Problems Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before October 15, 2013

- **5259:** *Proposed by Kenneth Korbin, New York, NY*

  Find a, b, and c such that with $a < b < c$,

$$\begin{cases} ab + bc + ca = -2 \\ a^2b^2 + b^2c^2 + c^2a^2 = 6 \\ a^3b^3 + b^3c^3 + c^3a^3 = -11. \end{cases}$$

- **5260:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  Find all primes p and q such that $a^{pq-1} \equiv a \pmod{pq}$, for all a relatively prime to pq.

- **5261:** *Proposed by Michael Brozinsky, Central Islip, NY*

  Show without calculus or trigonometric functions that the shortest focal chord of an ellipse is the latus rectum.

- **5262:** *Proposed by Pedro H.O. Pantoja, IMPA, Rio de Janeiro, Brazil*

  Prove that the equation $\varphi(10x^2) + \varphi(30x^3) + \varphi(34x^4) = y^2 + y^3 + y^4$ has infinitely many solutions for $x, y \in N$ where $\varphi(x)$ is the Euler-φ function.

- **5263:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

  Let a, b, c be positive numbers lying in the interval $(0, 1]$. Prove that

$$a \cdot \sqrt{\frac{bc}{1 + c + ab}} + b \cdot \sqrt{\frac{ca}{1 + a + bc}} + c \cdot \sqrt{\frac{ab}{1 + b + ca}} \leq \sqrt{3}.$$

- **5264:** *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia*

  Let x, y, z, α be positive real numbers. Show that if

$$\sum_{cyclic} \frac{(n+1)x^3 + nx}{x^2 + 1} = \alpha$$

then

$$\sum_{cyclic} \frac{1}{x} > \frac{3n}{\alpha} + \frac{(2n-1)\alpha}{3n} + \frac{3n\alpha}{9n^2+\alpha^2}$$

where n is a positive integer. Cyclic means the cyclic permutation of x, y, z (and not x, y, z and α).

Solutions

- **5242:** *Proposed by Kenneth Korbin, New York, NY*

Let N be any positive integer, and let $x = N(N+1)$. Find the value of

$$\sum_{K=0}^{x/2} \binom{x-K}{K} x^K.$$

Solution 1 by Anastasios Kotronis, Athens, Greece,

Using m instead of x for notation convenience we compute the generating function of $\sum_{k=0}^{m/2} \binom{m-k}{k} y^k$:

$$\begin{aligned}
\sum_{m\geq 0}\sum_{k=0}^{m/2} \binom{m-k}{k} y^k t^m &= \sum_{k\geq 0} y^k \sum_{m\geq 2k} \binom{m-k}{k} t^m \\
&= \sum_{k\geq 0} y^k \sum_{m\geq 0} \binom{m+k}{k} t^{m+2k} \\
&= \sum_{k\geq 0} y^k \sum_{m\geq 0} \binom{m+k}{m} t^{m+2k} \\
&= \sum_{k\geq 0} (yt^2)^k \sum_{m\geq 0} \binom{-k-1}{m} (-t)^m \\
&= \sum_{k\geq 0} (yt^2)^k (1-t)^{-k-1} \\
&= \frac{1}{1-t} \sum_{k\geq 0} \left(\frac{yt^2}{1-t}\right)^k \\
&= \frac{1}{1-t-yt^2}
\end{aligned}$$

It is easily shown, decomposing into partial fraction and expanding the geometric series, that if $ax^2 + by + c$ has two distinct non negative roots ρ_1, ρ_2, then

$$\frac{1}{ax^2 + by + c} = \sum_{m \geq 0} \frac{1}{a(\rho_1 - \rho_2)} \left(\rho_2^{-m-1} - \rho_1^{-m-1} \right) x^m,$$

so

$$\sum_{m \geq 0} \sum_{k=0}^{m/2} \binom{m-k}{k} y^k t^m = \sum_{m \geq 0} \frac{1}{\sqrt{1+4y}} \left(\left(\frac{-2y}{1-\sqrt{1+4y}} \right)^{m+1} - \left(\frac{-2y}{1+\sqrt{1+4y}} \right)^{m+1} \right) t^m$$

and hence

$$\sum_{k=0}^{m/2} \binom{m-k}{k} y^k = \frac{1}{\sqrt{1+4y}} \left(\left(\frac{-2y}{1-\sqrt{1+4y}} \right)^{m+1} - \left(\frac{-2y}{1+\sqrt{1+4y}} \right)^{m+1} \right).$$

Putting m in the place of y and then $N(N+1)$ in the place of m in the above relation, and since $N(N+1)+1$ is odd, we get

$$\sum_{K=0}^{N(N+1)/2} \binom{N(N+1)-K}{K} (N(N+1))^K = \frac{1}{2N+1} \left((N+1)^{N^2+N+1} + N^{N^2+N+1} \right).$$

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

We will attack this problem in four steps.

