Solutions to the problems stated in this issue should be posted before October 15, 2015

• 5355: Proposed by Kenneth Korbin, New York, NY
Find the area of the convex cyclic pentagon with sides

\((13, 13, 12\sqrt{3} + 5, 20\sqrt{3}, 12\sqrt{3} - 5)\).

• 5356: Proposed by Kenneth Korbin, New York, NY
For every prime number \(p\) there is a circle with diameter \(4p^4 + 1\). In each of these circles, it is possible to inscribe a triangle with integer length sides and with area

\((8p^3)(p + 1)(p - 1)(2p^2 - 1)\).

Find the sides of the triangles if \(p = 2\) and if \(p = 3\).

• 5357: Proposed by Neculai Stanciu, “George Emil Palade” School, Bușău, Romania and Titu Zvonaru, Comănăști, Romania
Determine all triangles whose side-lengths are positive integers (of which at least one is a prime number), whose semiperimeter is a positive integer, and whose area is equal to its perimeter.

• 5358: Proposed by Arkady Alt, San Jose, CA
Prove the identity

\[\sum_{k=1}^{m} k \left( \frac{m+1}{k+1} \right)^{r+1} = (r+1)^m (mr - 1) + 1.\]

• 5359: Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.
Let \(a, b, c\) be positive real numbers. Prove that

\[\sqrt[3]{15a^2b + 1} + \sqrt[3]{15b^2c + 1} + \sqrt[3]{15c^2a + 1} \leq \frac{63}{32} (a + b + c) + \frac{1}{32} \left( \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \right).\]
Let $n \geq 1$ be an integer and let
\[
I_n = \int_0^\infty \frac{\arctan x}{(1 + x^2)^n} \, dx.
\]
Prove that
\[
(a) \quad \sum_{n=1}^\infty \frac{I_n}{n} = \zeta(2);
\]
\[
(b) \quad \int_0^\infty \arctan x \ln \left( 1 + \frac{1}{x^2} \right) \, dx = \zeta(2).
\]

Solutions

\[
\sum_{n=1}^\infty \frac{I_n}{n} = \zeta(2);
\]
\[
\int_0^\infty \arctan x \ln \left( 1 + \frac{1}{x^2} \right) \, dx = \zeta(2).
\]

\[
\sum_{n=1}^\infty I_n^\frac{1}{n} = \zeta(2);
\]
\[
\int_0^\infty \arctan x \ln \left( 1 + \frac{1}{x^2} \right) \, dx = \zeta(2).
\]

\section*{Solution 2 by Albert Stadler, Herrliberg, Switzerland}

The cyclic quadrilateral has the maximal area among all quadrilaterals having the same sequence of side lengths. This is a corollary to Bretschneider’s formula (http://en.wikipedia.org/wiki/Bretschneider%27s_formula). It can also be proved using calculus (see([1])). The area of a cyclic quadrilateral with side $a, b, c, d$ is given by Brahmagupta’s formula
\[
A = \sqrt{(s-a)(s-b)(s-c)(s-d)} \text{ where } s = (a + b + c + d)/2.
\]
So if \( a = 1 + 3\sqrt{2} \), \( b = 6 + 4\sqrt{2} \), and \( c = -14 + 12\sqrt{2} \) then

\[
16A^2 = \left( d - 9 + 13\sqrt{2} \right) \left( d - 19 + 11\sqrt{2} \right) \left( d + 21 - 5\sqrt{2} \right) \left( -d - 7 + 19\sqrt{2} \right).
\]

This is a polynomial of degree four whose extremal points are located at the zeros of its derivative. Brute force shows that the extremal points are

\[
d_1 = 7 + 5\sqrt{2} > 0, \\
d_2 = \frac{-7 - 5\sqrt{2} + \sqrt{1987 - 1338\sqrt{2}}}{2} < 0, \\
d_3 = \frac{-7 - 5\sqrt{2} - \sqrt{1987 - 1338\sqrt{2}}}{2} < 0.
\]

So \( AD = d_1 = 7 + 5\sqrt{2} \)


Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that the area of the quadrilateral is maximum when \( AD = 7 + 5\sqrt{2} \).

Let \( AD = x, s \) be the semiperimeter and \( \Delta \) the area of the quadrilateral. Since the length of any side of a quadrilateral must be less than the sum of the lengths of the other three sides, we have \( 19 - 112\sqrt{2} < x < -7 + 19\sqrt{2} \). It is well known that

\[
\Delta \leq \sqrt{(s - AB) (s - BC) (s - AB) (s - AD)},
\]

so that \( 16\Delta^2 \leq f(x) \), where

\[
f(x) = -x^4 + 2\left( 571 - 282\sqrt{2} \right) x^2 + 32(27 + 13\sqrt{2})x - 454337 + 314940\sqrt{2}.
\]

It can be checked readily by differentiation that for \( 19 - 11\sqrt{2} < x < -7 + 19\sqrt{2} \), \( f(x) \) attains its unique maximum at \( x = 7 + 5\sqrt{2} \). Hence

\[
\Delta \leq \frac{\sqrt{f(7 + 5\sqrt{2})}}{4} = 14\sqrt{-137 + 106\sqrt{2}}.
\]

It can also be checked readily that the area of the quadrilateral with sides \( AB = 1 + 3\sqrt{2}, BC = 6 + 4\sqrt{2}, CD = -14 + 12\sqrt{2}, AD = 7 + 5\sqrt{2} \),

\[
\overline{AC} = \sqrt{7\left( -55 + 58\sqrt{2} \right)} \text{ in fact equals } 14\sqrt{-137 + 106\sqrt{2}}.
\]

This completes the solution.

Also solved by Arkardy Alt, San Jose, CA; Ed Gray, Highland Beach, FL; Henry Ricardo, New York Math Circle, NY; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.
Determine the maximum value of
\[
F(x, y, z) = \min \left\{ \frac{|y - z|}{|x|}, \frac{|z - x|}{|y|}, \frac{|x - y|}{|z|} \right\},
\]
where \(x, y, z\) are arbitrary nonzero real numbers.

**Solution 1 by Kee-Wai Lau, Hong Kong, China**

We show that the maximum value of \(F(x, y, z)\) is 1.

