Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
January 15, 2016

- **5367:** *Proposed by Kenneth Korbin, New York, NY*

  Given triangle ABC with integer length sides and integer area. The vertices have coordinates $A(0,0)$, $B(x,y)$ and $C(z,w)$ with $\sqrt{x^2+y^2}-\sqrt{z^2+w^2}=1$.

  Find positive integers x, y, z and w if the perimeter is 84.

- **5368:** *Proposed by Ed Gray, Highland Beach, FL*

  Let *abcd* be a four digit number in base 10, none of which are zero, such that the last four digits in the square of *abcd* are *abcd*, the number itself. Find the number *abcd*.

- **5369:** *Proposed by Chirita Marcel, Bucuresti, Romania*

  Let convex quadrilateral $ABCD$ have area S and side lengths $\overline{AB}=a, \overline{BC}=b, \overline{CD}=c, \overline{DA}=d$. Show that

$$2\left(a+b+c+d\right)^2+a^2+b^2+c^2+d^2 \geq 36\sqrt{\left(S^2+abcd\cos^2\frac{A+C}{2}\right)}.$$

- **5370:** *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

  Let $f(x)$ and $g(x)$ be arbitrary functions defined for all $x \in \Re$. Prove that there is a function $h(x)$ such that

$$\left(f(x)-h(x)\right)^{2015}\cdot\left(g(x)-h(x)\right)^{2015}$$

  is an odd function for all $x \in \Re$.

- **5371:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $a_1, a_2, \ldots, a_n$ be positive real numbers where $n \geq 4$. Prove that

$$\left(\frac{a_1}{a_n+a_2}\right)^2+\left(\frac{a_2}{a_1+a_3}\right)^2+\ldots+\left(\frac{a_n}{a_{n-1}+a_1}\right)^2 \geq \frac{4}{n}$$

- **5372:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  (a) Let $k \geq 2$ be an integer. Calculate

$$\int_0^\infty \frac{\ln(1+x)}{x\sqrt[k]{x}}dx.$$

  (b) Calculate

$$\int_0^\infty \frac{\ln(1-x+x^2)}{x\sqrt{x}}dx.$$

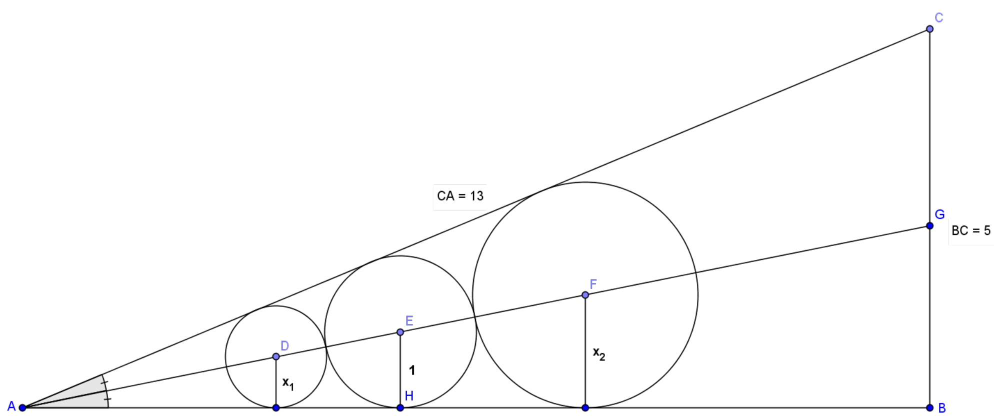
Solutions

- **5349:** *Proposed by Kenneth Korbin, New York, NY*

Given angle A with $\sin A = \dfrac{5}{13}$. A circle with radius 1 and a circle with radius x are each tangent to both sides of the angle. The circles are also tangent to each other. Find x.

Solution by Andrea Fanchini, Cantú, Italy

I) angle A is acute.



With the notations of the figure we have

$$AB = \sqrt{13^2 - 5^2} = 12$$

the centers of the circles are on the bisector of A and we know that the bisector divides the opposite side as the ratio of the lengths of the adjacent sides, so

$$\frac{BG}{GC} = \frac{12}{13} \quad \Rightarrow \quad \frac{BG}{5-BG} = \frac{12}{13} \quad \Rightarrow \quad BG = \frac{12}{5}$$

