Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before October 15, 2014

- **5307:** *Proposed by Haishen Yao and Howard Sporn, Queensborough Community College, Bayside, NY*

  Solve for x:
$$\sqrt{x^{15}} = \sqrt{x^{10}-1} + \sqrt{x^5-1}.$$

- **5308:** *Proposed by Kenneth Korbin, New York, NY*

  Given the sequence
$$t = (1, 7, 41, 239, \ldots)$$
  with $t_n = 6t_{n-1} - t_{n-2}$. Let (x, y, z) be a triple of consecutive terms in this sequence with $x < y < z$.

  Part 1) Express the value of x in terms of y and express the value of y in terms of x.

  Part 2) Express the value of x in terms of z and express the value of z in terms of x.

- **5309:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  Consider the expression $3^n + n^2$ for positive integers n. It is divisible by 13 for $n = 18$ and $n = 19$. Prove, however, that it is never divisible by 13 for three consecutive values of n.

- **5310:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

  Let $a > 0$ and a sequence $\{E_n\}_{n\geq 0}$, be defined by $\mathrm{E}_n = \sum_{k=0}^{n} \frac{1}{k!}$. Evaluate:
$$\lim_{n\to\infty} \sqrt[n]{n!}\left(a^{\sqrt[n]{E_n}-1} - 1\right).$$

- **5311:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let x, y, z be positive real numbers. Prove that

$$\sum_{cyclic} \sqrt{\left(\frac{x^2}{3}+3y^2\right)\left(\frac{2}{xy}+\frac{1}{z^2}\right)} \geq 3\sqrt{10}.$$

- **5312:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Calculate:

$$\int_0^1 \ln|\sqrt{x}-\sqrt{1-x}|dx.$$

Solutions

- **5289:** *Proposed by Kenneth Korbin, New York, NY*

  Part 1: Thirteen different triangles with integer length sides and with integer area each have a side with length 1131. The angle opposite 1131 is $Arcsin\left(\frac{3}{5}\right)$ in all 13 triangles.

  Find the sides of the triangles.

  Part 2: Fourteen different triangles with integer length sides and with integer area each have a side with length 6409. The size of the angle opposite 6409 is the same in all 14 triangles.

  Find the sides of the triangles.

  Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

  Part 1: If $\alpha = Arc\sin\left(\frac{3}{5}\right)$, then $\sin\alpha = \frac{3}{5}$ and $0 < \alpha < \frac{\pi}{2}$. It follows that

$$\cos\alpha = \sqrt{1-\frac{9}{25}} = \frac{4}{5}.$$

  Suppose x and y are the other sides of the triangle with $x \geq y$. The Law of Cosines implies that

$$\begin{aligned}(1131)^2 &= x^2+y^2-2xy\cos\alpha \\ &= x^2+y^2-\frac{8}{5}xy.\end{aligned}$$

  If we complete the square in x and simplify, we get

$$(5655)^2 = (5x-4y)^2+(3y)^2$$

  and hence, $(5x-4y, 3y, 5655)$ is a Pythagorean Triple. To solve for x and y, we must find all such triples and assign $5x-4y$ and $3y$ to the sides of each triple. E.g., for the triple $(2175, 5220, 5655)$, setting

$$5x-4y = 2175$$

$$3y = 5220$$

yields $x = 1827$ and $y = 1740$, while

$$\begin{aligned} 5x - 4y &= 5220 \\ 3y &= 2175 \end{aligned}$$

yields $x = 1624$ and $y = 725$. Some other triples give only one integral solution for x and y and a few give no integral solutions. In all, we found 14 solutions which are listed in the following table. (Repeated triples indicate multiple solutions as above.)

Pythagorean Triple	x	y
$(3393, 4524, 5655)$	1885	1508
$(2175, 5220, 5655)$	1827	1740
$(2175, 5220, 5655)$	1624	725
$(3900, 4095, 5655)$	1872	1365
$(3900, 4095, 5655)$	1859	1300
$(936, 5577, 5655)$	1365	312
$(663, 5616, 5655)$	1300	221
$(2280, 5175, 5655)$	1836	1725
$(2280, 5175, 5655)$	1643	760
$(2025, 5280, 5655)$	1813	1760
$(2025, 5280, 5655)$	1596	675
$(2772, 4929, 5655)$	1725	924
$(3009, 4788, 5655)$	1760	1003
$(2871, 4872, 5655)$	1740	957

.

It should be noted that in each case, the values of x, y, and 1131 satisfy the required triangle inequalities for the sides of a non-degenerate triangle. Also, the area of each triangle is $\frac{1}{2}xy\sin\alpha = \frac{3xy}{10}$. Since xy is a multiple of 10 in each case, the resulting triangle has integral area as well.

Part 2: If we once again use $\alpha = Arc\sin\left(\frac{3}{5}\right)$ for the angle opposite 6409, then by the same steps as described in Part 1, the remaining sides x and y (with $x \geq y$) must satisfy the equation

$$(32,045)^2 = (5x - 4y)^2 + (3y)^2 .$$

Following the same procedure as in Part 1, we found the 22 solutions listed in the following table. As before, each satisfies the required inequalities for the sides of a triangle and each yields an integral area.

