Problems Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
January 15, 2015

- **5319:** *Proposed by Kenneth Korbin, New York, NY*

  Let N be an odd integer greater than one. Then there will be a Primitive Pythagorean Triangle with perimeter equal to $\left(N^2+N\right)^2$. For example, if $N=3$, then the perimeter equals $\left(3^2+3\right)^2=144$.

  Find the sides of the PPT for perimeter $\left(15^2+15\right)^2$ and for perimeter $\left(99^2+99\right)^2$.

- **5320:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  It is fairly well known that if (a,b,c) is a Primitive Pythagorean Triple (PPT), then the product abc is divisible by 60. Find infinitely many PPT's (a,b,c) such that the sum $(a+b+c)$ is also divisible by 60.

- **5321:** *Proposed by Lawrence M. Lesser, University of Texas at El Paso, TX*

  On pop quizzes during the fall semester, Al gets 1 out of 3 questions correct, while Bob gets 3 of 8 correct. During the spring semester, Al gets 3/5 questions correct, while Bob gets 2/3 correct. So Bob did better each semester ($3/8>1/3$ and $2/3>3/5$) but worse for the overall academic year ($5/11<4/8$). The total number of questions involved in the above example was $3+8+5+3=19$, and the author conjectures (in his chapter in the 2001 Yearbook of the National Council of Teachers of Mathematics) that this is smallest dataset with nonzero numerators in which this reversal (Simpson's Paradox) happens. If we allow zeros, the smallest dataset is conjectured to be nine: $0/1<1/4$ and $2/3<1/1$, but $2/4>2/5$.

  Prove these conjectures or find counterexamples.

- **5322:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu "George Emil Palade" School, Buzău, Romania*

If $\lim_{n\to\infty}\left(-\frac{3}{2}\sqrt[3]{n^2}+\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}\right)=a>0$, then compute $\lim_{n\to\infty}\left(\frac{-\frac{3}{2}\sqrt[3]{n^2}+\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}}{a}\right)^{\sqrt[3]{n}}$.

- **5323:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be positive real numbers greater than or equal to one. Prove that

$$\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^{-2}+\left(\frac{1}{n^2}\prod_{k=1}^{n}a_k^{-2}\right)\left(\sum_{k=1}^{n}\left(a_k^2-1\right)^{1/2}\right)^2\le 1.$$

- **5324:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Calculate

$$\sum_{n=1}^{\infty}\left(n\ln\left(1+\frac{1}{n}\right)-1+\frac{1}{2n}\right).$$

Solutions

- **5301:** *Proposed by Kenneth Korbin, New York, NY*

  A convex cyclic quadrilateral with integer length sides is such that its area divided by its perimeter equals 2014.

  Find the maximum possible perimeter.

  Solution 1 by Proposer

- The figure is an isosceles trapezoid. Let b_1, b_2 be the bases, h the height, l the non-parallel sides, and let $N = 2014$.
- The bases are $b_1 = 2$ and $b_2 = 8N^2$.
- Each leg is equal to the arithmetic mean of the bases,

$$l=\frac{b_1+b_2}{2}=4N^2+1.$$

- The altitude h is equal to the geometric mean of the bases.

$$h=\sqrt{b_1b_2}=4N.$$

- The area equals,

$$\frac{1}{2}h\left(b_1+b_2\right)=hl=\frac{(b_1+b_2)(\sqrt{b_1b_2}}{2}=16N^3+4N.$$

- Perimeter $= b_1 + b_2 + 2l = 4l = 16N^2 + 4$.
- $\dfrac{\text{Area}}{\text{Perimeter}} = \dfrac{16N^3 + 4N}{16N^2 + 4} = N$.
- $l^2 - h^2 = (l-2)^2$, (sides of a PPP.)

Letting the sides be (a, b, c, d) and letting $a = 2$ and $\sqrt{ac} = 4 \cdot 2014 = 8056$ gives $c = 32,449,568$.

Letting $b = d = \dfrac{a+c}{2} = 16,224,785$.