We first prove that
\[
F(x, y, z) \leq 1, \tag{1}
\]
by showing that at least one of the numbers \(\frac{|y - z|}{|x|}, \frac{|z - x|}{|y|}, \frac{|x - y|}{|z|}\) is less than equal to 1.

Suppose, on the contrary, that all of them are greater than 1. From
\[
\frac{|y - z|}{|x|} > 1, \quad (y - z)^2 > x^2,
\]
or \((x + y - z)(x - y + z) < 0. \tag{2}\)

Similarly from \(\frac{|z - x|}{|y|} > 1, \quad (x - y - z)(x + y - z) > 0, \tag{3}\)
and
\(\frac{|x - y|}{|z|} > 1, \quad (x - y - z)(x - y + z) > 0. \tag{4}\)

Multiplying (2), (3) and (4) together, we obtain
\[(x + y - z)^2 (x - y + z)^2 (x - y - z)^2 < 0,
\]
which is false. Thus (1) holds. Since \(F(2, -1, 1) = 1\), we see that the maximum value of \(F(x, y, z)\) is 1 indeed.

**Solution 2 by Albert Stadler, Herrliberg, Switzerland**

We claim that the maximum value equals 1.

Let \(x > 0\). Then \(F(x, x + 1, -1) = \min \left\{ \frac{x + 2}{x}, \frac{x + 1}{x + 1}, \frac{1}{1} \right\} = 1. \)

So the maximum value is \(\geq 1. \)

Suppose the maximum value is \(> 1. \) Then there is a triple \((x, y, z)\) with
\[
|y - z| > |x|, \quad |z - x| > |y|, \quad |x - y| > |z|. \tag{1}
\]

By cyclic symmetry, we can assume that \(x \leq y \leq z. \)

Assume first that \(x \leq y \leq z. \) Then (1) reads as
\[z - y > |x|, \quad z - x > |y|, \quad y - x > |z|. \quad \text{So } z - x = (z - y) + (y - x) > |x| + |z| \geq z - x
\]
\[z - y > |x|, \quad z - x > |y|, \quad y - x > |z|. \quad z - x = (z - y) + (y - x) > |x| + |z| \geq z - x
\]
\[z - y > |x|, \quad z - x > |y|, \quad y - x > |z|. \quad z - x = (z - y) + (y - x) > |x| + |z| \geq z - x
\]
\[z - y > |x|, \quad z - x > |y|, \quad y - x > |z|. \quad z - x = (z - y) + (y - x) > |x| + |z| \geq z - x
\]
\[z - y > |x|, \quad z - x > |y|, \quad y - x > |z|. \quad z - x = (z - y) + (y - x) > |x| + |z| \geq z - x
\]
which is a contradiction.

Assume next that \( x \leq z \leq y \). Then (1) reads as

\[ y - z > |x|, \quad z - x > |y|, \quad y - x > |z|. \]

So \( y - x = (y - z) + (z - x) > |x| + |y| \geq y - x \),

which is a contradiction.

This concludes the proof.

Solution 3 by Paolo Perfetti, Department of Mathematics, “Tor Vergata” University, Rome Italy

Answer: 1

The symmetry of \( F(x, y, z) \) allows us to set \( x \leq z \leq y \). We have two cases:

1) \( 0 < z \leq y \leq x \) and

2) \( z < 0, \quad 0 < y \leq x \).

Moreover, by observing that \( F(x, y, z) = F(-x, -y, -z) \), the case \( z \leq y < 0, \quad x > 0 \)

is recovered by the case 2) simply changing sign to all the signs and the same happens if \( z \leq y \leq x < 0 \).

Now we study the case 1)

\[ \frac{|y - z|}{|x|} \leq \frac{|x - z|}{|y|} \iff \frac{y - z}{x} \leq \frac{x - z}{y} \iff z \leq x + y \]

which evidently holds true. Moreover,

\[ \frac{|y - z|}{|x|} \leq \frac{|x - y|}{|z|} \iff \frac{y - z}{x} \leq \frac{x - y}{z} \iff yx + yz \leq x^2 + z^2 \]

This generates two subcases.

1.1) \( 0 < z \leq y \leq x \) and \( yx + yz \leq x^2 + z^2 \). In this case we must find the maximum of the function \( \frac{y - z}{x} \). We have

\[ \frac{y - z}{x} \leq \frac{y - z}{y} = 1 - \frac{z}{y} < 1. \]

The value 1 is not attained because \( z \neq 0 \).

1.2) \( 0 < z \leq y \leq x \) and \( yx + yz > x^2 + z^2 \). In this case we must find the maximum of the function \( \frac{x - y}{z} \). We have

\[ \frac{x - y}{z} < \frac{y - z}{x} \leq \frac{y - z}{y} = 1 - \frac{z}{y} < 1. \]

Now we study case 2)

\[ F(x, y, z) = \min \left\{ \frac{y - z}{x}, \frac{x - z}{y}, \frac{x - y}{z} \right\} \]

and
\[
\frac{y - z}{x} \leq \frac{x - z}{y} \iff z \leq x + y
\]
which evidently holds true.
Moreover,
\[
\frac{y - z}{x} \leq \frac{z - y}{-z} \iff y \leq x + z.
\]
This generates two subcases.

2.1) \( z < 0, 0 < y < x, y \leq x + z \). In this case we must find the maximum of
\[
\frac{y - z}{x} \leq \frac{x}{x} = 1.
\]
The maximum achieved.

2.2) \( z < 0, 0 < y < x, y > x + z \). In this case we must find the maximum of
\[
\frac{x - y}{-z} \leq \frac{x - y}{x - y} = 1.
\]
The maximum achieved.

Also solved by Jerry Chu, (student at Saint George’s School), Spokane, WA; Ethan Gegner, (student, Taylor University), Upland, IN, and the proposer.

\[\text{5339: Proposed by D.M. Bătinențu-Giurgiu, “Matei Basarab” National College, Bucharest, Romania and Neculai Stanciu “George Emil Palade” School, Buzău, Romania}\]

Calculate: \( \int_{0}^{\pi/2} \frac{3 \sin x + 4 \cos x}{3 \cos x + 4 \sin x} \, dx \).