Now we have that

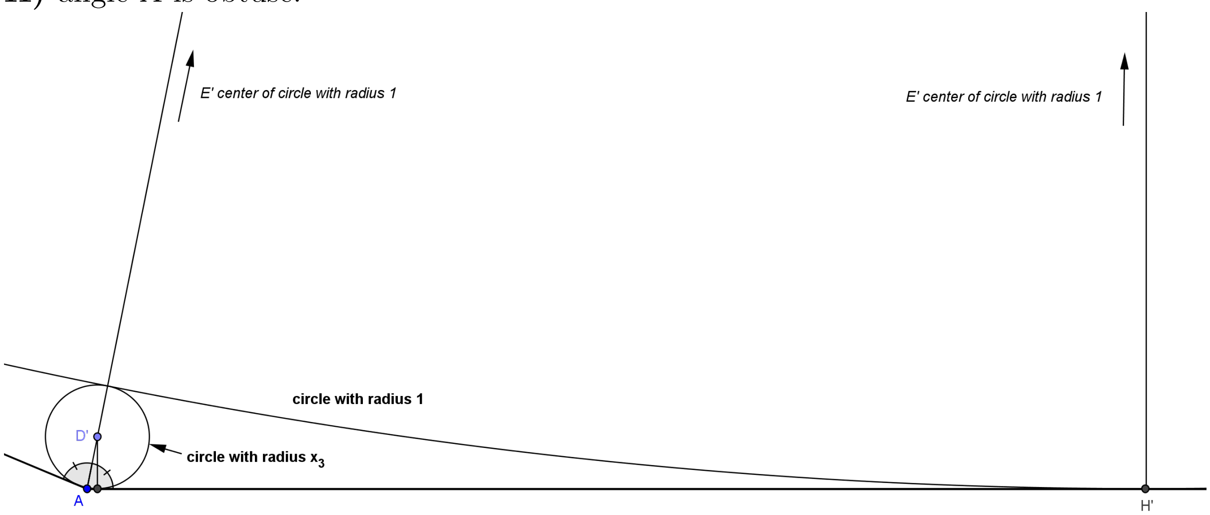
$$\tan\frac{A}{2} = \frac{\frac{12}{5}}{12} = \frac{1}{5} \quad \Rightarrow \quad AH = 5, \quad AE = \sqrt{5^2+1^2} = \sqrt{26}, \quad \sin\frac{A}{2} = \frac{1}{\sqrt{26}}$$

Finally, we obtain the two solutions

$$\sin\frac{A}{2}=\frac{x_1}{AD} \quad\Rightarrow\quad \frac{1}{\sqrt{26}}=\frac{x_1}{\sqrt{26}-1-x_1} \quad\Rightarrow\quad x_1=\frac{\sqrt{26}-1}{\sqrt{26}+1}$$

$$\sin\frac{A}{2}=\frac{x_2}{AF} \quad\Rightarrow\quad \frac{1}{\sqrt{26}}=\frac{x_2}{\sqrt{26}+1+x_2} \quad\Rightarrow\quad x_2=\frac{\sqrt{26}+1}{\sqrt{26}-1}$$

II) angle A is obtuse.



In this case $\angle E'AH'=90^\circ-\dfrac{A}{2}$, so with the notations of the figure we have

$$\tan\left(90^\circ-\frac{A}{2}\right)=\cot\frac{A}{2}=5 \quad\Rightarrow\quad AH'=\frac{1}{5},\quad AE'=\sqrt{\left(\frac{1}{5}\right)^2+1^2}=\frac{\sqrt{26}}{5}$$

Finally, we obtain the other two solutions (where F' is the center of circle with radius x_4)

$$\sin\left(90^\circ-\frac{A}{2}\right)=\frac{x_3}{AD'} \quad\Rightarrow\quad \frac{5}{\sqrt{26}}=\frac{x_3}{\frac{\sqrt{26}}{5}-1-x_3} \quad\Rightarrow\quad x_3=\frac{\sqrt{26}-5}{\sqrt{26}+5}$$

$$\sin\left(90^\circ-\frac{A}{2}\right)=\frac{x_4}{AF'} \quad\Rightarrow\quad \frac{5}{\sqrt{26}}=\frac{x_4}{\frac{\sqrt{26}}{5}+1+x_4} \quad\Rightarrow\quad x_4=\frac{\sqrt{26}+5}{\sqrt{26}-5}$$

Solution 2 by Brain D. Beasley, Presbyterian College, Clinton, SC

Given such a circle of radius 1, there are two circles which are tangent to both sides of angle A and to the original circle; one is smaller than the original, and the other is larger. We denote the radius of the smallest of these three circles by x and the radius of the largest circle by X. We bisect angle A to create three similar right triangles, each with acute angle $A/2$ and with opposite sides of lengths x, 1, and X, respectively. Using the half-angle formula for sine, we have two cases:

If $\sin(A/2)=1/\sqrt{26}$, then the "middle" triangle (which has opposite side of length 1) has a hypotenuse of length $\sqrt{26}$. Thus the hypotenuse of the smallest triangle has length $\sqrt{26}x$, and since the smallest circle is tangent to the "middle" circle, this yields

$$\sqrt{26} = \sqrt{26}x + x + 1.$$

Hence $x = \dfrac{\sqrt{26}-1}{\sqrt{26}+1}$. Similarly, since the largest circle is tangent to the "middle" circle and has a hypotenuse of length $\sqrt{26}X$, we obtain

$$\sqrt{26}X = \sqrt{26} + 1 + X.$$

Hence $X = \dfrac{\sqrt{26}+1}{\sqrt{26}-1} = \dfrac{1}{x}$.