Pythagorean Triple	x	y
$(15916, 27813, 32045)$	10600	9271
$(22244, 23067, 32045)$	10600	7689
$(8283, 30956, 32045)$	8400	2761
$(2277, 31964, 32045)$	7000	759
$(2400, 31955, 32045)$	7031	800
$(21000, 24205, 32045)$	10441	7000
$(19795, 25200, 32045)$	10679	8400
$(10192, 30381, 32045)$	10140	10127
$(18291, 26312, 32045)$	10140	6097
$(15708, 27931, 32045)$	9775	5236
$(7656, 31117, 32045)$	8265	2552
$(8580, 30875, 32045)$	8463	2860
$(12920, 29325, 32045)$	10404	9775
$(11475, 29920, 32045)$	9044	3825
$(20300, 24795, 32045)$	10672	8265
$(3045, 31900, 32045)$	7192	1015
$(5304, 31603, 32045)$	7735	1768
$(13572, 29029, 32045)$	9425	4524
$(22100, 23205, 32045)$	10608	7735
$(15080, 28275, 32045)$	10556	9425
$(12325, 29580, 32045)$	10353	9860
$(16269, 27608, 32045)$	9860	5423

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China, (part 1); David E. Manes, SUNY at Oneonta, Oneonta, NY, and the proposer.

- **5290:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

Someone wrongly remembered the description of an even perfect number as:
$N = 2^p\left(2^{p-1}-1\right)$, where p is a prime number. Classify these numbers correctly. Which are deficient and which are abundant?

Solution 1 by David E. Manes, SUNY College at Oneonta, Oneonta NY

We will show that if p is a prime, then $N = 2^p\left(2^{p-1}-1\right)$ is abundant except when $p = 2$ in which case N is deficient.

If $\sigma(n)$ is the sum of the positive divisors of n, then n is deficient when $\sigma(n) - n < n$ and abundant if $\sigma(n) - n > n$. If $p = 2$, then $N = 2^p\left(2^{p-1}-1\right) = 4$ and $\sigma(4) - 4 = 7 - 4 = 3$. Therefore $N = 4$ is deficient. If p is an odd prime, then $\gcd\left(2^p,\ 2^{p-1}-1\right) = 1$ implies

$$\sigma(N) = \sigma\left(2^p\left(2^{p-1}-1\right)\right) = \sigma\left(2^p\right)\sigma\left(2^{p-1}-1\right)$$

since σ is a multiplicative function. Moreover $\sigma\left(2^p\right) = 2^{p+1} - 1$ and
$\sigma\left(2^{p-1}-1\right) > \left(2^{p-1}-1\right) + 1 = 2^{p-1}$. Thus, $\sigma(N) > \left(2^{p+1}-1\right)2^{p-1}$. Therefore,

$$\sigma(N) - N \quad > \quad \left(2^{p+1}-1\right)2^{p-1} - 2^p\left(2^{p-1}-1\right)$$

$$\begin{aligned}
&= \left(2^{p-1}\right)\left(2^{p+1}-1-2\left(2^{p-1}-1\right)\right)\\
&= \left(2^{p-1}\right)\left(2^{p+1}-2^{p}+1\right)\\
&= 2^{p-1}\left(2^{p}+1\right)\\
&> \left(2^{p-1}-1\right)2^{p} = N.
\end{aligned}$$

Hence, N is an abundant integer.

Solution 2 by Paul M. Harms, North Newton, KS

I will use the theorem stating that proper multiples of perfect numbers and abundant numbers are abundant numbers.

When $p = 2, N = 4$ which is a deficient number.

When $p = 3, N = 2^2\left(2\left(2^2-1\right)\right) = 4(6) = 24$ which is 4 times the perfect number 6 and thus is an abundant number.

Consider p a prime number, $p \geq 3$. Then
$\left(2^{p-1}-1\right) = \left(2^2-1\right)\left(2^{p-3}+2^{p-5}+\ldots+2^2+1\right)$.

We now have $N = 2^p\left(2^{p-1}-1\right) = \left(2^{p-1}\left(2^{p-3}+2^{p-5}+\ldots+2^2+1\right)\right)\left(2\left(2^2-1\right)\right)$. Since N is a proper multiple of the perfect number $2\left(2^2-1\right) = 6$, N is an abundant number.

In conclusion, N is a deficient number when $p = 2$, but an abundant number for prime numbers $p > 2$.

Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC

We first establish that every nontrivial multiple of a perfect number is abundant (this result appears in most number theory texts, such as Burton's *Elementary Number Theory*). Given any positive integer n, we denote the sum of its positive divisors (including n itself) by $\sigma(n)$. The key observation is that for any positive integer n, we may sum over its positive divisors d to obtain

$$\sigma(n) = n\sum_{d|n}\frac{1}{d}.$$

Thus if n is perfect and m is a nontrivial multiple of n, then $\sigma(m)/m > \sigma(n)/n = 2$, so m is abundant. (In general, if we denote the *abundancy index* of n by $I(n) = \sigma(n)/n$, then the above observation establishes that $I(n) \leq I(m)$ whenever n divides m.)

Next, we solve the original problem based on the parity of the prime p. If $p = 2$, then $N = 4$ is deficient. If p is odd, then $2^{p-1}-1$ is divisible by 3, since $p-1$ is even and 2 raised to any even power is congruent to 1 modulo 3. Thus in this case N is a nontrivial multiple of 6, so N is abundant.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Georgia Southern

University, Statesboro, GA, and the proposer.

- **5291:** *Proposed by Arkady Alt, San Jose, CA*

Let $m_a m_b$ be the medians of a triangle with side lengths a, b, c. Prove that:

$$m_a m_b \leq \frac{2c^2 + ab}{4}.$$

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

We wish to prove that

$$\begin{aligned} 2c^2 + ab - 4m_a m_b &\geq 0 \text{ or equivalently,} \\ \left(2c^2 + ab + 4m_a m_b\right)\left(2c^2 + ab - 4m_a m_b\right) &\geq 0, \text{that is,} \\ \left(2c^2 + ab\right)^2 - 16m_a^2 m_b^2 &\geq 0. \end{aligned}$$

Since $m_a = \dfrac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$, and $m_b = \dfrac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$ we obtain:

$$(2c^2 + ab)^2 - 16m_a^2 m_b^2 = \left(2c^2 + ab\right)^2 - \left(2b^2 + 2c^2 - a^2\right)\left(2c^2 + 2a^2 - b^2\right)$$

$$\begin{aligned} &= 4c^4 + 4abc^2 + a^2b^2 - \left(4b^2c^2 + 4a^2b^2 - 2b^4 + 4c^4 + 4c^2a^2 - 2b^2c^2 - 2c^2a^2 - 2a^4 + a^2b^2\right) \\ &= 4abc^2 - 4a^2b^2 - 2b^2c^2 - 2c^2a^2 + 2a^4 + 2b^4 \\ &= 2a^4 + 2b^4 - 4a^2b^4 - 2b^2c^2 - 2c^2a^2 + 4abc^2 \\ &= 2\left(\left(a^2 - b^2\right)^2 - (bc - ca)^2\right) \\ &= 2\left((a + b)^2 (a - b)^2\right) - c^2 (b - a)^2 \\ &= 2(a - b)^2 \left((a + b)^2 - c^2\right) \\ &= 2(a - b)^2 (a + b + c)(a + b - c) \geq 0 \end{aligned}$$

By the triangle inequality $a + b - c > 0$, with equality if and only if $a = b$, that is , if and only if the triangle is isosceles with equal side lengths a and b.

Solution 2 by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain

Since the length of the medians of any triangle ABC with side lengths a, b, and c are given by the expression

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \quad \text{(cyclic)},$$

as it is well-known, then the inequality claimed becomes

$$\left(\frac{1}{2}\sqrt{2b^2+2c^2-a^2}\right)\left(\frac{1}{2}\sqrt{2c^2+2a^2-b^2}\right)\leq\frac{2c^2+ab}{4}$$

or

$$\sqrt{(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)}\leq 2c^2+ab$$

Squaring both sides of the above inequality and after canceling terms, we obtain

$$2a^4+2b^4-4c^2ab-4a^2b^2-2b^2c^2-2c^2a^2\geq 0$$

or equivalently,

$$2(a-b)^2(a+b+c)(a+b-c)\geq 0$$

which is true on account that in any non degenerate triangle ABC is $a+b>c$. Equality holds when $a=b$. That is when $\triangle ABC$ is isosceles, and we are done.

Also solved by D. M. Bătinetu-Giurgiu,"Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania, and Titu Zvonaru, Comănesti, Romania; Ed Gray, Highland Beach, FL; Kenneth Korbin, New York, NY; Paul M. Harms, North Newton, KS, Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Ecole Suppa, Teramo, Italy,and the proposer.

- **5292:** *Proposed by D.M. Bătinetu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania*

Let a and b be real numbers with $a<b$, and let c be a positive real number. If $f:R\longrightarrow R_+$ is a continuous function, calculate:

$$\int_a^b\frac{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}}{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}+e^{f(b-x)}\left(f(b-x)\right)^{\frac{1}{c}}}dx.$$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

If $f(x)=e^{f(x-a)}(f(x-a))^{\frac{1}{c}}$ and $g(x)=e^{f(b-x)}(f(b-x))^{\frac{1}{c}}$, then for $x\in(a,b)$, $f(x)=g(b-x+a)$ and hence the proposed integral, say I is equal to

$$I=\int_a^b\frac{e^{f(b-x)}(f(b-x))^{\frac{1}{c}}}{e^{f(x-a)}(f(x-a))^{\frac{1}{c}}+e^{f(b-x)}(f(b-x))^{\frac{1}{c}}}\ dx,$$

and so $I=\dfrac{b-a}{2}$.