Solution 1 by Haroun Meghaichi (student, University of Science and Technology Houari Boumediene), Algeria

Consider the general case for \( a, b > 0 \):
\[
I(a, b) = \int_{0}^{\pi/2} \frac{a \sin x + b \cos x}{b \sin x + a \cos x} \, dx,
\]
Note that the derivative of the denominator (with respect to \( x \)) is \( b \cos x - a \sin x \), and \( \{b \sin x + a \cos x, b \cos x - a \sin x\} \) form a base on \( R[\cos x, \sin x] \), then there are \( \alpha, \beta \in R \) such that
\[
a \sin x + b \cos x = \alpha (b \sin x + a \cos x) + \beta (b \cos x - a \sin x), \quad \forall x \in R
\]
\[
\iff b - a \alpha - b \beta = a - b \alpha + a \beta = 0.
\]
We can easily solve the system to get \( \alpha, \beta \) = \( \left( \frac{2ab}{a^2 + b^2}, \frac{b^2 - a^2}{a^2 + b^2} \right) \), then
\[
I(a, b) = \frac{1}{a^2 + b^2} \int_{0}^{\pi/2} 2ab + (b^2 - a^2) \frac{b \cos x - a \sin x}{b \sin x + a \cos x} \, dx.
\]
\[ = \frac{1}{a^2 + b^2} \left[ 2abx + (b^2 - a^2) \ln |a \cos x + b \sin x| \right]_0^{\pi/2} \]

\[ = \frac{1}{a^2 + b^2} \left( ab\pi + (b^2 - a^2) \ln \frac{b}{a} \right). \]

The proposed integral equals \( I(4, 3) = I(3, 4) = \frac{1}{25} \left( 12\pi + 7 \ln \frac{4}{3} \right). \)

**Solution 2 by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Andrew Siefker, Angelo State University, San Angelo, TX**

We attack the problem by using the classical technique for converting a rational function of \( \sin x \) and \( \cos x \) into an ordinary rational function. If we set

\[ u = \tan \left( \frac{x}{2} \right), \]

then the “half-angle” formulas imply that

\[ u^2 = \frac{\sin^2 \left( \frac{x}{2} \right)}{\cos^2 \left( \frac{x}{2} \right)} = \frac{1 - \cos x}{1 + \cos x} \]

and hence,

\[ \cos x = \frac{1 - u^2}{1 + u^2}. \]

(1)

Also, using (1) and the known identity

\[ u = \tan \left( \frac{x}{2} \right) = \frac{\sin x}{1 + \cos x}, \]

we get

\[ \sin x = \frac{2u}{1 + u^2}. \]

(2)

Finally,

\[ du = \sec^2 \left( \frac{x}{2} \right) \cdot \frac{1}{2} \, dx = \frac{1}{2} \left[ 1 + \tan^2 \left( \frac{x}{2} \right) \right] \, dx = \frac{1 + u^2}{2} \, dx, \]

i. e.,

\[ dx = \frac{2}{1 + u^2} \, du. \]

(3)

Since \( u = 0 \) when \( x = 0 \) and \( u = 1 \) when \( x = \frac{\pi}{2} \), (1), (2), and (3) yield (upon simplification)

\[ \int_0^{\pi/2} \frac{3 \sin x + 4 \cos x}{3 \cos x + 4 \sin x} \, dx = 4 \int_0^1 \frac{2u^2 - 3u - 2}{(3u^2 - 8u - 3)(1 + u^2)} \, du \]

\[ = 4 \int_0^1 \frac{2u^3 - 3u - 2}{(3u + 1)(u - 3)(1 + u^2)} \, du. \]

(4)
Then, (4) and the partial fraction expansion

\[
\frac{2u^2 - 3u - 2}{(3u + 1)(u - 3)(1 + u^2)} = \frac{12}{25} \cdot \frac{1}{1 + u^2} - \frac{7}{50} \cdot \frac{u}{1 + u^2} + \frac{21}{100} \cdot \frac{1}{3u + 1} + \frac{7}{100} \cdot \frac{1}{u - 3}
\]

imply that

\[
\int_0^{\pi/2} \frac{3 \sin x + 4 \cos x}{3 \cos x + 4 \sin x} \, dx = 4 \int_0^1 \frac{2u^2 - 3u - 2}{(3u + 1)(u - 3)(1 + u^2)} \, du
\]

\[
= \frac{48}{25} \tan^{-1} u \bigg|_0^1 - \frac{7}{25} \ln (1 + u^2) \bigg|_0^1 + \frac{7}{25} \ln |3u + 1| \bigg|_0^1
\]

\[
+ \frac{7}{25} \ln |u - 3| \bigg|_0^1
\]

\[
= \frac{12\pi}{25} - \frac{7}{25} \ln 2 + \frac{7}{25} \ln 4 + \frac{7}{25} \ln 2 - \frac{7}{25} \ln 3
\]

\[
= \frac{12\pi}{25} + \frac{7}{25} \ln \left(\frac{4}{3}\right)
\]

Solution 3 by Ethan Gegner, (student, Taylor University), Upland, IN

The value of the integral is \(\frac{1}{25} (12\pi + 7 \log(4/3))\).

Define

\[
I = \int_0^{\pi/2} \frac{3 \sin x + 4 \cos x}{3 \cos x + 4 \sin x} \, dx
\]

\[
A = \int_0^{\pi/2} \frac{\sin x}{3 \cos x + 4 \sin x} \, dx
\]

\[
B = \int_0^{\pi/2} \frac{\cos x}{3 \cos x + 4 \sin x} \, dx.
\]

Then

\[
I = 3A + 4B
\]
\[ I + A - B = \int_{0}^{\pi/2} \frac{3 \cos x + 4 \sin x}{3 \cos x + 4 \sin x} \, dx = \frac{\pi}{2} \]

\[ I - 6A = \int_{0}^{\pi/2} -3 \sin x + 4 \cos x \, dx = \int_{3}^{4} \frac{1}{u} \, du = \log(4/3) \]

Solving this system yields \( I = \frac{1}{25} (12\pi + 7 \log(4/3)) \).