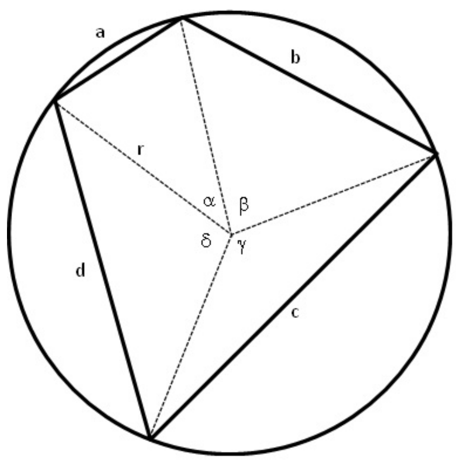
Then, Perimeter $= 4b = 4d$ and $\sqrt{bd} = b = d$.

Area $= K = \sqrt{abcd} = \sqrt{ac}\sqrt{bd} = 8056b$

$\dfrac{\text{Area}}{\text{Perimeter}} = \dfrac{8056b}{4b} = 2014$.

So, Perimeter$= P = 64,899,140$.

Solution 2 and Comments, jointly posted by Michael N. Fried of Kibbutz Revivim, Israel and Edwin Gray, Highland Beach, FL



We can begin to approach this problem in an obvious way. Let the sides be a, b, c, d, the area A, and the perimeter P. Let the quadrilateral be inscribed in a circle of radius r, and let the sides subtend the angles at the center $\alpha, \beta, \gamma, \delta$ (see figure). Then, we have:

$$A = \frac{1}{2}r^2(\sin\alpha + \sin\beta + \sin\gamma + \sin\delta)$$

And,

$$P = 2r\left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} + \sin\frac{\delta}{2}\right)$$

So that,

$$\frac{A}{P} = \frac{1}{4}\frac{\sin\alpha + \sin\beta + \sin\gamma + \sin\delta}{\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} + \sin\frac{\delta}{2}}r$$

Or, in terms of P,

$$\frac{A}{P} = \frac{1}{8}\frac{\sin\alpha + \sin\beta + \sin\gamma + \sin\delta}{\left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} + \sin\frac{\delta}{2}\right)^2}P = \sigma P$$

where σ is a function of α, β, γ (since $\delta = 360 - (\alpha + \beta + \gamma)$).

It is easy to see that σ tends to 0 as $\alpha, \beta, \gamma, \delta$ tend to 0. Consider a sequence of $\alpha, \beta, \gamma, \delta$ where $\alpha = \beta = \gamma$ and $\delta = 360 - 3\alpha$. For this sequence, we have:

$$\sigma(\alpha) = \frac{1}{8} \frac{3\sin\alpha - \sin 3\alpha}{\left(3\sin\frac{\alpha}{2} + \sin\frac{3\alpha}{2}\right)^2}$$

We can see clearly that (1) $\sigma(\alpha)$ is continuous in a right neighborhood of 0, and (2) if we write the Taylor expansions of the numerator and denominator of $\sigma(\alpha)$ we observe that the lowest power of α in the numerator is 3 while the lowest power is 2 in the denominator, so that $\sigma(\alpha)$ is $o(\alpha)$.

Therefore, if $P(\alpha)$ is the perimeter of the cyclic quadrilateral corresponding to $\alpha, \beta, \gamma, \delta$ where $\alpha = \beta = \gamma$ and $\delta = 360 - 3\alpha$ then since $\dfrac{2014}{\sigma(\alpha)} = P(\alpha)$, we find that $P(\alpha)$ increases without bound as α tends to 0.

This does not solve the problem; however, it does show that *if* the problem has a solution it depends entirely on the fact that the sides *each* have integer length (the situation is analogous to the fact that if (x, y) is an *integer* point on the hyperbola $x^2 - y^2 = 81$ then the sum $x + y$ has maximum, while if (x, y) is *any* point on the hyperbola then the sum $x + y$ has no maximum).

Now, Ken Korbin has shown the existence of a cyclic quadrilateral with integer sides and $\dfrac{A}{P} = 2014$, which he claims to be maximal. He maintains this is an isosceles trapezoid (which it must be if it is to be cyclic) with one base equal to 2 and the other 8×2014^2. He sets the remaining sides equal to the arithmetic mean of these values and asserts that the height must then be the geometric mean of the bases.