Solution 2 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

By letting $y=\dfrac{x-a}{b-a}$, the integral is equal to

$$I=(b-a)\int_0^1\frac{F((b-a)y)}{F((b-a)y)+F((b-a)(1-y))}dy$$

$$= (b-a)\int_0^1 dy - \frac{1}{b-a}\int_0^1 \frac{F((b-a)(1-y))}{F((b-a)y)+F((b-a)(1-y))}dy.$$

Letting $t = 1-y$ we obtain

$$I = (b-a) - (b-a)\int_0^1 \frac{F((b-a)(1-y))}{F((b-a)y)+F((b-a)(1-y))}dy$$

$$= (b-a) - (b-a)\int_0^1 \frac{F((b-a)t)}{F((b-a)(1-t))+F((b-a)t)}dy.$$

It follows that $2I = b-a \implies I = \dfrac{b-a}{2}$.

Solution 3 by Paul M. Harms, North Newton, KS

Let $A(x) = e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}$ and $B(x) = e^{f(b-x)}\left(f(b-x)\right)^{\frac{1}{c}}$. We see that

$$\int_a^b \frac{A(x)+B(x)}{A(x)+B(x)}dx = b-a = \int_a^b \frac{A(x)}{A(x)+B(x)}dx + \int_a^b \frac{B(x)}{A(x)+B(x)}dx.$$

For the definite integral from a to b of $\dfrac{B(x)}{A(x)+B(x)}$ consider the change of variables $x = a+b-u$. Then

$$f(x-a) = f(b-u)$$

$$f(b-u) = f(u-a)$$

$$B(x) = A(u) \text{ and}$$

$$A(x) = B(u).$$

With this change of variables,

$$\int_a^b \frac{B(x)}{A(x)+B(x)}dx = \int_b^a \frac{A(u)}{B(u)+A(u)}(-1)du = \int_a^b \frac{A(u)}{A(u)+B(u)}du.$$

Thus $\displaystyle\int_a^b \frac{A(x)}{A(x)+B(x)}dx$ and $\displaystyle\int_a^b \frac{B(x)}{A(x)+B(x)}dx$ have the same value. Since their sum is $(b-a)$, the value of $\displaystyle\int_a^b \frac{A(x)}{A(x)+B(x)}dx$ is $\dfrac{b-a}{2}$.

Also solved by Michael Brozinsky, Central Islip, NY; Kee-Wai Lau, Hong Kong, China; Titu Zvonaru, Comănesti, Romania, and the proposer.

- **5293:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let ABC be a triangle. Prove that

$$\sqrt[4]{\sin A\cos^2 B}+\sqrt[4]{\sin B\cos^2 C}+\sqrt[4]{\sin C\cos^2 A}\le 3\sqrt[8]{\frac{3}{64}}.$$

Comment: **Michael Brozinsky of Central Islip, NY and Kee-Wai Lau of Hong Kong China each** noticed that if $\triangle ABC$ has an obtuse angle, then the above inequality does not hold. This oversight can be corrected by restricting the statement of the problem to acute triangles.

Solution 1 by Michael Brozinsky of Central Islip, NY

The given inequality is proved for acute triangles. Without loss of generality let the diameter of the circumcircle be 1 so that by the law of sines, the sides corresponding to angle A, B, and C satisfy the following:

$$a=\sin A,\qquad b=\sin B,\ c=\sin(\pi-(A+B))=\sin(A+B),$$

$$\cos^2 C = (-\cos(A+B))^2=\cos^2(A+B)$$

$$\cos^2 B = (-\cos(A+C))^2=\cos^2(A+C)\text{ and}$$

$$\cos^2 A = (-\cos(C+B))^2=\cos^2(C+B).$$

We shall also use the identity $\cos(x+y)=\cos x\cos y-\sin x\sin y$ $(*)$.

We may also assume $A\le B\le C$ so that $a\le b\le c<1$ and by acuteness

$$\frac{\pi}{2}<A+B\le A+C\le B+C,\ \text{since } A+B+C=\pi.$$

We have using $(*)$ that
$\sin(A)\cdot\cos^2(B)=\sin(A)\cdot\cos^2(A+C)=a\cdot\left(\sqrt{1-a^2}\cdot\sqrt{1-c^2}-a\cdot c\right))^2.$

Now $\dfrac{\partial}{\partial a}\left(a\cdot\left(\sqrt{1-a^2}\cdot\sqrt{1-c^2}-a\cdot c\right)^2\right)=$

$\left(\sqrt{1-a^2}\cdot\sqrt{1-c^2}-ac\right)^2+2a\left(\sqrt{1-a^2}\sqrt{1-c^2}-ac\right)\left(-\dfrac{\sqrt{1-c^2}a}{\sqrt{1-a^2}}-c\right)$ is clearly positive when one notes that factor $\sqrt{1-a^2}\sqrt{1-c^2}-ac$ is negative being $\cos(A+C)$ where $A+C$ is obtuse. Hence the radicand in the first term on the left hand side of the given inequality increases with a and since $a\le b\le c$ has it maximum value when $a=b$.