**Solution 4 by Bruno Salgueiro Fanego, Viveiro, Spain**

Since \( \frac{d}{dx} (ax + b \ln(2 \cos x + 4 \sin x)) = \frac{(4a - 3b) \sin x + (3a + 4b) \cos x}{3 \cos x + 4 \sin x} \) when \( 3 \cos x + 4 \sin x > 0 \) and \( b \in \mathbb{R} \), if we take \( a, b, \in \mathbb{R} \) such that \( 4a - 3b = 3 \) and \( 3a + 4b = 4 \), that is, \( a = \frac{24}{25} \) and \( b = \frac{7}{25} \), we obtain that \( \frac{1}{25} (24x + 7 \ln(3 \cos x + 4 \sin x)) \) is a primitive of \( \frac{3 \sin x + 4 \cos x}{3 \cos x + 4 \sin x} \) in \([0, \pi/2]\), so, by Barrow’s rule,

\[ \int_{0}^{\pi/2} \frac{3 \sin x + 4 \cos x}{3 \cos x + 4 \sin x} \, dx = \frac{1}{25} (24x + 7 \ln(3 + 4 \cdot 1)) \bigg|_{0}^{\pi/2} \]

\[ = \frac{1}{25} (12x + 7 \ln(3 \cdot 0 + 4 \cdot 1)) - \frac{1}{25} (24 \cdot 0 + 7 \ln(31 + 4 \cdot 0)) \]

\[ = \frac{12\pi}{25} + \frac{7}{25} \ln \left( \frac{4}{3} \right). \]

**Solution 5 by Brian D. Beasely, Presbyterian College, Clinton, SC**

We let \( f(x) = 3 \sin x + 4 \cos x \) and \( g(x) = 3 \cos x + 4 \sin x \). Since \( g'(x) = -3 \sin x + 4 \cos x \), we seek constants \( A \) and \( B \) such that \( \frac{f(x)}{g(x)} = A \left( \frac{g'(x)}{g(x)} \right) + B \). This produces \( A = \frac{7}{25} \) and \( B = \frac{24}{25} \), so

\[ \int_{0}^{\pi/2} \frac{f(x)}{g(x)} \, dx = \int_{0}^{\pi/2} \left[ A \left( \frac{g'(x)}{g(x)} \right) + B \right] \, dx \]

\[ = A \ln(g(x)) + B \bigg|_{0}^{\pi/2} \]

\[ = A \ln \left( \frac{4}{3} \right) + B \left( \frac{\pi}{2} \right) \]

\[ = \frac{7}{25} \ln \left( \frac{4}{3} \right) + \frac{12\pi}{25}. \]
Addendum. We may generalize the above technique to show that
\[ \int_0^{\pi/2} \frac{m \sin x + n \cos x}{3 \cos x + 4 \sin x} \, dx = A \ln \left( \frac{4}{3} \right) + B \left( \frac{\pi}{2} \right), \]
where \( A = (-3m + 4n)/25 \) and \( B = (4m + 3n)/25 \).

We may further generalize to show that
\[ \int_0^{\pi/2} \frac{m \sin x + n \cos x}{p \cos x + q \sin x} \, dx = A \ln \left( \frac{q}{p} \right) + B \left( \frac{\pi}{2} \right), \]
where \( A = (-pm + qn)/(p^2 + q^2) \) and \( B = (qm + pn)/(p^2 + q^2) \), provided we place appropriate restrictions on the values of \( p \) and \( q \) (to keep \( p \cos x + q \sin x \neq 0 \) for each \( x \) in \([0, \pi/2] \), to avoid \( p = 0 \) or \( q = 0 \), etc.).

Also solved by Arkady Alt, San Jose, CA; Andrea Fanchini, Gantú, Italy; Paul M. Harms, North Newton, KS; Ed Gray, Highland Beach, FL; G.C. Greubel, Newport News, VA; Kee-Wai Lau, Hong Kong, China; Daniel López, Center for Mathematical Sciences, UNAM, Morelia, Mexico; Paolo Perfetti, Department of Mathematics, “Tor Vergata” University, Rome, Italy; Henry Ricardo (two solutions), New York Math Circle, NY; Albert Stadler, Herrliberg, Switzerland; Vu Tran (student, Purdue University), West Lafayette, IN; Nicusor Zlota, “Traian Vuia” Technical College, Focsani, Romania; Titu Zvonaru, Comănești, Romania, and the proposers.

5340: Proposed by Oleh Faynshteyn, Leipzig, Germany

Let \( a, b \) and \( c \) be the side-lengths, and \( s \) the semi-perimeter of a triangle. Show that
\[ \frac{a^2 + b^2}{(s - c)^2} + \frac{b^2 + c^2}{(s - a)^2} + \frac{c^2 + a^2}{(s - b)^2} \geq 24. \]

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Changing variables by letting \( s - a = x \), \( s - b = y \) and \( s - c = z \) the proposed inequality is equivalent to the following one, for \( x, y \) and \( z \) positive real numbers:
\[ \sum_{\text{cyclic}} \left( 1 + \frac{y}{z} \right)^2 + \left( 1 + \frac{x}{z} \right)^2 \geq 24. \]

The last inequality follows by the power-mean, arithmetic-mean, geometric-mean inequality:
\[
\sqrt[6]{\frac{\sum_{\text{cyclic}} \left( 1 + \frac{y}{z} \right)^2 + \left( 1 + \frac{x}{z} \right)^2}{6}} \geq \frac{1}{6} \sum_{\text{cyclic}} \left( 1 + \frac{y}{z} \right) + \left( 1 + \frac{x}{z} \right) \geq 1 + \sqrt[6]{\prod_{\text{cyclic}} \frac{y}{z} \cdot \frac{x}{z} \cdot \frac{y}{z}} = 1 + \sqrt[6]{\prod_{\text{cyclic}} \frac{y}{z} \cdot \frac{x}{z}} = 2
\]
from where the result follows, with equality if and only if \(x = y = z\), that is if \(a = b = c\).