If $\sin(A/2) = 5/\sqrt{26}$, then the "middle" triangle (which has opposite side of length 1) has a hypotenuse of length $\sqrt{26}/5$. Thus the hypotenuse of the smallest triangle has length $\sqrt{26}x/5$, and since the smallest circle is tangent to the "middle" circle, this yields

$$\frac{\sqrt{26}}{5} = \frac{\sqrt{26}x}{5} + x + 1.$$

Hence $x = \dfrac{\sqrt{26}-5}{\sqrt{26}+5}$. Similarly, since the largest circle is tangent to the "middle" circle and has a hypotenuse of length $\sqrt{26}X/5$, we obtain

$$\frac{\sqrt{26}X}{5} = \frac{\sqrt{26}}{5} + 1 + X.$$

Hence $X = \dfrac{\sqrt{26}+5}{\sqrt{26}-5} = \dfrac{1}{x}$.

Comment: David Stone and John Hawkins of Georgia Southern University in Statesboro, GA extended the conjecture of the problem. They solved the problem and then applied the conditions of the problem again, showing that there is a third larger circle of radius $\left(\dfrac{\sqrt{26}+1}{\sqrt{26}-1}\right)^2$, or in the obtuse case, $\left(\dfrac{\sqrt{26}+5}{\sqrt{26}-5}\right)^2$, lying outside the second one. Continuing on in this manner they noted that there is an infinite sequence of circles, growing larger geometrically, lying inside angle A, each one tangent to the sides of A and to its predecessor.

And similarly they noted that there is a infinite sequence of circles *below* the circle of radius 1, growing smaller geometrically, lying inside angle A, with each one being tangent to the sides of A and to its predecessor.

Also solved by Michael Brozinsky, Central Islip, NY; Jerry Chu (Student at George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Michael Fried, Kibbutz Revivim, Israel; Ed Gray, Highland Beach, FL; Paul M.Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; John Nord, Saint George's School, Spokane, WA; Neculai Stanciu, "George Emil Palade" School, Buză, Romania and Titu Zvonaru,Comănesti, Romania; Cassidy Wyse, Becca Gerig and Josh Stimmel (jointly, students at Taylor University), Upland, IN; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5350:** *Proposed by Kenneth Korbin, New York, NY*

The four roots of the equation

$$x^4 - 96x^3 + 206x^2 - 96x + 1 = 0$$

can be written in the form

$$x_{1,2} = \left(\frac{\sqrt{a} + \sqrt{b + \sqrt{c}}}{\sqrt{a} - \sqrt{b + \sqrt{c}}}\right)^{\pm 1}$$

$$x_{3,4} = \left(\frac{\sqrt{a} + \sqrt{b - \sqrt{c}}}{\sqrt{a} - \sqrt{b - \sqrt{c}}}\right)^{\pm 1}$$

where a, b, and c are positive integers.

Find a, b, and c if $(a, b, c) = 1$.

Solution 1 by David E. Manes, SUNY College at Oneonta, Oneonta, NY

The values a, b and c are $a = 10, b = 5$ and 21. One verifies that these values do yield the four roots of the polynomial equation. Also, note that
$(10, 5, 21) = ((10, 5), 21) = (5, 21) = 1$ as required.

Let $r = \dfrac{\sqrt{a} + \sqrt{b + \sqrt{c}}}{\sqrt{a} - \sqrt{b + \sqrt{c}}}$ and $s = \dfrac{\sqrt{a} + \sqrt{b - \sqrt{c}}}{\sqrt{a} - \sqrt{b - \sqrt{c}}}$. If $r, \dfrac{1}{r}, s$ and $\dfrac{1}{s}$ are the roots of the polynomial equation, then

$$(x - r)(x - \frac{1}{r})(x - s)(x - \frac{1}{s}) = x^4 - 96x^3 + 206x^2 - 96x + 1.$$

Expanding the left side of the equation and equating coefficients, one obtains

$$\begin{aligned} r + \frac{1}{r} + s + \frac{1}{s} &= 96 \\ \left(r + \frac{1}{r}\right)\left(s + \frac{1}{s}\right) &= 204. \end{aligned}$$

One calculates

$$r + \frac{1}{r} = \frac{\sqrt{a} + \sqrt{b + \sqrt{c}}}{\sqrt{a} - \sqrt{b + \sqrt{c}}} + \frac{\sqrt{a} - \sqrt{b + \sqrt{c}}}{\sqrt{a} + \sqrt{b + \sqrt{c}}} = \frac{2(a + b + \sqrt{c})}{a - b - \sqrt{c}},$$

$$s + \frac{1}{s} = \frac{\sqrt{a} + \sqrt{b - \sqrt{c}}}{\sqrt{a} - \sqrt{b - \sqrt{c}}} + \frac{\sqrt{a} - \sqrt{b - \sqrt{c}}}{\sqrt{a} + \sqrt{b - \sqrt{c}}} = \frac{2(a + b - \sqrt{c})}{a - b + \sqrt{c}}.$$