Similarly we have using $(*)$ that
$\sin(B)\cdot\cos^2(C)=\sin(B)\cdot\cos^2(A+B)=b\cdot\left(\sqrt{1-a^2}\sqrt{1-b^2}-ab\right)^2.$

Now $\dfrac{\partial}{\partial b}\left(b\cdot\left(\sqrt{1-a^2}\cdot\sqrt{1-b^2}-a\cdot b\right)^2\right)=$

$\left(\sqrt{1-a^2}\cdot\sqrt{1-b^2}-ab\right)^2+2b\left(\sqrt{1-a^2}\sqrt{1-b^2}-ab\right)\left(-\dfrac{\sqrt{1-a^2}b}{\sqrt{1-b^2}}-a\right)$ is clearly positive when one notes that factor $\sqrt{1-a^2}\sqrt{1-b^2}-ab$ is negative being $\cos(A+B)$

where $A+B$ is obtuse. Hence the radicand in the second term on the left hand side of the given inequality increases with b and since $a \le b \le c$ has it maximum value when $b=c$.

And similarly we have using $(*)$ that
$\sin(C)\cdot\cos^2(A) = \sin(C)\cdot\cos^2(C+B) = c\cdot\left(\sqrt{1-c^2}\sqrt{1-b^2}-cb\right)^2$ and
$\dfrac{\partial}{\partial b}\left(c\cdot\left(\sqrt{1-c^2}\cdot\sqrt{1-b^2}-c\cdot b\right)^2\right) =$
$2c\left(\sqrt{1-b^2}\cdot\sqrt{1-c^2}-bc\right)^2 + 2b\left(-\dfrac{\sqrt{1-c^2}b}{\sqrt{1-b^2}}-c\right)$ is clearly positive when one notes that factor $\sqrt{1-b^2}\sqrt{1-c^2}-b\cdot c$ is negative being $\cos(C+B)$ where $C+B$ is obtuse. Hence the radicand in the third term on the left hand side of the given inequality increases with b and since $a\le b\le c$ has its maximum value with $b=c$.

Thus the first three radicands are maximized simultaneously when $a=b=c$ and since A, B and C are acute, we have $A=B=C=\dfrac{\pi}{3}$ and the left hand side of the given inequality has its maximum value $3\cdot\sqrt[4]{\left(\dfrac{\sqrt{3}}{2}\right)\cdot\left(\dfrac{1}{2}\right)^2} = 3\cdot\sqrt[4]{\dfrac{\sqrt{3}}{8}} = 3\cdot\sqrt[8]{\dfrac{3}{64}}$ as was to be shown.

Solution 2 by Arkady Alt, San Jose, CA

Since by AM-GM Inequality

$$\sqrt[4]{\frac{1}{2}\cdot\frac{\sin A}{\sqrt{3}}\cdot\cos^2 B} \quad\le\quad \frac{\frac{1}{2}+\frac{\sin A}{\sqrt{3}}+2\cos B}{4} \text{ then}$$

$$\frac{1}{\sqrt[8]{12}}\sum_{cyc}\sqrt[4]{\sin A\cos^2 B} \quad=\quad \sum_{cyc}\sqrt[4]{\frac{1}{2}\cdot\frac{\sin A}{\sqrt{3}}\cdot\cos^2 B} \le \sum_{cyc}\frac{\frac{1}{2}+\frac{\sin A}{\sqrt{3}}+2\cos B}{4}$$

$$= \frac{3}{8}+\frac{1}{\sqrt{3}}(\sin A+\sin B+\sin C)+2(\cos A+\cos B+\cos C).$$

Since $R\ge 2r$ (Euler Inequality) we have $\cos A+\cos B+\cos C = 1+\dfrac{r}{R}\le\dfrac{3}{2}$.
Also, since $\sin x$ is concave down on $[0,\pi]$ then

$$\frac{\sin A+\sin B+\sin C}{3} \quad\le\quad \sin\frac{A+B+C}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \iff \sin A+\sin B+\sin C\le\frac{3\sqrt{3}}{2}.$$

Thus,

$$\frac{1}{\sqrt[8]{12}}\sum_{cyc}\sqrt[4]{\sin A\cos^2 B} \quad\le\quad \frac{1}{4}\left(\frac{3}{2}+\frac{1}{\sqrt{3}}\cdot\frac{3\sqrt{3}}{2}+2\cdot\frac{3}{2}\right) = \frac{3}{2}$$

$$\iff \sum_{cyc} \sqrt[4]{\sin A \cos^2 B} \le \frac{3}{2} \cdot \sqrt[8]{12} = 3\sqrt[8]{\frac{3}{64}}.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; D.M. Bătinetu−Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania, and the proposer.