**Solution 2 by Nikos Kalapodis, Patras, Greece**

\[a + b + c = 2s \implies a^2 = (s - b + s - c)^2.\]

Using the well-known inequality \((x + u)^2 \geq 4xy\) for \(x = s - b\) and \(y = s - c\) we have

\[(s - b + s - c)^2 \geq 4(s - b)(s - c), \text{ i.e.,}\]

\[a^2 \geq 4(s - b)(s - c) \quad (1)\]

Similarly we obtain,

\[b^2 \geq 4(s - c)(s - a) \quad (2)\]

\[c^2 \geq 4(s - a)(s - b). \quad (3)\]

Applying the well known inequality \(x^2 + y^2 \geq 2xy\), to (1), (2), and (3) we have

\[
\frac{a^2 + b^2}{(s - c)^2} + \frac{b^2 + c^2}{(s - a)^2} + \frac{c^2 + a^2}{(s - b)^2} =
\]

\[
\left[\left(\frac{a}{s - b}\right)^2 + \left(\frac{a}{s - c}\right)^2\right] + \left[\left(\frac{b}{s - c}\right)^2 + \left(\frac{b}{s - a}\right)^2\right] + \left[\left(\frac{c}{s - a}\right)^2 + \left(\frac{c}{s - b}\right)^2\right] \geq
\]

\[
\frac{2a^2}{(s - b)(s - c)} + \frac{2b^2}{(s - c)(s - a)} + \frac{2c^2}{(s - a)(s - b)} \geq 2(4 + 4 + 4) = 24.
\]

**Solution 3 by Arkady Alt, San Jose, CA**

Note that \(\sum_{cyc} \frac{a^2 + b^2}{(s - c)^2} \geq 24 \iff \sum_{cyc} \frac{a^2 + b^2}{(a + b - c)^2} \geq 6.\)

Since \(a^2 \geq a^2 - (b - c)^2 \iff \frac{a^2}{a + b - c} \geq c + a - b\)

and

\[b^2 \geq b^2 - (c - a)^2 \iff \frac{b^2}{a + b - c} \geq b + c - a\]

then by AM-GM Inequality we have

\[\sum_{cyc} \frac{a^2}{(a + b - c)^2} \geq \sum_{cyc} \frac{c + a - b}{a + b - c} \geq 3\sqrt[3]{\frac{c + a - b}{a + b - c} \cdot \frac{a + b - c}{b + c - a} \cdot \frac{b + c - a}{c + a - b}} = 3\]

and

\[\sum_{cyc} \frac{b^2}{(a + b - c)^2} \geq \sum_{cyc} \frac{b + c - a}{a + b - c} \geq 3\sqrt[3]{\frac{b + c - a}{a + b - c} \cdot \frac{c + a - b}{b + c - a} \cdot \frac{a + b - c}{c + a - b}} = 3.\]

Thus, \(\sum_{cyc} \frac{a^2 + b^2}{(a + b - c)^2} \geq 6.\)
We shall prove that
\[ \frac{x a^m}{(s - c)^m} + \frac{x b^m}{(s - a)^m} + \frac{y c^m}{(s - b)^m} + \frac{y a^m}{(s - c)^m} \geq 3 \sqrt{xy} \cdot 2^{m+1}, \text{ where } m, x, y > 0. \]

Proof: We denote the area of the triangle by \( F \), its circumradius by \( R \) and its inradius by \( r \).

By the AM-GM inequality and taking into account that \( F = sr = \sqrt{s(s-a)(s-b)(s-c)} \) we have that

\[ \sum_{cyclic} \frac{x a^m + y b^m}{(s - c)^m} \geq 2 \sqrt{xy} \sum_{cyclic} \frac{(\sqrt{ab})^m}{(s - c)^m} \geq 2 \sqrt{xy} \cdot 3 \cdot \prod_{cyclic} \frac{(\sqrt{ab})^m}{(s - c)^m} \]

\[ = 6 \sqrt{xy} \cdot 3 \sqrt[3]{\frac{abc}{(s - a)(s - b)(s - c)}}^m \]

\[ = 6 \sqrt{xy} \cdot 3 \sqrt[3]{\frac{(4RF)^m s^m}{(s - a)(s - b)(s - c)^m}} \]

\[ = 6 \sqrt{xy} \cdot 3 \sqrt[3]{\frac{4^m R^m F^m s^m}{F^2 m}} \]

\[ = 6 \sqrt{xy} \cdot 3 \sqrt[3]{\frac{4^m R^m s^m}{s^m m}} \]

\[ = 6 \sqrt{xy} \cdot 3 \sqrt[3]{\frac{4^m R^m}{s m^m}} \]

\[ = 6 \sqrt{xy} \cdot 3 \sqrt[3]{4^m \left( \frac{R}{r} \right)^m} \]

\[ \geq \text{Euler} (R \geq 2r) \cdot 6 \sqrt{xy} \cdot \sqrt[3]{4^m 2^m} \]

\[ = 6 \sqrt{xy} \cdot 3 \sqrt[3]{2^{3m}} = 6 \sqrt{xy} \cdot 3 \sqrt[3]{2^{3m}} = 3 \sqrt{xy} 2^{m+1} \]

If we take \( m = 2 \) we obtain a solution to problem 5340.

Solution 5 by Paul M. Harms, North Newton, KS
If $x > 0$, then using calculus we can show that the minimum value of both expressions

$$\begin{cases} x + \frac{1}{x} \\ x^2 + \frac{1}{x^2} \end{cases}$$

is 2 and occurs at $x = 1$. I will use several substitutions to get the left side of the problem inequality into a form easier to use.

First let $t > 0$ and $r > 0$ such that $a = rc$ and $b = tc$. Then $s = \frac{c}{2}(r + t + 1)$ and the left side of the problem inequality is

$$\frac{(r^2 + t^2)}{(t + r - 1)}^2 + \frac{(t^2 + 1)}{(t - r + 1)}^2 + \frac{(r^2 + 1)}{(r - t + 1)}^2.$$

Now let

$$\begin{cases} 2H = r + t - 1, \\ 2L = t - r + 1 \end{cases} \quad \text{then} \quad \begin{cases} r = H + K \\ t = H + L \end{cases} \quad \text{with } H, L \text{ and } K \text{ positive since}$$

$s - a, s - b$ and $s - c$ are positive.

