1$. Expanding $(X+1)^m - X^m$ and $(Y+1)^m - Y^m$ we see that

$$(X+1)^m - X^m = mX^{m-1} + \frac{m(m-1)}{2}X^{m-2} + \cdots + mX + 1$$

and

$$(Y+1)^m - Y^m = mY^{m-1} + \frac{m(m-1)}{2}Y^{m-2} + \cdots + mY + 1.$$

Since $0 \leq X < Y$, we can compare these two sums term-by-term and conclude that each term involving Y is larger than the corresponding term involving X. Therefore,

$$(X+1)^m - X^m < (Y+1)^m - Y^m.$$

Since $x = X + 1$ and $y = Y + 1$, we have shown that

$$x^m - (x-1)^m < y^m - (y-1)^m.$$

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Solution 3 by José Luis Díaz-Barrero, Barcelona, Spain.

We will argue by induction. The case when $m = 2$ trivially holds because $x^2 - (x-1)^2 = 2x - 1 < 2y - 1 = y^2 - (y-1)^2$. Suppose that

$$x^m - (x-1)^m < y^m - (y-1)^m$$

holds and we have to see that

$$x^{m+1} - (x-1)^{m+1} < y^{m+1} - (y-1)^{m+1}$$

holds. In fact, multiplying by $m+1$ both sides of $x^m - (x-1)^m < y^m - (y-1)^m$ yields

$$(m+1)(x^m - (x-1)^m) < (m+1)(y^m - (y-1)^m)$$

and

$$\int_1^x (m+1)(x^m - (x-1)^m)\, dx < \int_1^y (m+1)(y^m - (y-1)^m)\, dy$$

from which immediately follows

$$x^{m+1} - (x-1)^{m+1} < y^{m+1} - (y-1)^{m+1}$$

Therefore, by the PMI the statement is proved and we are done.

Solution 4 by Kenneth Korbin, New York, NY.

Let $m \geq 2$ be a positive integer, and let $1 \leq x < y$. Then,

$$\begin{array}{rcl} (y-1)^m & < & y^m, \text{ and} \\ (y-1)^{m-1}(x-1) & < & y^{m-1}(x), \text{ and} \\ (y-1)^{m-2}(x-1)^2 & < & y^{m-2}(x^2), \text{ and} \\ & . & \\ & . & \\ & . & \\ y^0 = 1 & \leq & x^m. \end{array}$$

Adding gives

$$\Big[(y-1)^m + (y-1)^{m-1}(x-1) + \cdots + 1\Big] < \Big[y^m + y^{m-1}x + y^{m-2}x^2 + \cdots + x^m\Big].$$

Multiplying both sides by $[(y-1)-(x-1)] = [y-x]$ gives

$$(y-1)^m - (x-1)^m < y^m - x^m.$$

Therefore

$$x^m - (x-1)^m < y^m - (y-1)^m.$$

Also solved by Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; Raul A. Simon, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA; various teams of students at Taylor University in Upland, IN:

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