1. If $q > 0$, let

$$x_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} q^k$$

for $n \geq 1$. Then, $x_1 = 1$, $x_2 = 1 + q$, and for $n \geq 1$,

$$\begin{aligned} x_{n+1} + qx_n &= \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-k}{k} q^k + q \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} q^k \\ &= \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-k}{k} q^k + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} q^{k+1} \\ &= \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-k}{k} q^k + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor + 1} \binom{n-k+1}{k-1} q^k. \end{aligned}$$

Note that if n is odd, then $\left\lfloor \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{n+2}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \frac{n+1}{2}$, while if n is even, then $\left\lfloor \frac{n+1}{2} \right\rfloor = \frac{n}{2}$ and $\left\lfloor \frac{n+2}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \frac{n}{2} + 1$. It follows that if n is odd,

$$x_{n+1} + qx_n = \sum_{k=0}^{\frac{n+1}{2}} \binom{n+1-k}{k} q^k + \sum_{k=1}^{\frac{n+1}{2}} \binom{n-k+1}{k-1} q^k$$

$$
\begin{aligned}
&= 1 + \sum_{k=1}^{\frac{n+1}{2}} \left[\binom{n+1-k}{k} + \binom{n+1-k}{k-1} \right] q^k \\
&= 1 + \sum_{k=1}^{\frac{n+1}{2}} \binom{n+2-k}{k} q^k \\
&= \sum_{k=0}^{\frac{n+1}{2}} \binom{n+2-k}{k} q^k \\
&= \sum_{k=0}^{\lfloor \frac{n+2}{2} \rfloor} \binom{n+2-k}{k} q^k \\
&= x_{n+2}
\end{aligned}
$$

while if n is even,

$$
\begin{aligned}
x_{n+1} + qx_n &= \sum_{k=0}^{\frac{n}{2}} \binom{n+1-k}{k} q^k + \sum_{k=1}^{\frac{n}{2}+1} \binom{n-k+1}{k-1} q^k \\
&= 1 + \sum_{k=1}^{\frac{n}{2}} \left[\binom{n+1-k}{k} + \binom{n+1-k}{k-1} \right] q^k + q^{\frac{n}{2}+1} \\
&= 1 + \sum_{k=1}^{\frac{n}{2}} \binom{n+2-k}{k} q^k + q^{\frac{n}{2}+1} \\
&= \sum_{k=0}^{\frac{n}{2}+1} \binom{n+2-k}{k} q^k \\
&= \sum_{k=0}^{\lfloor \frac{n+2}{2} \rfloor} \binom{n+2-k}{k} q^k \\
&= x_{n+2}.
\end{aligned}
$$

Therefore, $\{x_n\}$ can also be described by the recursive definition $x_1 = 1$, $x_2 = 1 + q$, and $x_{n+2} = x_{n+1} + qx_n$ for all $n \geq 1$.

2. We can now find a closed form formula for $\{x_n\}$ by following the usual method for solving homogeneous linear difference equations with constant coefficients. This entails considering solutions of the form $x_n = t^n$ for some $t \neq 0$. Then, the recurrence relation $x_{n+2} = x_{n+1} + qx_n$ becomes

$$t^{n+2} = t^{n+1} + qt^n$$

or

$$t^2 = t + q \qquad (1)$$

since $t \neq 0$. Further, $q > 0$ guarantees that (1) has two distinct real solutions

$$t_1 = \frac{1 + \sqrt{1+4q}}{2} \quad \text{and} \quad t_2 = \frac{1 - \sqrt{1+4q}}{2}.$$

In this situation, the general solution is

$$x_n = c_1 t_1^n + c_2 t_2^n$$

for some constants c_1 and c_2. Finally, the initial conditions $x_1 = 1$ and $x_2 = 1 + q$ imply that

$$c_1 = \frac{t_1}{\sqrt{1+4q}} \quad \text{and} \quad c_2 = \frac{-t_2}{\sqrt{1+4q}}.$$

As a result, we have

$$x_n = \frac{t_1^{n+1} - t_2^{n+1}}{\sqrt{1+4q}}$$

for $n \geq 1$.