Therefore,

$$r + \frac{1}{r} + s + \frac{1}{s} = \frac{a^2 - b^2 + c}{(a - b)^2 - c} = 24 \quad (\mathbf{1})$$

$$\left(r+\frac{1}{r}\right)\left(s+\frac{1}{s}\right)=\frac{(a+b)^2-c}{(a-b)^2-c}=51 \quad (\mathbf{2})$$

Equation ($\mathbf{1}$) written $a^2-b^2+c=24(a-b)^2-c$ when expanded and simplified yields

$$23a^2+25b^2-48ab-25c=0. \qquad (\mathbf{3})$$

Rewrite equations ($\mathbf{1}$) and ($\mathbf{2}$) as follows:

$$\begin{aligned}(a-b)^2-c &= \frac{a^2=b^2+c}{24}\\ (a-b)^2-c &= \frac{(a+b)^2-c}{51}.\end{aligned}$$

Then $\dfrac{a^2-b^2+c}{24}=\dfrac{(a+b)^2-c}{51}$ or

$$9a^2-25b^2-16ab+25c=0. \qquad (\mathbf{4})$$

Adding equations ($\mathbf{3}$) and ($\mathbf{4}$) we get $32a^2-64ab=0$ or $a=2b$ since $a\neq 0$. Substituting $2b$ for a in equation ($\mathbf{4}$) one obtains $25c=2b^2$ or $c=\dfrac{21}{25}b^2$. Since b and c are positive integers, it follows that $b=5k$ for some integer k. Therefore, $c=21k^2$ and $a=2b=10k$. Hence, $b=5, a=10$, and $c=21$ since $(a,b,c)=1$.

Solution 2 by Jerry Chu (student, Saint George's School), Spokane, WA

Obviously, $x_1x_2=x_3x_4=1$. So we can factor the equation into $(x^2+kx+1)(x^2+lx+1)$; expanding this and equating its coefficients to those in the given equation we obtain $\begin{cases}k+l=-96\\ kl=204.\end{cases}$

Let $k=x_1+x_2=\dfrac{2(a+b+\sqrt{c}\,)}{a-b-\sqrt{c}}$, and similarly $l=x_3+x_4=\dfrac{2(a+b-\sqrt{c}\,)}{a-b+\sqrt{c}}$.

Subtracting, we get $k-l=\dfrac{8a\sqrt{c}}{(a-b)^2-c}$. And also from the above system of equations $k-l=\sqrt{(k+l)^2-4kl}=20\sqrt{21}$.

So $c=21$ because $a,b,c\in Z^+$ and $(a,b,c)=1$, therefore $5\left((a-b)^2-21\right)=2a$. Call this Equation $\mathbf{1}$.

On the other hand, $kl=\dfrac{4\left((a+b)^2-c\right)}{(a-b)^2-c}=204$. Therefore, $5\left((a+b)^2-21\right)=(51)(2a)$. Call this Equation ($\mathbf{2}$). Subtracting Equation $\mathbf{1}$ from Equation $\mathbf{2}$ gives us that

$$(a+b)^2-(a-b)^2 = \frac{50(2a)}{5}$$

$$\begin{aligned} 4ab &= 20a \\ b &= 5 \end{aligned}$$

Plugging $b = 5$ into equation 1 we obtain that $a = 10$. Therefore, $\begin{cases} a = 10 \\ b = 5 \\ c = 21. \end{cases}$

Solution 3 by Haroun Meghaichi (student, University of Science and Technology, Houari Boumediene), Algiers, Algeria

Note that the equation in the statement of the problem is equivalent to

$$\left(x + \frac{1}{x}\right)^2 - 96\left(x + \frac{1}{x}\right) + 204 = 0.$$

If x is a solution to this equation, then x^{-1} is also a solution. Take

$$x = \frac{\sqrt{a} + \sqrt{b + \sqrt{c}}}{\sqrt{a} - \sqrt{b + \sqrt{c}}},$$

where $a, b, c \in N$ and c is not a perfect square and $(a, b, c) = 1$, which means that

$$\begin{aligned} x + \frac{1}{x} &= \frac{\sqrt{a} + \sqrt{b + \sqrt{c}}}{\sqrt{a} - \sqrt{b + \sqrt{c}}} + \frac{\sqrt{a} - \sqrt{b + \sqrt{c}}}{\sqrt{a} + \sqrt{b + \sqrt{c}}} \\ &= \frac{2\left(a + b + \sqrt{c}\right)}{a - b - \sqrt{c}}. \end{aligned}$$

with some basic algebraic manipulations we get

$$\left(x + \frac{1}{x}\right)^2 - 96\left(x + \frac{1}{x}\right) + 204 = \frac{16\left(a^2 + 25\left(b^2 - ab + c + \sqrt{c}\,(2b - a)\right)\right)}{\left(a - b - \sqrt{c}\right)^2}.$$

therefore $2b = a$, the equation becomes $25c = 21b^2$. Since $(b, c) = 1$ then $b = 5k, c = 21n$ for some coprime positive integers k, n, and so $n = k^2$, but $(n, k) = 1$ so $n = k = 1$, and

$$(a, b, c) = (10, 5, 21).$$

The same technique works on $x_{3,4}$, so the solution to the problem is $(10, 5, 21)$.