From this, he shows easily enough that this quadrilateral with sides 2, $8 \times 2014^2 = 32,449,568$, and 16,224,785 taken twice satisfies the condition that $\dfrac{A}{P} = 2014$. But of course this does not prove that the perimeter is maximal (even if it is). I might also mention that the sides of the equilateral trapezoid can be permuted without making the resulting quadrilateral non-cyclic or changing the perimeter and area, being an equilateral trapezoid is not essential

Ed Gray, however, has explained clearly why the equal sides should be the arithmetic mean of the other sides when we take the height to be the geometric mean and why, in this special case, Ken's solution is maximal. Ed writes as follows:

I have looked at Ken's solution to the problem, and while the answer may be correct, I don't see any proof that the answer is a maximum. It is easy to buy into the shape of an isosceles trapezoid, and we shall do that in general terms.

Let the trapezoid have an "upper" base of a, a "lower" base of c, with $c > a$. Let the trapezoid have equal lateral sides be b and d, with b on the right, d on the left, so the figure is $abcd$ reading clockwise.

From the right-most end of a, we drop an altitude h perpendicular to a down to c, where it also meets at right angles. Call the intersection point F. Since $c > a$, there is a part of c to the right of $F = \dfrac{c-a}{2}$ or $\dfrac{c}{2} - \dfrac{a}{2}$. We now have a right triangle with hypotenuse b and legs h and $\dfrac{c}{2} - \dfrac{a}{2}$.

By the Pythagorean Theorem,

$$b^2 = h^2 + (c/2 - a/2)^2 \tag{1}$$

$$b^2 = h^2 + c^2/4 - ac/2 + a^2/4. \tag{2}$$

By letting $h^2 = ac$, we have

$$b^2 = ac + c^2/4 - ac/2 + a^2/4 = c^2/4 + ac/2 + a^2/4 = (c/2 + a/2)^2 \tag{3}$$

or

$$b = (a + c)/2 \tag{4}$$

The area is:

$$A = (1/2)(a + c)\sqrt{ac} \tag{5}$$

The perimeter is:

$$P = a + c + 2(a + c)/2 = 2(a + c) \tag{6}$$

By hypothesis,

$$A = 2014P \tag{7}$$

Substituting (5) and (6) into (7),

$$(1/2)(a + c)\sqrt{ac} = 2014(2a + 2c) = 4028(a + c) \tag{8}$$

Multiplying by $2/(a + c)$,

$$\sqrt{ac} = 8056 \tag{9}$$

Squaring,

$$ac = 8056^2 \tag{10}$$

Side a must be even in order for b to be an integer. Since $b = d = (c + a)/2$, to maximize the perimeter $P = 2(a + c)$, we should like a to be the smallest integer possible (this is because ac is constant). Since it must also be even, let $a = 2$. Then (10) becomes:

$$2c = 8056^2 \tag{11}$$

So that, $c = 32,449,568$ and $b = d = (c + a)/2 = 16,224,785$. Thus the largest perimeter in this case is:

$$p = 2 + 16,224,785 + 32,449,568 + 16,224,785 = 64,899,140$$

Q.E.D.

Michael continues on as follows:

I would only add one clarification to Ed's explanation. It is that seemingly arbitrary assumption that $h^2 = ac$. The point is this. Since $A = 2014P$, A is an integer and h is rational. On the other hand if we multiply equation (1) by 4, we obtain $(2b)^2 = (2h)^2 + (c - a)^2$. From this it follows that $2h$ is an integer and $2b$, $c - a$, and $2h$ are a Pythagorean triple. Accordingly, $2b = k(m^2 + n^2)$, $c - a = k(m^2 - n^2)$ and $2h = 2kmn$ or $h = kmn$. Thus, taking $c = km^2$ and $a = kn^2$, we have, $h^2 = k^2m^2n^2 = km^2kn^2 = ac$.

- **5302:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  If n is an even perfect number, $n > 6$, and $\phi(n)$ is the Euler phi-function, then show that $n - \phi(n)$ is a fourth power of an integer. Find infinitely many integers n such that $n - \phi(n)$ is a fourth power.

  Solution 1 by Brian D. Beasley, Presbyterian College, Clinton, SC

  (i) If n is an even perfect number with $n > 6$, then $n = 2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are both odd primes. Since ϕ is multiplicative, we have $\phi(n) = 2^{p-2}(2^p - 2)$, which implies

  $$n - \phi(n) = 2^{p-1}(2^p - 1) - 2^{p-2}(2^p - 2) = 2^{2p-1} - 2^{2p-2} = 2^{2p-2} = (2^{(p-1)/2})^4,$$

  where $2^{(p-1)/2}$ is an integer since p is odd.