3. By Parts 1 and 2,

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} q^k$$

$$= \frac{1}{\sqrt{1+4q}} \left[\left(\frac{1 + \sqrt{1+4q}}{2} \right)^{n+1} - \left(\frac{1 - \sqrt{1+4q}}{2} \right)^{n+1} \right] \qquad (2)$$

for all $n \geq 1$. In particular, since $n(n+1)$ is always even, we have

$$\left\lfloor \frac{n(n+1)}{2} \right\rfloor = \frac{n(n+1)}{2}$$

and (2) yields

$$\sum_{k=0}^{\frac{n(n+1)}{2}} \binom{n(n+1)-k}{k} q^k$$

$$= \frac{1}{\sqrt{1+4q}} \left[\left(\frac{1 + \sqrt{1+4q}}{2} \right)^{n(n+1)+1} - \left(\frac{1 - \sqrt{1+4q}}{2} \right)^{n(n+1)+1} \right] \qquad (3)$$

for $n \geq 1$.

4. Finally, if we substitute $q = n(n+1)$ in (3), then $\sqrt{1+4q} = 2n+1$ and for all $n \geq 1$, we get

$$\sum_{k=0}^{\frac{n(n+1)}{2}} \binom{n(n+1)-k}{k} [n(n+1)]^k = \frac{(n+1)^{n(n+1)+1} - (-n)^{n(n+1)+1}}{2n+1}$$

$$= \frac{(n+1)^{n(n+1)+1} + n^{n(n+1)+1}}{2n+1}$$

(since $n(n+1)+1$ is odd for all $n \geq 1$).

Solution 3 by Adrian Naco, Polytechnic University, Tirana, Albania

Based on the *Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX. to problem 4919, SSMA, February 2007* ,for $n \in Z^{*+}$ and $0 \leq k \leq n+2$, we have that

$$\binom{2n+4-k}{k} = \binom{2n+2-k}{k} + 2\binom{2n+3-k}{k-1} - \binom{2n+2-k}{k-2}.$$

Let $S(n) = \sum_{k=0}^{n} \binom{2n-k}{k} z^k$, (z is constant) $\forall n \geq 1$. Then we have that,

$$\begin{aligned}
S(n+2) &= \sum_{k=0}^{n+2} \binom{2n+4-k}{k} z^k = 1 + (2n+3)z + \sum_{k=2}^{n+1} \binom{2n+4-k}{k} z^k + z^{n+2} \\
&= 1 + (2n+3)z + \sum_{k=2}^{n+1} \left[\binom{2n+2-k}{k} + 2\binom{2n+3-k}{k-1} - \binom{2n+2-k}{k-2} \right] z^k + z^{n+2} \\
&= 2z + \sum_{k=0}^{n+1} \binom{2n+2-k}{k} z^k + 2\sum_{k=2}^{n+1} \binom{2n+3-k}{k-1} z^k - \sum_{k=2}^{n+1} \binom{2n+2-k}{k-2} z^k + z^{n+2} \\
&= 2z + S(n+1) + 2\sum_{k=1}^{n} \binom{2n+2-k}{k} z^{k+1} - \sum_{k=0}^{n-1} \binom{2n-k}{k} z^{k+2} + z^{n+2} \\
&= 2z + S(n+1) + 2z\sum_{k=1}^{n} \binom{2n+2-k}{k} z^{k} - z^2\sum_{k=0}^{n-1} \binom{2n-k}{k} z^{k} + z^{n+2} \\
&= 2z + S(n+1) + 2z\,[S(n+1) - 1 - z^{n+1}\,] - z^2\,[S(n) - z^n\,] + z^{n+2} \\
&= (1+2z)S(n+1) - z^2 S(n).
\end{aligned}$$

As a result, we get the following homogeneous linear difference equation with constant coefficients,

$$S(n+2) - (1+2z)S(n+1) + z^2 S(n) = 0.$$

Solving the respective characteristic equation (considering z as constant),

$$r^2 - (1+2z)r + z^2 = 0$$

we get the solutions

$$r_1 = \frac{(1+2z) + \sqrt{1+4z}}{2}, \qquad \text{and} \qquad r_2 = \frac{(1+2z) - \sqrt{1+4z}}{2}.$$

The general formula for $S(n)$ is

$$S(n) = C_1 r_1^n + C_2 r_2^n, \qquad n \in Z^{*+}.$$

Considering the fact that $S(1) = 1 + z$ and $S(2) = 1 + 3z + z^2$ we have that

$$\begin{aligned} S(1) &= C_1 r_1 + C_2 r_2 = 1 + z = \sum_{k=0}^{1} \binom{2-k}{k} z^k \quad \text{and} \\ S(2) &= C_1 r_1^2 + C_2 r_2^2 = 1 + 3z + z^2 = \sum_{k=0}^{2} \binom{4-k}{k} z^k \end{aligned}$$

from where it implies that

$$C_1 = \frac{(1+3z+z^2) - r_2(1+z)}{r_1(r_1 - r_2)} \qquad \text{and} \qquad C_2 = \frac{(1+3z+z^2) - r_1(1+z)}{r_2(r_2 - r_1)}.$$