Solution 4 by Brian D. Beasley, Presbyterian College, Clinton, SC

Since $x^4 - 96x^3 + 206x^2 - 96x + 1 = (x^2 + ux + 1)(x^2 + vx + 1)$ with $u = -48 + 10\sqrt{21}$ and $v = -48 - 10\sqrt{21}$, the four roots are

$$x_{1,2} = 24 + 5\sqrt{21} \pm \sqrt{1100 + 240\sqrt{21}} = 24 + 5\sqrt{21} \pm (4\sqrt{35} + 6\sqrt{15})$$

and

$$x_{3,4} = 24 - 5\sqrt{21} \pm \sqrt{1100 - 240\sqrt{21}} = 24 - 5\sqrt{21} \pm (4\sqrt{35} - 6\sqrt{15}),$$

with $x_1 > x_2$ and $x_3 > x_4$. We also note that the roots in each pair are reciprocals, since $(24 + 5\sqrt{21})^2 - (4\sqrt{35} + 6\sqrt{15})^2 = 1$ and $(24 - 5\sqrt{21})^2 - (4\sqrt{35} - 6\sqrt{15})^2 = 1$.

To write the four roots in the desired form, we first set $d_1 = \sqrt{a} + \sqrt{b + \sqrt{c}}$, $d_2 = \sqrt{a} - \sqrt{b + \sqrt{c}}$, $d_3 = \sqrt{a} + \sqrt{b - \sqrt{c}}$, and $d_4 = \sqrt{a} - \sqrt{b - \sqrt{c}}$. Since

$d_1 > d_3 > d_4 > d_2$, this justifies our designating x_1 as the largest root above, with $x_1 > x_3 > x_4 > x_2$. As a result, we require $x_1 + x_2 = d_1/d_2 + d_2/d_1 = 48 + 10\sqrt{21}$ and $x_3 + x_4 = d_3/d_4 + d_4/d_3 = 48 - 10\sqrt{21}$. Then rationalizing produces

$$x_1 + x_2 = \frac{2(a^2 - b^2 + c + 2a\sqrt{c})}{(a-b)^2 - c} = 48 + 10\sqrt{21},$$

so we set $a^2 - b^2 + c = 24[(a-b)^2 - c]$ and $2a = 5[(a-b)^2 - c]$. Letting $c = 21$, we obtain $48ab - 23a^2 = 5(5b^2 - 105)$ and $2a + 10ab - 5a^2 = 5b^2 - 105$. Thus $10a + 2ab - 2a^2 = 0$, so $a - b = 5$, which yields $a = 10$ and $b = 5$. Similarly, we note that $(a, b, c) = (10, 5, 21)$ produces $x_3 + x_4 = 48 - 10\sqrt{21}$ as needed.

Finally, we observe that since there is a unique real number $x > 1$ with $x + 1/x = 48 + 10\sqrt{21}$, we may conclude

$$x_1 = 24 + 5\sqrt{21} + 4\sqrt{35} + 6\sqrt{15} = \frac{\sqrt{10} + \sqrt{5 + \sqrt{21}}}{\sqrt{10} - \sqrt{5 + \sqrt{21}}}.$$

Similarly, we have the corresponding results for x_2, x_3, and x_4.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Neculai Stanciu, "George Emil Palade" School, Buză, Romania and Titu Zvonaru, Comănesti, Romania; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro GA; Vu Tran (student, Purdue University), West Lafayette, IN, and the proposer.

- **5351:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

Let x, y, z be positive real numbers. Show that

$$\frac{xy}{x^3 + y^3 + xyz} + \frac{yz}{y^3 + z^3 + xyz} + \frac{zx}{z^3 + x^3 + xyz} \leq \frac{3}{x + y + z}.$$

Solution 1 by Ed Gray, Highland Beach, FL

Divide the numerator and denominator of the first term on the left side of the inequality by xy, and the numerator and denominator of the second term by yz and similarly the third term by zx. Thus, the left hand side becomes

$$\frac{1}{\dfrac{x^3 + y^3}{xy} + z} + \frac{1}{\dfrac{y^3 + z^3}{yz} + x} + \frac{1}{\dfrac{z^3 + x^3}{zx} + y}.$$

$$\begin{aligned} \frac{x^3 + y^3}{xy} + z &= \frac{(x+y)(x^2 - xy + y^2)}{xy} + z \\ &= (x+y)\left(\frac{x^2}{xy} - 1 + \frac{y^2}{xy}\right) + z \end{aligned}$$

$$= \quad (x+y)\left(\frac{x}{y}-1+\frac{y}{x}\right)+z$$

But $\frac{x}{y}+\frac{y}{z}-1 \geq 1$, so $\frac{x^3+y^3}{xy}+z \geq (x+y+z)$, and $\frac{1}{\frac{x^3+y^3}{xy}+z} \leq \frac{1}{x+y+z}$.