  (ii) One trivial solution is to let n be any prime. Then $n - \phi(n) = 1$. A less trivial solution is to take $n = 2^{4k+1}$ for any nonnegative integer k. Then

  $$n - \phi(n) = 2^{4k+1} - 2^{4k} = 2^{4k} = (2^k)^4.$$

  Solution 2 by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX

  We will begin with the following facts for the phi-function:
  1. If p is prime, $\phi(p) = p - 1$.
  2. If p is prime and a is a positive integer, $\phi(p^a) = p^{a-1}(p-1)$.
  3. If the gcd (a,b)=1, $\phi(ab) = \phi(a)\phi(b)$.

  We also note that an even perfect number $n > 6$ can be written in the form $n = 2^{k-1}(2^k - 1)$, where k and 2^{k-1} are prime and $k > 2$. Then, since $\gcd(2^{k-1}, 2^k - 1) = 1$ and $2^k - 1$ is prime,

  $$\begin{aligned} \phi(n) &= \phi[2^{k-1}(2^k - 1)] \\ &= \phi(2^{k-1})\phi(2^k - 1) \\ &= 2^{k-2}(2^k - 2), \\ &= 2^{k-1}(2^{k-1} - 1). \end{aligned}$$

  Further, since k must be an odd prime,

  $$\begin{aligned} n - \phi(n) &= 2^{k-1}(2^k - 1) - 2^{k-1}(2^{k-1} - 1) \\ &= 2^{k-1}(2^k - 2^{k-1}) \\ &= 2^{k-1}[2^{k-1}(2-1)] \\ &= 2^{2(k-1)} \\ &= (2^{k-1})^2 \\ &= \left[2^{2\left(\frac{k-1}{2}\right)}\right]^2 \\ &= \left(2^{\frac{k-1}{2}}\right)^4. \end{aligned}$$

  Therefore, $n - \phi(n)$ is a fourth power of an integer. If $k = 4m + 1$ for $m \geq 1$, and p is an arbitrary prime,

$$\begin{aligned}\phi(n) &= \phi(p^{4m+1})\\ &= p^{4m}(p-1)\\ &= p^{4m+1}-p^{4m}.\end{aligned}$$

Then,

$$\begin{aligned}n-\phi(n) &= p^{4m+1}-\phi(p^{4m+1})\\ &= p^{4m+1}-(p^{4m+1}-p^{4m})\\ &= p^{4m}\\ &= (p^m)^4.\end{aligned}$$

Since there are an infinite number of choices for p and m, this provides an example of infinitely many integers n such that $n-\phi(n)$ is a fourth power.

Also solved by Pat Costello, Eastern Kentucky University, Richmond. KY; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5303:** *Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

Let a, b, c, d be positive real numbers Prove that

$$a^4+b^4+c^4+d^4+4 \ \geq\ 4\left(\left(a^2b^2+1\right)\left(b^2c^2+1\right)\left(c^2d^2+1\right)\left(d^2a^2+1\right)\right)^{1/4}.$$

Solution 1 by David E. Manes, SUNY College at Oneonta, Oneonta, NY By the Arithmetic Mean-Geometric Mean inequality,

$$\left(\left(a^2b^2+1\right)\left(b^2c^2+1\right)\left(c^2d^2+1\right)\left(d^2a^2+1\right)\right)^{1/4} \leq \frac{a^2b^2+b^2c^2+c^2d^2+d^2a^2+4}{4}$$

with equality if and only if $a=b=c=d$. Therefore,

$$4\left(\left(a^2b^2+1\right)\left(b^2c^2+1\right)\left(c^2d^2+1\right)\left(d^2a^2+1\right)\right)^{1/4} \leq a^2b^2+b^2c^2+c^2d^2+d^2a^2+4.$$

Define vectors $\vec{u}$ and $\vec{v}$ such that $\vec{u}=\left\langle a^2, b^2, c^2, d^2\right\rangle$ and $\vec{v}=\left\langle b^2, c^2, d^2, a^2\right\rangle$,

Then the Cauchy-Schwarz inequality implies $\vec{u}\bullet\vec{v}=||\vec{u}||\cdot||\vec{v}||$ so that

$$\begin{aligned}a^2b^2+b^2c^2+c^2d^2+d^2a^2 &\leq \sqrt{a^4+b^4+c^4+d^4}\sqrt{b^4+c^4+d^4+a^4}\\ &= a^4+b^4+c^4+d^4.\end{aligned}$$