Finally,

$$\begin{aligned} S(n) &= \sum_{k=0}^{n} \binom{2n-k}{k} z^k = C_1 \cdot r_1^n + C_2 \cdot r_2^n \\ &= \frac{(1+z)\sqrt{1+4z} + (1+3z)}{2\sqrt{1+4z}} \cdot r_1^{n-1} + \frac{(1+z)\sqrt{1+4z} - (1+3z)}{2\sqrt{1+4z}} \cdot r_2^{n-1} \\ &= \frac{1}{2\sqrt{1+4z}} \Big\{ [(1+z)\sqrt{1+4z} + (1+3z)] r_1^{n-1} + [(1+z)\sqrt{1+4z} - (1+3z)] r_2^{n-1} \Big\}. \end{aligned}$$

Thus, the general formula is,

$$S(n) = \frac{1}{2\sqrt{1+4z}} \Big\{ [(1+z)\sqrt{1+4z} + (1+3z)] r_1^{n-1} + [(1+z)\sqrt{1+4z} - (1+3z)] r_2^{n-1} \Big\}.$$

Applying the above formula for $z = x = 2n = N(N+1)$, (since $N(N+1)$ is an even number for $N \in Z^{*+}$), and after making some manipulations, we have that,

$$r_1 = (N+1)^2, \quad r_2 = N^2, \quad C_1 = \frac{N+1}{2N+1}, \quad C_2 = \frac{N}{2N+1}, \quad N = \frac{\sqrt{1+4x} - 1}{2}$$

$$\Rightarrow \quad \sum_{k=0}^{x/2} \binom{x-k}{k} x^k = \frac{N+1}{2N+1} \cdot (N+1)^{N(N+1)} + \frac{N}{2N+1} \cdot N^{N(N+1)}$$

$$\Rightarrow \quad \sum_{k=0}^{x/2} \binom{x-k}{k} x^k = \frac{1}{2N+1} \Big[(N+1)^{N(N+1)+1} + N^{N(N+1)+1} \Big]$$

or related to $x, (x = N(N+1))$, we get the formula,

$$\Rightarrow \quad \sum_{k=0}^{x/2} \binom{x-k}{k} x^k = \frac{1}{2^{x+1}\sqrt{1+4x}} \left[(\sqrt{1+4x}+1)^{x+1} + (\sqrt{1+4x}-1)^{x+1} \right].$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China, and the proposer.

- **5243:** *Proposed by Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

If a, b, c are consecutive Pythagorean numbers, then solve in the integers the equation:

$$\frac{x^2+bx}{a^y-1} = c.$$

(A consecutive Pythagorean triple is a Pythagorean triple that is composed of consecutive integers.)

Solution by David E. Manes, SUNY College at Oneonta, Oneonta, NY

There are no solutions to the equation for a consecutive Pythagorean triple.

Assume that a is a positive integer and $b = a+1, c = a+2$ so that a, b, c is a consecutive Pythagorean triple. Then $a^2 + (a+1)^2 = (a+2)^2$ reduces to the quadratic equation $a^2 - 2a - 3 = 0$ whose only positive integer solution is $a = 3$. Therefore $a = 3, b = 4, c = 5$ is the only positive consecutive Pythagorean triple and the given equations becomes

$$\frac{x^2+4x}{3^y-1} = 5.$$

Note that if $y = 0$ the the equation is undefined. If $y < 0$, then $y = -n$ for some positive integer n. The equation then reduces to $3^n(x^2+4x) = 5(1-3^n)$. Since 3 is a prime, it follows that either 3 divides 5 or 3 divides $1 - 3^n$, both contradictions.

Hence, $y > 0$ and $x^2 + 4x = 5(3^y - 1)$ or $x^2 + 4x + 5 = 3^y = 5$. Let $p(x) = x^2 + 4x + 5$. If $x \equiv 0 (\text{mod } 3)$, then $p(x) \equiv 2 (\text{mod } 3)$. Therefore, $p(x) = x^2 + 4x + 5$ is never congruent to 0 module 3 for any integer x. However, $3^y 5 \equiv 0 (\text{mod } 3)$ for each integer $y > 0$. Hence, there are no nonzero solutions, where $y \neq 0$ to the equation $x^2 + 4x + 5 = 3^y 5$ and this completes the solution.