- **5294:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

a) Calculate $\sum_{n=2}^{\infty} (n - \zeta(2) - \zeta(3) - \cdots - \zeta(n))$.

b) More generally, for $k \ge 2$ an integer, find the value of the multiple series

$$\sum_{n_1,n_2,\cdots,n_k=1}^{\infty} (n_1 + n_2 + \cdots + n_k - \zeta(2) - \zeta(3) - \cdots - \zeta(n_1 + n_2 + n_3 + \cdots + n_k)),$$

where ζ denotes the Riemann Zeta function.

Solution 1 by Anastasios Kotronis, Athens, Greece

We will answer b) which answers both questions. At first, it is rather straightforward using induction and the sum of geometric series that for $k \ge 1$ and $m \ge 2$ integers we have

$$\sum_{n_1,n_2,\ldots,n_k=1}^{+\infty} \frac{1}{m^{n_1+n_2+\cdots+n_k}} = \frac{1}{(m-1)^k}.$$

Now with the change of the summation order, whenever takes place, being justified by the constant sign of the summands, we have

$$\sum_{n_1,n_2,\ldots,n_k=1}^{+\infty} (n_1 + n_2 + \cdots + n_k - \zeta(2) - \zeta(3) - \cdots - \zeta(n_1 + n_2 + \cdots + n_k))$$

$$= \sum_{n_1,n_2,\ldots,n_k=1}^{+\infty} \left(1 - \sum_{k=2}^{n_1+n_2+\cdots+n_k} \sum_{m\ge 2} \frac{1}{m^k}\right)$$

$$= \sum_{n_1,n_2,\ldots,n_k=1}^{+\infty} \left(1 - \sum_{m\ge 2} \sum_{k=2}^{n_1+n_2+\cdots+n_k} \frac{1}{m^k}\right)$$

$$= \sum_{n_1,n_2,\ldots,n_k=1}^{+\infty} \left(1 - \sum_{m\geq 2}\left(\frac{1}{m-1} - \frac{1}{m}\right) + \sum_{m\geq 2} \frac{1}{m-1}\cdot\frac{1}{m^{n_1+n_2+\cdots+n_k}}\right)$$

$$= \sum_{n_1,n_2,\ldots,n_k=1}^{+\infty} \sum_{m\geq 2} \frac{1}{m-1}\cdot\frac{1}{m^{n_1+n_2+\cdots+n_k}}$$

$$= \sum_{m\geq 2} \frac{1}{m-1} \sum_{n_1,n_2,\ldots,n_k=1}^{+\infty} \frac{1}{m^{n_1+n_2+\cdots+n_k}}$$

$$= \sum_{m\geq 2} \frac{1}{(m-1)^{k+1}} = \zeta(k+1).$$

Solution 2 by Kee-Wai Lau, Hong Kong, China

For $k \geq 2$, we have

$$\sum_{n_1,n_2,\ldots n_k=1}^{\infty} (n+1+n_2+\cdots+n_k - \zeta(2) - \zeta(3) - \cdots - \zeta(n_1+n_2+\cdot+n_k))$$

$$= \sum_{n_1,n_2,\ldots n_k=1}^{\infty} \left(n_1+n_2+\cdots+n_k - \sum_{s=2}^{n_1+n_2+\cdots+n_k} \sum_{m=1}^{\infty} \frac{1}{m^s}\right)$$

$$= \sum_{n_1,n_2,\ldots n_k=1}^{\infty} \left(1 - \sum_{m=2}^{\infty} \sum_{s=2}^{n_1+n_2+\cdots+n_k} \frac{1}{m^s}\right)$$

$$= \sum_{n_1,n_2,\ldots n_k=1}^{\infty} \left(1 - \sum_{m=2}^{\infty}\left(\frac{1}{(m-1)m} - \frac{1}{(m-1)m^{n_1+n_2+\cdot+n_k}}\right)\right)$$

$$= \sum_{n_1,n_2,\ldots n_k=1}^{\infty} \left(1 - \sum_{m=2}^{\infty}\left(\frac{1}{m-1} - \frac{1}{m}\right) + \sum_{m+2}^{\infty} \frac{1}{(m-1)m^{n+1+n_2+\cdots+n_k}}\right)$$

$$= \sum_{n_1,n_2,\ldots n_k=1}^{\infty} \sum_{m=2}^{\infty} \frac{1}{(m-1)m^{n+1+n_2+\cdots+n_k}}$$

$$= \sum_{m=2}^{\infty} \frac{1}{m-1} \left(\sum_{n_1=1}^{\infty} \frac{1}{m^{n_1}}\right)\left(\sum_{n_2=1}^{\infty} \frac{1}{m^{n_2}}\right)\cdots\left(\sum_{n_k=1}^{\infty} \frac{1}{m^{n_k}}\right)$$

$$= \sum_{m=2}^{\infty} \frac{1}{(m-1)^{k+1}}.$$

So the answer to (b) is $\zeta(k+1)$. From the steps above, we see that the sum in (a) equals

$$\begin{aligned}
& \sum_{m=2}^{\infty} \frac{1}{m-1} \sum_{n=2}^{\infty} \frac{1}{m^n} \\
= & \sum_{m=2}^{\infty} \frac{1}{(m-1)^2 m} \\
= & \sum_{m=2}^{\infty} \frac{1}{(m-1)^2} - \sum_{m=2}^{\infty} \left(\frac{1}{m-1} - \frac{1}{m} \right) \\
= & \frac{\pi^2}{6} - 1.
\end{aligned}$$