Hence,

$$4\left(\left(a^2b^2+1\right)\left(b^2c^2+1\right)\left(c^2d^2+1\right)\left(d^2a^2+1\right)\right)^{1/4} \leq a^4+b^4+c^4+d^4+4$$

with equality if and only if $a = b = c = d$.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

$$\begin{aligned} a^4+b^4+c^4+d^4+4 &= \Big((a^2)^2+(b^2)^2+(c^2)^2+(d^2)^2\Big)^{1/2}\Big((a^2)^2+(b^2)^2+(c^2)^2+(d^2)^2\Big)^{1/2}+4 \\ &\geq a^2b^2+b^2c^2+c^2d^2+d^2a^2+4 \\ &= (a^2b^2+1)+(b^2c^2+1)+(c^2d^2+1)+(d^2a^2+1) \\ &\geq 4\Big((a^2b^2+1)(b^2c^2+1)(c^2d^2+1)(d^2a^2+1)\Big)^{1/4}, \end{aligned}$$

where we have used the Cauchy-Schwarz and the arithmetic mean-geometric mean inequalities.

Equality occurs if, and only if, it occurs in both inequalities, that is if, and only if, $a^2/b^2 = b^2/c^2 = c^2/d^2 = d^2/a^2$ and $a^2b^2+1 = b^2c^2+1 = c^2d^2+1 = d^2a^2+1$.

That is, inequality holds if, and only if, $a = b = c = d$.

Solution 3 by Albert Stadler, Herrliberg, Switzerland

By the AM-GM inequality,

$$\begin{aligned} \frac{a^4+b^4}{2} &\geq a^2b^2 \\ \frac{b^4+c^4}{2} &\geq b^2c^2 \\ \frac{c^4+d^4}{2} &\geq c^2d^2 \\ \frac{d^4+a^4}{2} &\geq d^2a^2. \end{aligned}$$

Adding these inequalities we obtain

$$a^4+b^4+c^4+d^4+4 \geq (a^2b^2+1)+(b^2c^2+1)+(c^2d^2+1)+(d^2a^2+1).$$

We apply once more the AM-GM inequality to obtain

$$(a^2b^2+1)+(b^2c^2+1)+(c^2d^2+1)+(d^2a^2+1) \geq 4\Big((a^2b^2+1)+(b^2c^2+1)+(c^2d^2+1)+(d^2a^2+1)\Big)^{1/4},$$

and the claimed statement follows.

Comment by editor: **Titu Zvonaru, Comăesti, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania** jointly solved the problem in the manner of solution 3, and noted that the statement of the problem can be made stronger for it also holds for all real numbers, not just the positive ones.

Also solved by Arkady Alt, San Jose, CA; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX; José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, and the proposer.

- **5304:** *Proposed by Michael Brozninsky, Central Islip, NY*

Determine whether or not there exist nonzero constants a and b such that the conic whose polar equation is

$$r = \sqrt{\frac{a}{\sin(2\theta) - b\cdot\cos(2\theta)}}$$

has a rational eccentricity.

Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX

We will begin with the use of the transformation formulas and the following trigonometric identities to change the polar form into the rectangular form of the hyperbola:

$$\begin{aligned} x &= r\cos\theta) && (1)\\ y &= r\sin\theta && (2)\\ \sin(2\theta) &= 2\sin\theta\cos\theta, && (3)\\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta. && (4) \end{aligned}$$

Then, using (1), (2), (3), and (4),

$$\begin{aligned} r &= \sqrt{\frac{a}{\sin(2\theta) - b\cos(2\theta)}}\\ r^2 &= \frac{a}{\sin(2\theta) - b\cos(2\theta)}\\ 2r^2\sin\theta\cos\theta - br^2(\cos^2\theta - \sin^2\theta) &= a\\ 2(r\sin\theta)(r\cos\theta) - b(r\cos\theta)^2 + b(r\sin\theta)^2 &= a\\ 2xy - bx^2 + by^2 &= a\\ x^2 - \frac{2}{b}xy - y^2 - \frac{a}{b} &= 0. \end{aligned}$$