The inequality in terms of the positive numbers $H, K$ and $L$ can be written as

$$\frac{(H + K)^2}{H^2} + \frac{(H + L)^2 + 1}{L^2} + \frac{(H + K)^2 + 1}{K^2} \geq 24.$$

Working with the left side of the inequality we can obtain

$$2\left(\frac{K}{H} + \frac{H}{K}\right) + 2\frac{L}{H} + 2\left(\frac{L}{H}\right)^2 + \frac{(\frac{H}{L})^2}{2} + \frac{2L}{H} + 1 + \frac{1}{L^2} + \frac{(\frac{H}{L})^2 + \frac{2H}{K} + 1 + \frac{1}{K^2}}{2}$$

$$= 2\left(\frac{K}{H} + \frac{H}{K}\right) + 2\left(\frac{L}{H} + \frac{H}{L}\right) + 2\left(\frac{H}{K}\right)^2 + \left(\frac{K}{H}\right)^2 + \left(\frac{L}{H}\right)^2 + \left(\frac{H}{L}\right)^2 + 4 + \frac{1}{K^2} + \frac{1}{L^2}.$$

Each of the brackets in the last expression has the form $\left(x + \frac{1}{x}\right)$ or $\left(x^2 + \frac{1}{x^2}\right)$ so the minimum value of each bracket is 2. Then the left side of the original problem inequality is greater than or equal to $2(2) + 2(2) + 2 + 2 + 4 + \frac{1}{K^2} + \frac{1}{L^2}$. If we can show that this expression is greater than or equal 24, the original inequality is correct.

We must show that $\frac{1}{K^2} + \frac{1}{L^2}$ is at least 8. Since $K$ and $L$ are positive numbers such that $L = 1 - K$, the derivative of the two terms is $\frac{-2}{K^3} - \frac{2}{L^3}(-1)$. Letting the derivative equal to zero, we obtain $K = L = \frac{1}{2}$. The value of 8 is clearly a minimum for $\frac{1}{K^2} + \frac{1}{L^2}$.

Thus the problem inequality is correct.

Solution 6 by Henry Ricardo, New York Math Circle, NY
It is a known consequence of the arithmetic-geometric mean inequality that the side-lengths of a triangle satisfy the inequality

\[(b + c - a)(c + a - b)(a + b - c) \leq abc.\]

Using this fact and the arithmetic-geometric mean inequality twice more, we have

\[
a^2 + b^2 \frac{(s - c)^2}{2} + b^2 + c^2 \frac{(s - a)^2}{2} + c^2 + a^2 \frac{(s - b)^2}{2} \geq 3 \left( \frac{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)}{(s - a)^2(s - b)^2(s - c)^2} \right)^{1/3} \geq 3 \left( \frac{(2ab)(2bc)(2ac)}{((b + c - a)(a + c - b)(a + b - c))^2/64} \right)^{1/3} \geq 3 \left( \frac{8a^2b^2c^2}{(abc)^2/64} \right)^{1/3} = 24.
\]

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; D. M.Btinetu-Giurgiu, Bucharest, Romania; Bruno Salgueiro Fanego, Viveiro, Spain; Ethan Gegner (student, Taylor University), Upland, IN; Ed Gray, Highland Beach, FL; Nikos Kalapodis (two additional solutions to #2 above), Patras, Greece; Kee-Wai Lau, Hong Kong, China; Haroun Meghaichi (student, University of Science and Technology Houari Boumediene), Algeria; Albert Stadler, Herrliberg, Switzerland; Nicusor Zlota, “Traian Vuia” Technical College, Focsani, Romania; Titu Zvonaru and Neculai Stanciu, Romania, and the proposer.

- **5341**: Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain

Let \(z_1, z_2, \ldots, z_n\), and \(w_1, w_2, \ldots, w_n\) be sequences of complex numbers. Prove that

\[\text{Re} \left( \sum_{k=1}^{n} z_kw_k \right) \leq \frac{3}{(n+1)(n+2)} \sum_{k=1}^{n} |z_k|^2 + \frac{3n^2 + 6n + 1}{20} \sum_{k=1}^{n} |w_k|^2.
\]

**Solution 1 by Kee-Wai Lau, Hong Kong, China**

We have

\[
\text{Re} \left( \sum_{k=1}^{n} z_kw_k \right) \leq \left| \sum_{k=1}^{n} z_kw_k \right| \leq \sum_{k=1}^{n} |z_k| |w_k| = \sum_{k=1}^{n} \left| \frac{\sqrt{6}z_k}{\sqrt{(n+1)(n+2)}} \right| \left| \frac{\sqrt{(n+1)(n+2)}w_k}{\sqrt{6}} \right| \leq \frac{1}{2} \left( \sum_{k=1}^{n} \left| \frac{\sqrt{6}z_k}{\sqrt{(n+1)(n+2)}} \right|^2 + \left| \frac{\sqrt{(n+1)(n+2)}w_k}{\sqrt{6}} \right|^2 \right) \right) \leq \frac{3}{(n+1)(n+2)} \sum_{k=1}^{n} |z_k|^2 + \frac{(n+1)(n+2)}{12} \sum_{k=1}^{n} |w_k|^2.
\]
Since
\[
\frac{(n + 1)(n + 2)}{12} = \frac{3n^2 + 6n + 1}{20} - \frac{(n - 1)(4n + 7)}{60} \leq \frac{3n^2 + 6n + 1}{20},
\]
so the inequality of the problem holds.

**Solution 2 by Ethan Gegner (student, Taylor University), Upland, IN**

For \( n \in \mathbb{N} \), define
\[
f(n) = \left( \frac{3}{(n + 1)(n + 2)} \right) \left( \frac{3n^2 + 6n + 1}{20} \right)
\]
and observe that \( f \) is an increasing function of \( n \); thus, \( f(n) \geq f(1) = 1/4 \) for all \( n \in \mathbb{N} \).