Solution 3 by G.C. Greubel, Newport News, VA

First note that

$$\sum_{k=2}^{n} x^k = \frac{x(x - x^n)}{1-x}. \qquad (1)$$

Now, the first series to consider is that of

$$S_1 = \sum_{n=2}^{\infty} \left(n - \zeta(2) - \zeta(3) - \cdots - \zeta(n) \right). \qquad (2)$$

The Zeta function is given by

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \qquad (3)$$

and helps lead the series S_1 to the form

$$\begin{aligned}
S_1 &= \sum_{n=2}^{\infty} \left[n - \sum_{k=2}^{n} \zeta(k) \right] \\
&= \sum_{n=2}^{\infty} \left[n - \sum_{r=1}^{\infty} \left(\sum_{k=2}^{n} \frac{1}{k^r} \right) \right] \\
&= \sum_{n=2}^{\infty} \left[n - \sum_{r=1}^{\infty} \frac{1}{r-1} \left(\frac{1}{r} - \frac{1}{r^n} \right) \right], \qquad (4)
\end{aligned}$$

where (1) was used. It is seen that the first term of the series summed by r is problematic. To handle the difficulty consider the limit of the terms as $r \to 1$. This limit is

$$\lim_{r \to 1} \left\{ \frac{1}{r-1} \left(\frac{1}{r} - \frac{1}{r^n} \right) \right\} \to \frac{0}{0}. \qquad (5)$$

Use of L'Hospital's rule applies and leads to

$$\lim_{r\to 1}\left\{\frac{1}{r-1}\left(\frac{1}{r}-\frac{1}{r^n}\right)\right\} = \lim_{r\to 1}\left\{\frac{-1}{s^2}+\frac{n}{s^{n+1}}\right\} = n-1. \quad (6)$$

With this term the series of (4) now becomes

$$\begin{aligned}
S_1 &= \sum_{n=2}^{\infty}\left[1-\sum_{r=2}^{\infty}\frac{1}{r-1}\left(\frac{1}{r}-\frac{1}{r^n}\right)\right] \\
&= \sum_{n=2}^{\infty}\left[1-\sum_{r=2}^{\infty}\left(\frac{1}{r-1}-\frac{1}{r}-\frac{1}{r^n(r-1)}\right)\right] \\
&= \sum_{n=2}^{\infty}\sum_{r=2}^{\infty}\frac{1}{r^n(r-1)} \\
&= \sum_{r=2}^{\infty}\frac{1}{r-1}\cdot\sum_{n=2}^{\infty}\frac{1}{r^n} \\
&= \sum_{r=2}^{\infty}\frac{2r-1}{r(r-1)^2} \\
&= \sum_{r=2}^{\infty}\left(\frac{1}{(r-1)^2}-\frac{1}{r(r-1)}\right) \\
&= \zeta(2)-\sum_{r=2}^{\infty}\left(\frac{1}{r-1}-\frac{1}{r}\right) \\
S_1 &= \zeta(2)-1. \qquad (7)
\end{aligned}$$

This is the value of the first series in question.

The second series to consider is that of

$$S_2 = \sum_{n_1,n_2,\cdots,n_k=1}^{\infty}\left(\sum_{p=1}^{k}n_p-\sum_{s=2}^{n_1+n_2+\cdots+n_k}\zeta(s)\right). \quad (8)$$

In a similar manor as in the evaluation of the first series the second follows here.

$$\begin{aligned}
S_2 &= \sum_{n_k=1}^{\infty}\left[\sum_{p=1}^{k}n_p-\sum_{s=2}^{n_1+\cdots n_k}\sum_{r=1}^{\infty}\frac{1}{r^s}\right] \\
&= \sum_{n_k=1}^{\infty}\left[\sum_{p=1}^{k}n_p-\sum_{r=1}^{\infty}\frac{1}{r-1}\left(\frac{1}{r}-\frac{1}{r^{n_1+\cdots+n_k}}\right)\right]. \qquad (9)
\end{aligned}$$

As in the case before the first term of the series summed over r is problematic and is dealt with by use of L'Hospital's rule and leads to the result

$$\lim_{r\to 1}\left\{\frac{1}{r-1}\left(\frac{1}{r}+\frac{1}{r^{n_1+\cdots+n_k}}\right)\right\}=\sum_{p=1}^{k} n_p-1. \tag{10}$$