With the general form of the hyperbola being

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

we have $A = 1$, $B = -\dfrac{2}{b}$, $C = -1$, and $F = -\dfrac{a}{b}$. The usual methods of rotation of axes in analytic geometry can be used to ascertain the eccentricity of the hyperbola, or the following formula **[1]** gives the eccentricity in a straightforward manner.

$$e = \sqrt{\frac{2\sqrt{(A-C)^2 + B^2}}{\eta(A+C) + \sqrt{(A-C)^2 + B^2}}}\quad, \qquad (12)$$

where $\eta = 1$ if the determinant of the 3x3 matrix

$$\begin{pmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{pmatrix}$$

is negative, or $\eta = -1$ if the determinant is positive. Thus, using (6)

$$\begin{aligned} e &= \sqrt{\frac{2\sqrt{4+\dfrac{4}{b^2}}}{\eta(0)+\sqrt{4+\dfrac{4}{b^2}}}} \\ &= \sqrt{2}. \end{aligned}$$

Thus, the eccentricity is irrational for all values of a and b.

Reference:
[1]Ayoub, Ayoub B., "The Eccentricity of a Conic Section," *The College Mathematics Journal* 34(2), March 2003, 116-121.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5305:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let x be a positive real number. Prove that

$$\frac{[x]}{2x+\{x\}} + \frac{[x]\{x\}}{3x^2} + \frac{\{x\}}{2x+[x]} \leq \frac{1}{2},$$

where $[x]$ is the greatest integer function and $\{x\}$ is the fractional part of the real number. I.e., $\{x\} = x - [x]$.

Solution 1 by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Since, $x = \lfloor x \rfloor + \{x\}$, then $x^2 = \lfloor x \rfloor^2 + \{x\}^2 + 2\lfloor x \rfloor \{x\}$. Now,

$$\frac{\lfloor x \rfloor}{2x+\{x\}} + \frac{\{x\}}{2x+\lfloor x \rfloor} = \frac{2x^2+\lfloor x \rfloor^2+\{x\}^2}{6x^2+\lfloor x \rfloor\{x\}} = \frac{3x^2-2\lfloor x \rfloor\{x\}}{6x^2+\lfloor x \rfloor\{x\}}.$$

Therefore, the left-hand side of the proposed inequality, LHS is

$$\begin{aligned} LHS &= \frac{3x^2-2\lfloor x \rfloor\{x\}}{6x^2+\lfloor x \rfloor\{x\}} + \frac{\lfloor x \rfloor\{x\}}{3x^2} \\ &= \frac{3A-2B}{6A+B} + \frac{B}{3A} = \frac{9A^2-B^2}{18A^2+3AB} \\ &\leq \frac{1}{2} \end{aligned}$$

where $A = x^2$ and $B = \lfloor x \rfloor \{x\}$.

Solution 2 by Titu Zvonaru, Comănesti, and Neculai Stanciu "George Emil Palade" School, Buzău, Romania

We denote $a = [x]$ and $b = \{x\}$, so $a \geq 0,\ b \geq 0$ and $x = a + b$.

Because
$$(2a+3b)(3a+2b) = 6a^2 + 13ab + 6b^2 \geq 6(a+b)^2,$$
we have
$$\begin{aligned} \frac{a}{2a+3b} + \frac{ab}{3(a+b)^2} + \frac{b}{3a+2b} &= \frac{3a^2+4ab+3b^2}{(2a+3b)(3a+2b)} + \frac{ab}{3(a+b)} \\ &\leq \frac{3a^2+4ab+3b^2}{6(a+b)^2} + \frac{ab}{3(a+b)^2} \\ &= \frac{3a^2+6ab+3b^2}{6(a+b)^2} \\ &= \frac{1}{2}. \end{aligned}$$

Because we only used the inequality $ab \geq 0$, we obtain that equality holds if, and only if $ab = 0$, i.e., if, and only if x is an integer or if $x \in (0,1)$.

Solution 3 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

For convenience, let $n \leq x \leq n+1$, so $[x] = n$ and $\{x\} = x - n$.

Then the given inequality becomes $\dfrac{n}{2x+(x-n)} + \dfrac{n(x-n)}{3x^2} + \dfrac{x-n}{2x+n} \leq \dfrac{1}{2}$.

Upon clearing fractions and simplifying, this becomes $0 \leq n\left(3x^3 - 5nx^2 + 4n^2x - 2n^3\right)$.