Each of the other two terms are handled in precisely the same manner, so, to avoid repetition,

$$\frac{1}{\frac{x^3+y^3}{xy}+z}+\frac{1}{\frac{y^3+z^3}{yz}+x}+\frac{1}{\frac{z^3+x^3}{zx}+y} \leq \frac{1}{x+y+z}+\frac{1}{y+z+x}+\frac{1}{z+x+y}=\frac{3}{x+y+z}.$$

Note that equality holds if, and only if, $x=y=z$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

We have

$$\frac{xy}{x^3+y^3+xyz}+\frac{yz}{y^3+z^3+xyz}+\frac{zx}{z^3+x^3+xyz}$$

$$= \quad \frac{1}{x+y+z+\frac{(x+y)(x-y)^2}{xy}}+\frac{1}{x+y+z+\frac{(y+z)(y-z)^2}{yz}}+\frac{1}{x+y+z+\frac{(z+x)(z-x)^2}{zx}}$$

$$\leq \quad \frac{1}{x+y+z}+\frac{1}{x+y+z}+\frac{1}{x+y+z}$$

$$= \quad \frac{3}{x+y+z}, \text{ as required.}$$

Solution 3 by Arkady Alt, San Jose, CA

Since $x^3+y^3 \geq xy\,(x+y) \iff x^3+y^3-xy\,(x+y)=(x+y)\,(x-y)^2 \geq 0$ then

$$\sum_{cyc}\frac{xy}{x^3+y^3+xyz} \leq \sum_{cyc}\frac{xy}{xy\,(x+y)+xyz}=\sum_{cyc}\frac{1}{x+y+z}=\frac{3}{x+y+z}.$$

Also solved by Dionne T. Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Jerry Chu (student, Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ethan Gegner (student, Taylor University), Upland, IN; Nikos Kalapodis, Patras, Greece; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria Spain; Henry Ricardo, New York Math Circle, NY; Albert Stadler, Herrliberg, Switzerland; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania; Titu Zvonaru, Comănesti, Romania; and the proposers.

- **5352:** *Proposed by Arkady Alt, San Jose, CA*

  Evaluate $\sum\limits_{k=0}^{n} x^k - (x-1)\sum\limits_{k=0}^{n-1}(k+1)x^{n-1-k}$.

  Solution 1 by G.C. Greubel, Newport News, VA

  Consider the series $\displaystyle\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$ for which the series in question becomes

$$\begin{aligned} S &= \sum_{k=0}^{n} x^k - (x-1)\sum_{k=0}^{n-1}(k+1)\,x^{n-k-1} \\ &= \frac{1-x^{n+1}}{1-x} + (1-x)\left[\sum_{k=0}^{n-1} x^{n-k-1} + \sum_{k=0}^{n} k\,x^{n-k-1}\right] \\ &= \frac{1-x^{n+1}}{1-x} + (1-x)\,x^{n-1}\left[\frac{1-\left(\frac{1}{x}\right)^n}{1-\frac{1}{x}} + x\,\partial_x\left(\frac{1-\left(\frac{1}{x}\right)^n}{1-\frac{1}{x}}\right)\right] \\ &= \frac{1-x^{n+1}}{1-x} + (1-x)\cdot\frac{1-x^n}{1-x} + (x-1)\,x^{n+2}\left[\frac{n(x-1)+1-x^n}{x^{n+2}}\right] \\ &= \frac{1-x^{n+1}}{1-x} + 1 - x^n + n - \frac{1-x^n}{1-x} \\ &= n+1. \end{aligned}$$

  From this it can be stated that

$$\sum_{k=0}^{n} x^k - (x-1)\sum_{k=0}^{n-1}(k+1)\,x^{n-k-1} = n+1.$$

  Solution 2 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

  Since $\displaystyle\sum_{k=0}^{n-1}(k+1)x^{n-1-k} = \sum_{k=1}^{n} kx^{n-k}$, then $\displaystyle(x-1)\sum_{k=0}^{n-1}(k+1)x^{n-1-k} =$
  $\displaystyle\sum_{k=1}^{n} kx^{n-k+1} - \sum_{k=2}^{n+1}(k-1)x^{n-k+1} = x^n + \sum_{k=2}^{n} x^{n-k+1} - n = -n + \sum_{k=1}^{n} x^k$, and therefore

$$\sum_{k=0}^{n} x^k - (x-1)\sum_{k=0}^{n-1}(k+1)x^{n-1-k} = 1+n.$$

  Solution 3 by Henry Ricardo, New York Math Circle, NY

  Denote the given expression as $F_n(x)$, where we assume that $n \geq 1$ and $x \neq 0$. Since
  $F_1(x) = 1 + x - (x-1)(0) = 2 = 1+1$ and
  $F_2(x) = (1+x+x^2) - (x-1)(x+2) = 3 = 2+1$, we conjecture that $F_n(x) = n+1$ for all nonzero values of x and prove this by induction.