Applying AM-GM inequality and then Cauchy’s inequality, we obtain
\[
\frac{3}{(n + 1)(n + 2)} \sum_{k=1}^{n} |z_k|^2 + \frac{3n^2 + 6n + 1}{20} \sum_{k=1}^{n} |w_k|^2 \geq 2 \sqrt{f(n) \left( \sum_{k=1}^{n} |z_k|^2 \right) \left( \sum_{k=1}^{n} |w_k|^2 \right)^2}
\]
\[
\geq \left( \sum_{k=1}^{n} |z_k|^2 \right)^{1/2} \left( \sum_{k=1}^{n} |w_k|^2 \right)^{1/2}
\]
\[
\geq \sum_{k=1}^{n} |z_k| |w_k|
\]
\[
\geq \text{Re} \left( \sum_{k=1}^{n} z_k w_k \right).
\]

**Solution 3 by Paolo Perfetti, Department of Mathematics, “Tor Vergata” University, Rome, Italy**

The AGM yields
\[
\frac{3}{(n + 1)(n + 2)} \sum_{k=1}^{n} |z_x|^2 + \frac{3n^2 + 6n + 1}{20} \sum_{k=1}^{n} |w_x|^2 \geq 2 \sqrt{\frac{3}{20} \frac{3n^2 + 6n + 1}{n^2 + 3n + 2} \sum_{k=1}^{n} |z_x|^2 \cdot \sum_{r=1}^{n} |w_r|^2}.
\]

Then we use Cauchy–Schwarz
\[
\sqrt{\sum_{k=1}^{n} |z_x|^2 \cdot \sum_{r=1}^{n} |w_r|^2} \geq \sum_{k=1}^{n} |z_x| \cdot |w_k|
\]

Moreover
\[ \text{Re} \left( \sum_{k=1}^{n} z_k w_k \right) \leq \left| \sum_{k=1}^{n} z_k w_k \right| \leq \sum_{k=1}^{n} |z_k w_k| , \]

and the inequality amounts to show that

\[ 2 \sqrt{\frac{3}{20} \frac{3n^2 + 6n + 1}{n^2 + 3n + 2}} \geq 1 \iff n \leq -\frac{7}{4}, \quad n \geq 1. \]

This completes the proof.

Solution 4 by Nicusor Zlota, “Traian Vuia” Technical College, Focsani, Romania

Let \( z_k = x_k + iy_k \) and \( w_k = a_k + ib_k \), for \( 0 \leq k \leq n \). We can assume that \( x_k, y_k, a_k, b_k \geq 0 \), because we can increase the left hand side of the statement of the problem by using absolute values.

We wish to prove the inequality:

\[ \sum_{k=1}^{n} (a_k x_k - b_k y_k) \leq \frac{3}{(n+1)(n+2)} \sum_{k=1}^{n} (x_k^2 + y_k^2) + \frac{3n^2 + 6n + 1}{20} \sum_{k=1}^{n} (a_k^2 + b_k^2) . \]

Because of symmetry, we need only show that:

\[ a_k x_k \leq \frac{3}{(n+1)(n+2)} x_k^2 + \frac{3n^2 + 6n + 1}{20} a_k^2 . \]

Considering this as a quadratic inequality for the variable \( x_k \), we see that the discriminant is negative.

\[ \Delta = a_k^2 - \frac{3}{(n+1)(n+2)} \frac{3n^2 + 6n + 1}{20} a_k = a_k^2 \left( \frac{-4n^2 + 3n + 7}{5(n+1)(n+2)} \right) < 0. \]

Hence, the problem is solved.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL, and the proposer.

\[ \bullet \text{5342: Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania} \]

Let \( a, b, c, \alpha > 0 \), be real numbers. Study the convergence of the integral

\[ I(a, b, c, \alpha) = \int_{1}^{\infty} \left( a^{1/x} - \frac{b^{1/x} + c^{1/x}}{2} \right)^{\alpha} dx. \]

The problem is about studying the conditions which the four parameters, \( a, b, c, \) and \( \alpha \), should verify such that the improper integral would converge.

Solution 1 by Arkady Alt, San Jose, CA
Case 1. If \( a = b = c \), then for any nonzero \( x \), \( a^{\frac{1}{x}} - \frac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2} = 0 \), so \( I(a, b, c, \alpha) = 0 \) for any real \( \alpha > 0 \).

Case 2. Suppose \( \alpha \) isn’t an integer. Then \( a^{\frac{1}{x}} - \frac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2} \) must be nonnegative for any \( x \) and in particular, it must be positive for \( x = 1 \), that is \( a > \frac{b + c}{2} \).

Since \( \begin{cases} 2a = b + c & \iff \ a = b = c \\ b = c & \end{cases} \) then, to avoid the trivial case 1, we will consider \( a, b, c \) such that
\[
\frac{b + c}{2} \quad \text{or} \quad \begin{cases} 2a = b + c \\ b \neq c. \end{cases}
\]

Then, by the AM-PM inequality, for \( x > 1 \) we have
\[
\frac{b + c}{2} > \left( \frac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2} \right)^x \iff \left( \frac{b + c}{2} \right)^{\frac{1}{x}} > \frac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2},
\]
and we obtain \( a^{\frac{1}{x}} > \frac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2} \) for any \( x > 1 \) and that the integral is defined.

For any real \( p > 0 \) we have \( \lim_{t \to 0} \frac{p^t - 1}{t} = \ln p \). So, \( \lim_{x \to \infty} x \left( a^{\frac{1}{x}} - \frac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2} \right) = \ln a - \frac{\ln b + \ln c}{2} = \ln a \sqrt{bc} > 0 \), because \( a > \sqrt{bc} \) if \( b \neq c \) or if \( a > \frac{b + c}{2} \).

Therefore, \( \lim_{x \to \infty} \frac{1}{x^\alpha} \left( a^{\frac{1}{x}} - \frac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2} \right)^\alpha \) converges if \( \frac{1}{x^\alpha} \) converges; that is, \( I(a, b, c, \alpha) \) converges if \( \alpha > 1 \) and diverges if \( \alpha \in (0, 1] \).

Case 3. Let \( \alpha \) be a positive integer. Then the expression \( \left( a^{\frac{1}{x}} - \frac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2} \right)^\alpha \) is defined for any positive \( a, b, c \) and since
\[
\lim_{x \to \infty} \left( a^{\frac{1}{x}} - \frac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2} \right)^\alpha = \ln^\alpha \frac{a}{\sqrt{bc}} > 0
\]
is the limit of \( I(a, b, c, \alpha) \) for \( a > \sqrt{bc} \) and when \( \alpha > 1 \). So the situation of \( a = \sqrt{bc} \) must be analyzed.
Then \( \left( \frac{1}{a^x} - \frac{b}{2^x} + \frac{c}{2^x} \right)^\alpha \) = \( (-1)^\alpha \left( \frac{1}{b^x} - \frac{1}{c^x} \right)^{2\alpha} \).