Editor's comment: Some readers gave $(0, 0)$ and $(-4, 0)$ as solutions to the equation $x^2 + 4x + 5 = 3^y 5$. This certainly true, but the expression $x^2 + 4x + 5 = 3^y 5$ was obtained from the original statement of the problem under the assumption that $y \neq 0$.

$$\left(\frac{x^2+4x}{3^y-1} = 5 \right) \iff \left(x^2 + 4x + 5 = 3^y 5 \right) \text{ if, and only if } y \neq 0.$$

In this case, multiplication by the denominator is not valid. Stated otherwise, the equation $\dfrac{x^2+4x}{3^y-1} = 5$ has no solution, but the equation $x^2 + 4x + 5 = 3^y 5$ has two integer solutions, $(0,0)$ and $(-4,0)$. The two equations are not equivalent to one another because they have different domains of definition.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Ed Gray, Highland Beach, FL; David Stone and John Hawkins (jointly), Georgia Southern University, Statesboro, GA, and the proposer.

- **5244:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

Let T_a and S_b denote the a^{th} triangular and the b^{th} square number, respectively. Find explicit instances of such numbers to prove that every Fibonacci number F_n occurs among the values $\gcd(T_a, S_b)$.

Solution 1 by David Diminnie, Texas Instruments, Inc., Dallas, TX

Recall that $T_a = \dfrac{a\,(a+1)}{2}$ and $S_b = b^2$. If we set $a = 2F_n$ and $b = F_n$ then by applying the identity $\gcd(p,\, q) = \gcd(p-q,\, q)\,,\ p > q$ we may evaluate $\gcd(T_a,\, S_b)$ as follows:

$$\begin{aligned}
\gcd\left(T_{2F_n},\, S_{F_n}\right) &= \gcd\left(\frac{2F_n\,(2F_n+1)}{2},\, F_n^2\right) \\
&= \gcd\left(2F_n^2 + F_n,\, F_n^2\right) \\
&= \gcd\left(F_n^2 + F_n,\, F_n^2\right) \\
&= \gcd\left(F_n,\, F_n^2\right) \\
&= F_n.
\end{aligned}$$

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, (jointly), Angelo State University, San Angelo, TX

More generally, we will show that for every positive integer n, $\gcd(T_{2n}, S_{2n}) = n$. The desired result then follows as an easy application of this property. To do so, we will use the following elementary results from number theory.

Lemma 1. If m and n are positive integers and d is a positive common divisor of m and n such that $\gcd\left(\dfrac{m}{d}, nd\right) = 1$, then $d = \gcd(m,n)$.

Proof. Since $\gcd(md, nd) = 1$, there are integers a and b such that

$$1 = a\left(\frac{m}{d}\right) + b\left(\frac{n}{d}\right)$$

or

$$d = am + bn.$$

Then, any positive common divisor of m and n must also divide d and it follows that $d = \gcd(m,n)$.

Lemma 2. For every positive integer n, $\gcd(2n+1, 4n) = 1$.

Proof. If $d = \gcd(2n+1, 4n)$, then d divides $(2n+1)$ and hence, d is odd. Further, since d is odd and d divides $4n$, d must divide n. Finally, d is a common divisor of n and $(2n+1)$ implies that d divides $(2n+1) - 2n = 1$. Therefore, $d = 1$.

For any positive integer n,

$$T_{2n} = \frac{2n\,(2n+1)}{2} = n\,(2n+1) \text{ and } S_{2n} = 4n^2.$$

Then, n is a positive common divisor of T_{2n} and S_{2n} and Lemma 2 implies that

$$\gcd\left(\frac{T_{2n}}{n}, \frac{S_{2n}}{n}\right) = \gcd\,(2n+1, 4n) = 1.$$

By Lemma 1, we have $\gcd\,(T_{2n}, S_{2n}) = n$ and our solution is complete.

Solution 3 by Paul M. Harms, North Newton, KS

We have $T_a = a(a+1)/2$ and $S_b = b^2$. When the Fibonacci number F_n is an odd integer let $a = F_n = b$. Then $a+1$ is even and the number $a = F_n$ does not have any common factor (except 1) with $a+1$ or $(a+1)/2$.

With $S_b = b^2 = F_n^2$, the $\gcd(T_a, S_b)$ =$\gcd\big(F_n\,(F_n+1)\,, F_n^2\big) = F_n$. When F_n is an even integer let $a = 2F_n$ and $b = F_n$. Then $a+1$ is odd and has no common factors with $a/2 = F_n$. Again we have $\gcd(T_a, S_b)$ =$\gcd\big(F_n\,(2F_n+1)\,, F_n^2\big) = F_n$.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; David E. Manes, SUNY College at Oneonta, Oneonta, NY; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA and the proposer.

- **5245:** *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany*

  Determine all real valued functions $f : \Re - \{-2, -\frac{1}{2}, -1, 0, \frac{1}{2}, 1, 2\} \to \Re$, which satisfy the relation

  $$f(x) + f\left(\frac{-x-5}{2x+1}\right) + f\left(\frac{4x+5}{-2x+2}\right) = ax + b$$

  where $a, b, \in \Re$.