Suppose that $F_N(x) = N+1$ for some integer $N \geq 3$ and all $x \neq 0$. Then

$$\begin{aligned}
F_{N+1}(x) &= \sum_{k=0}^{N+1} x^k - (x-1)\sum_{k=0}^{N}(k+1)x^{N-k} \\
&= x\sum_{k=0}^{N} x^k + 1 - (x-1)\left(\sum_{k=0}^{N-1}(k+1)x^{N-k} + N + 1\right) \\
&= x\sum_{k=0}^{N} x^k + 1 - (x-1)\left(x\sum_{k=0}^{N-1}(k+1)x^{N-k-1} + N + 1\right) \\
&= 1 + x\left(\sum_{k=0}^{N} x^k - (x-1)\sum_{k=0}^{N-1}(k+1)x^{N-k-1}\right) - (N+1)(x-1) \\
&= 1 + x(N+1) - (N+1)(x-1) = N+2 = (N+1)+1.
\end{aligned}$$

Also solved by Dionne T. Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Jerry Chu (student, Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ethan Gegner (student, Taylor University), Upland, IN; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Haroun Meghaichi (student, University of Science and Technology, Houari Boumediene), Algiers, Algeria; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Henry Ricardo (two additional solutions to his one above), New York Math Circle, New York; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins of Georgia Southern University in Statesboro, GA, and the proposer.

5353: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Let $A(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n with complex coefficients. Prove that all its zeros lie in the disk $\mathcal{D} = \{z \in C \,:\, |z| < r\}$, where

$$r = \left\{1 + \left(\sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right|^3\right)^{1/2}\right\}^{2/3}$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

$A(z)$ is a polynomial of degree n. So $a_n \neq 0$. Let $|z| \geq r$. Then, by Hölder's inequality,

$$\frac{1}{|a_n|}|A(z)| \geq |z|^n - \sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right||z|^k \geq |z|^n - \left(\sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right|^3\right)^{\frac{1}{3}}\left(\sum_{k=0}^{n-1}|z|^{\frac{3k}{2}}\right)^{\frac{2}{3}} = |z|^n - \left(r^{\frac{3}{2}}-1\right)^{\frac{2}{3}}\left(\frac{|z|^{\frac{3n}{2}}-1}{|z|^{\frac{3}{2}}-1}\right)^{\frac{2}{3}}$$

$$\geq |z|^n - \left(r^{\frac{3}{2}}-1\right)^{\frac{2}{3}}\left(\frac{|z|^{\frac{3n}{2}}-1}{|r|^{\frac{3}{2}}-1}\right)^{\frac{2}{3}}$$

$$= |z|^n - \left(|z|^{\frac{3n}{2}} - 1\right)^{\frac{2}{3}} > |z|^n - \left(|z|^{\frac{3n}{2}}\right)^{\frac{2}{3}} = 0.$$

So all zeros lie in the open disk $\mathcal{D}$

Solution 2 by Kee-Wai Lau, Hong Kong, China

According to Theorem (27.4) on p. 124 of [1], we have the following result:

For any p and q such that $p > 1, q > 1, \dfrac{1}{p} + \dfrac{1}{q} = 1$, the polynomial $f(x) = a_0 + a_1 x + \cdots a_n z^n, a_n \neq 0$ has all of its zeros in the circle

$$|z| < \left\{ 1 + \left(\sum_{j=0}^{n-1} \left| \frac{a_j}{a_n} \right|^p \right)^{q/p} \right\}^{1/q} \leq \left(1 + n^{q/p} M^q\right)^{1/q},$$

where $M = \max \left| \dfrac{a_j}{a_n} \right|, j = 0, 1, \cdots, n-1.$

In particular, when $p = 3$, the result of the above problem follows.

Reference: 1. M. Marden: *Geometry of Polynomials,* Mathematical Surveys and Monographs Number 3, American Mathematical Society, (1966).

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain, and the proposer.