Assume, without loss of generality, \( b > c \). Since \( \lim_{x \to \infty} x \left( \frac{1}{b^x} - \frac{1}{a^x} \right) = \frac{1}{2} \ln \frac{b}{c} > 0 \),
then \( \lim_{x \to \infty} \frac{x \left( \frac{1}{b^x} - \frac{1}{a^x} \right)^{2\alpha}}{1} = \left( \frac{1}{2} \ln \frac{b}{c} \right)^{2\alpha} > 0 \), and by the Limit Comparison Test

\( I(a, b, c, \alpha) \) is convergent iff \( \frac{1}{x^{2\alpha}} \) convergent, that is \( I(a, b, c, \alpha) \) convergent if \( \alpha > 1/2 \) and divergent if \( \alpha \in (0, 1/2] \).

In summary,

- If \( a = b = c \) then \( I(a, b, c, \alpha) = 0 \) is convergent for any real \( \alpha \);
- If \( \alpha \in \mathbb{R}_+/\mathbb{N} \) and \( a > \frac{b + c}{2} \) or \( \begin{cases} 2a = b + c \\ b \neq c \end{cases} \) then \( I(a, b, c, \alpha) \) is convergent for \( \alpha > 1 \) and divergent for \( \alpha \in (0, 1] \);
- If \( \alpha \in \mathbb{R}_+/\mathbb{N} \) and \( a > \sqrt{bc} \) then \( I(a, b, c, \alpha) \) is convergent for \( \alpha > 1 \) and divergent for \( \alpha \in (0, 1] \);
- If \( \alpha \in \mathbb{N} \) and \( a = \sqrt{bc} \) then \( I(a, b, c, \alpha) \) is convergent for \( \alpha > 1/2 \) and divergent for \( \alpha \in (0, 1/2] \).

**Solution 2 by Paolo Perfetti, Department of Mathematics, “Tor Vergata” University, Rome, Italy**

To have the integral well defined, a necessary condition is \( 2a \geq b + c \).

The convergence occurs in one of the following cases:

1) if \( a = b = c \) we have convergence for any value of \( \alpha \)
2) if \( \alpha > 1 \) we have convergence regardless the values of \( a, b, c \)
3) if \( 1/2 < \alpha \leq 1 \) and \( a = \sqrt{bc} \) we have convergence.

**Proof**

If \( \alpha \) is irrational or it is a rational \( p/q \) reduced to the lowest terms with \( q \) even, we must impose

\[ 2a^{1/x} - b^{1/x} - c^{1/x} \geq 0 \]

but this doesn’t seem to me easy to prove. A necessary condition is \( 2a \geq b + c \)
corresponding to \( x = 1 \).

If \( a = b = c \) the integrand is identically zero and then the integral converges regardless the value of \( \alpha \).

From now on, \( a \neq b \) or \( b \neq c \) or \( a \neq c \).

We have \( a^{1/x} = e^{\ln a/x} = 1 + \frac{\ln a}{x} + \frac{\ln^2 a}{2x^2} + \frac{\ln^3 a}{6x^3} + O(x^{-4}) \) whence
\[ \left[ \frac{a^{1/x} - b^{1/x} + c^{1/x}}{2} \right]^\alpha = \left\{ \frac{1 + \ln a}{x} + \frac{\ln^2 a}{2x^2} + \frac{\ln^3 a}{6x^3} + \frac{1 + \ln b}{2x^2} + \frac{\ln^2 b}{2x^3} + \frac{\ln^3 b}{6x^3} + \frac{1 + \ln c}{2x^2} + \frac{\ln^2 c}{2x^3} + \frac{\ln^3 c}{6x^3} + O(x^{-4}) \right\}^\alpha = \]

\[ = \frac{1}{x^\alpha} \left( \ln \frac{a}{\sqrt{bc}} + \frac{\ln^2 a - \frac{\ln^2 b}{2} - \frac{\ln^2 c}{2}}{2x} + xA \right)^\alpha \]

\[ A = \frac{1}{6} \left( \ln^3 a - \frac{\ln^3 b}{2x^3} - \frac{\ln^3 c}{2x^3} \right) + O(x^{-4}) \]

The positivity of \( \ln \frac{a}{\sqrt{bc}} + \frac{\ln^2 a - \frac{\ln^2 b}{2} - \frac{\ln^2 c}{2}}{2x} + xA \) for \( x \) large enough, imposes \( \ln \frac{a}{\sqrt{bc}} > 0 \) that is \( a^2 \geq bc \) which in turn follows by \( 2a \geq b + c \). Indeed

\[ a^2 \geq \frac{(b + c)^2}{4} = \frac{b^2 + c^2 + 2bc}{4} \geq \frac{4bc}{4} = bc \]

Let \( \alpha > 1 \). Since for any \( x \) large enough it is

\[ \left( \ln \frac{a}{\sqrt{bc}} + \frac{\ln^2 a - \frac{\ln^2 b}{2} - \frac{\ln^2 c}{2}}{2x} + xA \right)^\alpha \leq C \]

if \( \alpha > 1 \) the integral \( \int_1^\infty \frac{1}{x^\alpha} \left( \ln \frac{a}{\sqrt{bc}} + \frac{\ln^2 a - \frac{\ln^2 b}{2} - \frac{\ln^2 c}{2}}{2x} + xA \right)^\alpha \) converges.

Let \( 1/2 < \alpha \leq 1 \) and \( a = \sqrt{bc} \).

\[ 0 \leq \left( a^{1/x} - \frac{b^{1/x} + c^{1/x}}{2} \right)^\alpha = \frac{1}{x^{2\alpha}} \left( \frac{1}{4} (\ln b - \ln c)^2 + x^2 A \right)^\alpha \leq \frac{C_1}{x^{2\alpha}} \]

whence convergence.

Let \( 0 < \alpha \leq 1/2 \), and \( a = \sqrt{bc} \). To have convergence we need \( \ln b = \ln c \) that is \( b = c \), but this would yield \( a = b = c \), a forbidden condition.

Also solved by the proposer.