Further algebra simplifies the inequality:
$$\begin{aligned} n(x-n)\left(3x^2 - 2nx + 2n^2\right) &\geq 0 \\ n(x-n)\left((x-n)^2 + 2x^2 + n^2\right) &\geq 0. \end{aligned}$$

Because $x \geq n \geq 0$, this is certainly true.

The final version of the inequality also reveals that equality holds if and only if $n = 0$ (that is, $0 \leq x < 1$ so$\{x\} = x$) or $x = n = [x]$ (that is, x is an integer.)

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, and the proposer.

- **5306:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate: $\displaystyle\int_0^1 \frac{\ln\left(1-x+x^2\right)}{x-x^2}dx.$

Solution 1 by Kee-Wai Lau, Hong Kong, China

Let

$$\begin{aligned} I_1 &= \int_0^1 \frac{\ln(1-x+x^2)}{x}dx \\ I_2 &= \int_0^1 \frac{\ln(1-x+x^2)}{1-x}dx \\ I_3 &= \int_0^1 \frac{\ln(1+x^3)}{x}dx \text{ and} \\ I_4 &= \int_0^1 \frac{\ln(1+x)}{x}dx. \end{aligned}$$

Clearly, $I = I_1 + I_2$ and $I_1 = I_3 - I_4$.

By the substitution $x = 1-y$ into I_2, we easily see that $I_2 = I_1$.

By the substitution $x = y^{1/3}$ into I_3, we obtain $I_3 = \dfrac{1}{3}I_4$.

It follows that $I = 2I_1 = 2(I_3 - I_4) = \dfrac{-4}{3}I_4$. But I_4 is a well-known integral with value $\dfrac{\pi^2}{12}$ and so $I = \dfrac{-\pi^2}{9}$.

Solution 2 by Albert Stadler, Herrliberg Switzerland

We have

$$\int_0^1 \frac{\ln\left(1-x+x^2\right)}{x-x^2}dx = \int_0^1 \left(\frac{1}{x} - \frac{1}{1-x}\right)\ln(1-x+x^2)dx$$

$$\begin{aligned} &= \int_0^1 \frac{\ln\left(1-x+x^2\right)}{x}dx + \int_0^1 \frac{\left(\ln\left(1-(1-x)+(1-x)^2\right)\right.}{x}dx \\ &= 2\int_0^1 \frac{\ln(1-x+x^2)}{x}dx \\ &= 2\int_0^1 \frac{\ln\left(\dfrac{1+x^3}{1+x}\right)}{x}dx \\ &= 2\int_0^1 \frac{\ln(1+x^3)}{x}dx - 2\int_0^1 \frac{\ln(1+x)}{x}dx \\ &= 2\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\int_0^1 x^{3k-1}dx - 2\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\int_0^1 x^{k-1}dx \end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{3k^2} - 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}\\
&= -\frac{4}{3}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}\\
&= -\frac{4}{3}\left(\sum_{k=1}^{\infty}\frac{1}{k^2} - 2\sum_{k=1}^{\infty}\frac{1}{(2k)^2}\right)\\
&= -\frac{2}{3}\sum_{k=1}^{\infty}\frac{1}{k^2} = -\frac{2}{3}\cdot\frac{\pi^2}{6} = -\frac{\pi^2}{9}.
\end{aligned}
$$

The interchange of summation and integration is permitted because of uniform convergence of the series $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}x^{3k-1}$ and $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}x^{k-1}$ in the interval $[0,1]$.

Addendum: It is noteworthy to mention that the famous relation $\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}} = \frac{\zeta(2)}{3} = \frac{\pi^2}{18}$ is easily derived from the above integral (see for instance http://en.wikipedia.org/wiki/Ap%C3%A9ry's theorem for reference). Indeed,

$$
\begin{aligned}
\frac{\pi^2}{9} = -\int_0^1 \frac{\ln(1-x+x^2)}{x-x^2}dx &= \sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 \left(x-x^2\right)^{k-1}dx\\
&= \sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 x^{k-1}(1-x)^{k-1}dx = \sum_{k=1}^{\infty}\frac{1}{k}\frac{\Gamma(k)\Gamma(k)}{\Gamma(2k)}\\
&= \sum_{k=1}^{\infty}\frac{1}{k}\frac{(k-1)!(k-1)!}{(2k-1)!} = 2\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{k!k!}{(2k)!} = 2\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}.
\end{aligned}
$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain, and the proposer.