Solution 1 by Adrian Naco, Polytechnic University, Tirana, Albania

If we let, $g(x) = \dfrac{-x-5}{2x+1}$, $h(x) = \dfrac{4x+5}{-2x+2}$, then we have that,

$$\begin{aligned} (g \circ g)(x) &= x && (1) \\ (g \circ h)(x) &= (h \circ g)(x) && (2) \\ \text{and} \qquad (h \circ h)(x) &= g(x) && (3) \end{aligned}$$

Thus the given problem can be expressed as,

$$f(x) + (f \circ g)(x) + (f \circ h)(x) = ax + b, \quad (4)$$

Considering equation (4) and applying for $g(x)$, it implies that,

$$\begin{aligned}(f\circ g)(x)+[f\circ(g\circ g)](x)+(f\circ h\circ g)(x) &= ag(x)+b, \text{ or equvalently}\\ (f\circ g)(x)+f(x)+(f\circ h\circ g)(x) &= ag(x)+b, \qquad (5)\end{aligned}$$

Considering typos (4) and applying for $h(x)$, it implies that,

$$\begin{aligned}(f\circ h)(x)+[f\circ(g\circ h)](x)+(f\circ h\circ h)(x) &= ah(x)+b, \text{ or equivalently}\\ (f\circ h)(x)+[f\circ(g\circ h)](x)+(f\circ g)(x) &= ah(x)+b, \qquad (6)\end{aligned}$$

Considering equation (5) and applying for $h(x)$, it implies that,

$$\begin{aligned}(f\circ g\circ h)(x)+(f\circ h)(x)+(f\circ h\circ g\circ h)(x) &= a(g\circ h)(x)+b, \text{ or equivalently}\\ (f\circ g\circ h)(x)+(f\circ h)(x)+f(x) &= a(g\circ h)(x)+b, \qquad (7)\end{aligned}$$

Adding (simultaneously) side by side equations in (4), (5), and (6) to equation (7), results in,

$$3[f(x)+(f\circ g)(x)+(f\circ h)(x)+(f\circ g\circ h)(x)] = ax+ag(x)+ah(x)+a(g\circ h)(x)+4b,$$

$$f(x)+(f\circ g)(x)+(f\circ h)(x)+(f\circ g\circ h)(x) = \frac{1}{3}[ax+ag(x)+ah(x)+a(g\circ h)(x)+4b], \qquad (8)$$

Finally, if we subtract equation (6) from equation (8), then,

$$\begin{aligned} f(x) &= \frac{1}{3}a[x+g(x)-2h(x)+(g\circ h)(x)]+\frac{1}{3}b\\ \Leftrightarrow\quad f(x) &= \frac{1}{3}a\left[x+\frac{-x-5}{2x+1}-2\frac{4x+5}{-2x+2}+\frac{2x-5}{2x+4}\right]+\frac{1}{3}b\\ \Leftrightarrow\quad f(x) &= \frac{1}{3}a\left[x-\frac{x+5}{2x+1}+\frac{4x+5}{x-1}+\frac{2x-5}{2(x+2)}\right]+\frac{1}{3}b\\ \Leftrightarrow\quad f(x) &= \frac{a}{3}\cdot\frac{4x^4+24x^3+30x^2+59x+45}{2(2x^3+3x^2-3x-2)}+\frac{1}{3}b\end{aligned}$$

Solution 2 by David Diminnie, Texas Instruments, Inc., Dallas, TX, and Charles Diminnie, Angelo State University, San Angelo, TX

The restrictions on the domain and range in the problem statement appear to be swapped, and the domain restriction appears to be both overly stringent and missing a critical value. For the discussion below we will assume that $f : \Re - \left\{-2,\ -\frac{1}{2},\ 1\right\} \to \Re$ satisfies

$$f(x) + f\left(\frac{-x-5}{2x+1}\right) + f\left(\frac{4x+5}{-2x+2}\right) = ax + b \quad (1)$$

for given $a, b \in \Re$. 8pt
Consider the function $g : \Re - \left\{-2,\ -\dfrac{1}{2},\ 1\right\} \to \Re$ with definition

$$g(x) = \frac{4x+5}{-2x+2}.$$

Since $g(x) \neq -2,\ -\frac{1}{2},\ 1$ when $x \neq -2,\ -\frac{1}{2},\ 1$ it follows that

$$g^2(x) = (g \circ g)(x) = \frac{4g(x)+5}{-2g(x)+2} = \frac{-x-5}{2x+1}.$$

Similarly, $g^2(x) \neq -2,\ -\dfrac{1}{2},\ 1$ when $x \neq -2,\ -\dfrac{1}{2},\ 1$ and we see that

$$g^3(x) = (g \circ g \circ g)(x) = \frac{-g(x)-5}{2g(x)+1} = \frac{2x-5}{2x+4}.$$