This then leads to

$$\begin{aligned}
S_2 &= \sum_{n_k=1}^{\infty}\left[1-\sum_{r=2}^{\infty}\frac{1}{r-1}\left(\frac{1}{r}-\frac{1}{r^{n_1+\cdots+n_k}}\right)\right]\\
&= \sum_{n_k=1}^{\infty}\left[1-\sum_{r=2}^{\infty}\left(\frac{1}{r-1}-\frac{1}{r}\right)+\sum_{r=2}^{\infty}\frac{1}{(r-1)r^{n_1+\cdots+n_k}}\right]\\
&= \sum_{n_k=1}^{\infty}\sum_{r=2}^{\infty}\frac{1}{r-1}\left(\frac{1}{r}\right)^{n_1+\cdots+n_k}\\
&= \sum_{r=2}^{\infty}\frac{1}{r-1}\left(\sum_{n=1}^{\infty}\frac{1}{r^n}\right)^{k}\\
&= \sum_{r=2}^{\infty}\frac{1}{(r-1)^{k+1}}\\
S_2 &= \zeta(k+1). \qquad (11)
\end{aligned}$$

This is the desired value of the second series.

Solution 4 by the proposer

First, we prove that

$$S_n=\sum_{k=1}^{\infty}\frac{1}{k(k+1)^n}=n-\zeta(2)-\zeta(3)-\cdots-\zeta(n).$$

We have, since

$$\frac{1}{k(k+1)^n}=\frac{1}{k(k+1)^{n-1}}-\frac{1}{(k+1)^n},$$

that

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)^n}=\sum_{k=1}^{\infty}\frac{1}{k(k+1)^{n-1}}-\sum_{k=1}^{\infty}\frac{1}{(k+1)^n},$$

and hence, $S_n=S_{n-1}-(\zeta(n)-1)$. Iterating this equality we obtain that

$$S_n=S_1-(\zeta(2)+\zeta(3)+\cdots+\zeta(n)-(n-1)),$$

and, since $S_1=\sum_{k=1}^{\infty}1/(k(k+1))=1$, we get that $S_n=n-\zeta(2)-\zeta(3)-\cdots-\zeta(n)$. Now we are ready to solve the problem.

a) The series equals $\zeta(2)-1$. We have,

$$\begin{aligned}\sum_{n=2}^{\infty}(n-\zeta(2)-\zeta(3)-\cdots-\zeta(n)) &= \sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(k+1)^n}\\ &= \sum_{k=1}^{\infty}\frac{1}{k}\left(\sum_{n=2}^{\infty}\frac{1}{(k+1)^n}\right)\\ &= \sum_{k=1}^{\infty}\frac{1}{k^2(k+1)}\\ &= \frac{\pi^2}{6}-1,\end{aligned}$$

and the first part of the problem is solved.

b) The series equals $\zeta(k+1)$. Let T_k be the value of the multiple series. We have,

$$\begin{aligned}T_k &= \sum_{n_1,n_2,\cdots,n_k=1}^{\infty}\left(\sum_{p=1}^{\infty}\frac{1}{p(p+1)^{n_1+n_2+\cdots+n_k}}\right)\\ &= \sum_{p=1}^{\infty}\frac{1}{p}\left(\left(\sum_{n_1=1}^{\infty}\frac{1}{(p+1)^{n_1}}\right)\cdots\left(\sum_{n_k=1}^{\infty}\frac{1}{(p+1)^{n_k}}\right)\right)\\ &= \sum_{k=1}^{\infty}\frac{1}{p}\left(\sum_{m=1}^{\infty}\frac{1}{(p+1)^m}\right)^k\\ &= \sum_{k=1}^{\infty}\frac{1}{p^{k+1}}\\ &= \zeta(k+1),\end{aligned}$$

and the problem is solved.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed. Gray, Highland Beach, FL (part a), and Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy.

Comments

Kenneth Korbin's problem 5283 challenged us to find the sides of two different isosceles triangles for which each has a perimeter of 162 and an area 1008.

Brian D. Beasley's solution was one of those featured in the April issue of the column and in it he stated: "In general, if we seek all isosceles triangles of the form $(x, x, P-2x)$ that have perimeter P and area A, then we obtain the equation

$$16Px^3-20P^2x^2+8P^3x-(P^4+16A^2)=0.$$

The given values $P = 162$ and $A = 1008$ produce exactly two such triangles. For what values of P and A would we find no triangles, one triangle, two triangles, or three triangles?"

Ken Korbin answered this question.

- If $A > \dfrac{P^2\sqrt{3}}{36}$, then no triangle is possible.
- If $A = \dfrac{P^2\sqrt{3}}{36}$, the exactly one triangle is possible and that triangle is equilateral.
- If $0 < A < \dfrac{P^2\sqrt{3}}{36}$ then exactly two different isosceles triangles have perimeter $=P$, and area $=A$.

Late Solutions

G. C. Greubel of Newport News, VA solved 5283.