- **5354:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a, b, c > 0$ be real numbers. Prove that the series

$$\sum_{n=1}^{\infty} \left[n \cdot \left(a^{\frac{1}{n}} - \frac{b^{\frac{1}{n}} + c^{\frac{1}{n}}}{2} \right) - \ln \frac{a}{\sqrt{bc}} \right],$$

converges if and only if $2 \ln^2 a = \ln^2 b + \ln^2 c$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

Let x be real. By Taylor's theorem there is a number $h = h(x),\ 0 \leq h \leq 1$, such that $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} e^{hx}$. We choose $x = \dfrac{\ln a}{n},\ x = \dfrac{\ln b}{n},\ x = \dfrac{\ln c}{n}$ and get

$$\begin{aligned}
a^{\frac{1}{n}} &= 1 + \frac{\ln a}{n} + \frac{\ln^2 a}{2n^2} + \frac{\ln^3 a}{6n^3} a^{\frac{h}{n}},\ 0 \leq h = h(a, n) \leq 1, \\
b^{\frac{1}{n}} &= 1 + \frac{\ln b}{n} + \frac{\ln^2 b}{2n^2} + \frac{\ln^3 b}{6n^3} b^{\frac{h}{n}},\ 0 \leq h = h(b, n) \leq 1, \\
c^{\frac{1}{n}} &= 1 + \frac{\ln c}{n} + \frac{\ln^2 c}{2n^2} + \frac{\ln^3 c}{6n^3} c^{\frac{h}{n}},\ 0 \leq h = h(c, n) \leq 1.
\end{aligned}$$

So

$$\sum_{n=1}^{\infty}\left[n\left(a^{\frac{1}{n}}-\frac{b^{\frac{1}{n}}+c^{\frac{1}{n}}}{2}\right)-\ln\frac{a}{\sqrt{bc}}\right]$$

$$=\sum_{n=1}^{\infty}\frac{1}{2n}\left(\ln^2 a-\frac{\ln^2 b+\ln^2 c}{2}\right)+\sum_{n=1}^{\infty}\frac{1}{6n^2}\left((\ln^3 a)a^{\frac{h(a,n)}{n}}-\frac{(\ln^3 b)b^{\frac{h(b,n)}{n}}+(\ln^3 c)b^{\frac{h(c,n)}{n}}}{2}\right).$$

The second sum is convergent. The first sum equals 0 if $\ln^2 a=\dfrac{\ln^2 b+\ln^2 c}{2}$ and it diverges if $\ln^2 a\neq\dfrac{\ln^2 b+\ln^2 c}{2}$.

Solution 2 by Anastasios Kotronis, Athens, Greece

For $x>0$ real number it is

$$x^{\frac{1}{n}}=\exp\left(\frac{\ln x}{n}\right)=1+\frac{\ln x}{n}+\frac{\ln^2 x}{2n^2}+\mathcal{O}(n^{-3}). \tag{1}$$

Setting

$$A_n=n\cdot\left(a^{\frac{1}{n}}-\frac{b^{\frac{1}{n}}+c^{\frac{1}{n}}}{2}\right)-\ln\frac{a}{\sqrt{bc}}$$

and

$$A=\frac{\ln^2 a}{2}-\frac{\ln^2 b}{4}-\frac{\ln^2 c}{4},$$

so that $A=0\iff 2\ln^2 a=\ln^2 b+\ln^2 c$, with a,b and c respectively in the place of x in (1) we get

$$A_n=\frac{A}{n}+\mathcal{O}(n^{-2}). \tag{2}$$

- If $A=0$, (2) gives that for some real $c>0$ and positive integer n_0,

$$0\le |A_n|\le\frac{c}{n^2},\qquad n\ge n_0$$

so $\sum_{n\ge n_0}A_n$ converges absolutely and hence the given series converges.

- If $A\neq 0$, (2) gives that for some real $c>0$ and positive integer n_0,

$$-\frac{c}{n^2}+A\le A_n\le A+\frac{c}{n^2},\qquad n\ge n_0$$

so $\sum_{n\ge n_0}A_n=\operatorname{sgn}(A)\cdot\infty$ and hence the given series diverges.

Solution 3 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

The general term of the series is

$$
\begin{aligned}
&n\Big(1+\frac{\ln a}{n}+\frac{\ln^2 a}{4n^2}+O\left(\frac{1}{n^3}\right)-\frac{1}{2}-\frac{\ln b}{2n}-\frac{\ln^2 a}{8n^2}+O\left(\frac{1}{n^3}\right)+\\
&-\frac{1}{2}-\frac{\ln c}{2n}-\frac{\ln^2 c}{8n^2}+O\left(\frac{1}{n^3}\right)-\ln\frac{a}{\sqrt{bc}}\Big)=\\
&=n(1-\frac{1}{2}-\frac{1}{2})+(\ln\frac{a}{\sqrt{bc}}-\ln\frac{a}{\sqrt{bc}})+n\frac{1}{8n}\left(2\ln^2 a-\ln^2 b-\ln^2 c\right)+O\left(\frac{1}{n^2}\right)=\\
&=O\left(\frac{1}{n^2}\right)
\end{aligned}
$$

whence the absolute convergence.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Haroun Meghaichi (student, University of Science and Technology, Houari Boumediene), Algiers, Algeria, and the proposer.

Mea Culpa

Apologies to **Arkady Alt of San Jose, CA** for inadvertently not acknowledging his solutions to problems 5343, 5344 and 5346.