Finally, $g^3(x) \neq -2,\ -\dfrac{1}{2},\ 1$ when $x \neq -2,\ -\dfrac{1}{2},\ 1$ implies that

$$g^4(x) = (g \circ g \circ g \circ g)(x) = \frac{2g(x)-5}{2g(x)+4} = x.$$

As a result, we can see (by Comment [1]) that $g^n(x) = g^{n \bmod 4}(x)$ may therefore be re-expressed as

$$f(x) + f\left(g^2(x)\right) + f\left(g(x)\right) = ax + b. \quad (2)$$

If we substitute $g(x)$, $g^2(x)$, $g^3(x)$ into (2), taking advantage of the fact that $g^{i+j}(x) = g^{i+j \bmod 4}(x)$ (with $g^0(x) \equiv x$), we obtain the following additional relations (respectively):

$$f\left(g(x)\right) + f\left(g^3(x)\right) + f\left(g^2(x)\right) = ag(x) + b \quad (3)$$

$$f\left(g^2(x)\right) + f\left(x\right) + f\left(g^3(x)\right) = ag^2(x) + b \quad (4)$$

$$f\left(g^3(x)\right) + f\left(g(x)\right) + f(x) = ag^3(x) + b. \quad (5)$$

By adding (2), (4), and (5) and subtracting two times (3) from the result (again, with $x \neq -2,\ -\dfrac{1}{2},\ 1$), we may find an expression for $f(x)$:

$$3f(x) = a\left(x + g^2(x) + g^3(x) - 2g(x)\right) + b$$

$$f(x) = \frac{a}{3}\Big(x + g^2(x) + g^3(x) - 2g(x)\Big) + \frac{b}{3}$$

$$f(x) = \frac{a}{3}\left(x - \frac{x+5}{2x+1} + \frac{2x-5}{2x+4} - \frac{4x+5}{-x+1}\right) + \frac{b}{3}. \qquad (6)$$

To verify (6) is a solution, note that

$$f\left(\frac{4x+5}{-2x+2}\right) = \frac{a}{3}\left(\frac{4x+5}{-2x+2} + \frac{2x-5}{2x+4} + x + \frac{2x+10}{2x+1}\right) + \frac{b}{3}$$

$$f\left(\frac{-x-5}{2x+1}\right) = \frac{a}{3}\left(\frac{-x-5}{2x+1} + x + \frac{4x+5}{-2x+2} - \frac{2x-5}{x+2}\right) + \frac{b}{3}$$

and therefore

$$f(x) + f\left(\frac{4x+5}{-2x+2}\right) + f\left(\frac{-x-5}{2x+1}\right) = ax + b.$$

Comment [1]. Note that $\left\{x,\ \frac{4x+5}{-2x+2},\ \frac{-x-5}{2x+1},\ \frac{2x-5}{2x+4}\right\} = \{g^0(x),\ g(x),\ g^2(x),\ g^3(x)\}$ forms a cyclic group of order 4 under function composition, with generator $g(x)$: Function composition is an associative operation, the identity element is $g^0(x) = g^4(x) = x$ (and hence $g^n(x) = g^{n \bmod 4}(x)$, as claimed above, so the set is closed under composition), and $g^k \circ g^{4-k}(x) = g^{4-k} \circ g^k(x) = x$ for $k = 0,\ 1,\ 2,\ 3$.

Solution 3 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

Let $L(x) = ax + b, h(x) = \frac{-x-5}{2x+1} = -\frac{x+5}{2x+1}$ and $k(x) = \frac{4x+5}{-2x+2} = -\frac{4x+5}{2(x-1)}$.

Then the given condition becomes

$$(1)\quad f(x) + f\left(h(x)\right) + f\left(k(x)\right) = L(x).$$

Suppressing the argument x and adopting concatenation to represent composition, this becomes a functional condition:

$$(1a)\quad f + fh + fk = L.$$

Straightforward computation shows that $h^2(x) = h(h(x)) = x$, that

$$k^2(x){=}h(x) \text{ and } h(k(x)) = k(h(x)) = \frac{2x-5}{2(x+2)}.$$

That is, with i denoting the identity function,

$$(2)\quad h^2 = i, k^2 = h \text{ and } kh = hk.$$

It follows that $k^4 = i$ and $khk = hk^2 = hk = i$.

Applying both sides of $(1a)$ to $h(x)$ yields $fh + fh^2 + fkh = Lh$, or

$$(3)\quad fh + f + fkh = Lh.$$

Applying both sides of $(1a)$ to $k(x)$ yields $fk + fhk + fkk = Lk$ or

$$(4) \quad fk + fhk + fh = Lk.$$

Finally, applying both sides of (3) to $k(x)$ yields $fhk + fk + fkhk = Lhk$, or

$$(5) \quad fhk + fk + f = Lhk.$$

Thus we have a system of 4 equations in the 4 unknowns, f, fh, fk, fhk:

$$\begin{cases} f + fh + fk \qquad = L \\ f + fh \qquad + fhk = Lh \\ \qquad fh + fk + fhk = Lk \\ f \qquad + fk + fhk = Lhk \end{cases}$$

Calculations reveal that

$$(6) \quad f = \frac{1}{3}\{L + Lh + Lhk - 2Lk\}.$$

That is,

$$\begin{aligned} f(x) &= \frac{1}{3}\{ax + b + ah(x) + b + ah(k(x)) + b - 2ak(x) - 2b\} \\ &= \frac{a}{3}\left\{x + h(x) + h(k(x)) - 2k(x)\right\} + \frac{b}{3} \\ &= \frac{a}{3}\left\{x + \frac{-x-5}{2x+1} + \frac{2x-5}{2(x+2)} - 2\frac{4x+5}{-2x+2}\right\} + \frac{b}{3} \\ &= \frac{a}{3}\left\{\frac{4x^4 + 24x^3 + 30x^3 + 59x + 45}{2(2x+1)(x-1)(x+2)}\right\} + \frac{b}{3}. \end{aligned}$$

Comment 1. More generally, note that if h and k are any two functions such that h has order $2, k^2 = h$ and h commute with k, then (6) gives the function f satisfying (1).

Comment 2. We believe that he domain and codomain of f, as stated in in the problem, are a typo. The conditions on the domain and codomain of f (and h and k and kh) are probably best summarized as "for all x for which everything makes sense." The domain of f consists of all reals except the obvious ones: $1, -2$ and $-\frac{1}{2}$.

Then fh is well defined because h is defined for all x except $-\frac{1}{2}$ and does not map any real to $-\frac{1}{2}$. Similarly fk is defined because k is defined for all reals except 1 and has range all reals except -2.

The composed functions $kh = hk$ both map from $\Re - \{-2\}$ to $\Re - \{1\}$ despite the technical concerns with domains.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Boris Rays, Brooklyn, NY, and the proposers.

- **5246:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

  Let $a_1, a_2, \ldots a_n, (n \geq 3)$ be distinct complex numbers. Compute the sum

$$\sum_{k=1}^{n} s_k \prod_{j \neq k} \frac{(-1)^n}{a_j - a_k},$$

where $s_k = \left(\sum_{i=1}^{n} a_i \right) - a_k,\ 1 \leq k \leq n.$

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Let $f : C \to C$ be the function defined by $f(a) = \left(\sum_{i=1}^{n} a_j \right) - a, a \in C$. Then f is a polynomial function such that $f(a_k) = s_k, 1 \leq k \leq n$. It is also known that there is only one polynomial function $p : C \to C$ with degree less than n and such that $p(a_k) = s_k, 1 \leq k \leq n$ which can be obtained for example with the Lagrange interpolation formula:

$$p(a) = \sum_{k=1}^{n} s_k \prod_{j \neq k} \frac{a - a_j}{a_k - a_j} = \sum_{k=1}^{n} s_k \frac{\prod\limits_{j \neq k} (a - a_j)}{(-1)^{n-1} \prod\limits_{j \neq k} (a_j - a_k)} = \sum \frac{(-1)^{n-1} s_k}{\prod\limits_{j \neq k} (a_j - a_k)} \prod_{j \neq k} (a - a_j)\,.$$

So, both polynomial functions p and f, must be equal; in particular, their respective leading coefficients must coincide, that is $\sum\limits_{k=1}^{n} \frac{(-1)^{n-1} s_k}{\prod\limits_{j \neq k} (a_j - a_k)} = 0$. Thus, the required sum is

$$\sum_{k=1}^{n} \frac{(-1)^n}{\prod\limits_{j \neq k} (a_j - a_k)} = 0.$$

Solution 2 by Paul M. Harms, North Newton, KS

Consider the polynomial

$$\begin{aligned} P(x) &= \frac{(x - a_2)(x - a_3) \cdots (x - a_n)}{(a_1 - a_2)(a_1 - a_3) \cdots (a_1 - a_n)} + \frac{(x - a_1)(x - a_3) \cdots (x - a_n)}{(a_2 - a_1)(a_2 - a_3) \cdots (a_2 - a_n)} \\ &+ \cdots \frac{(x - a_1)(x - a_2) \cdots (x - a_{n-1})}{(a_n - a_1)(a_n - a_2) \cdots (a_n - a_{n-1})} - 1. \end{aligned}$$

We see that the degree of $p(x)$ is $n - 1$. Note that $0 = p(a_1) = p_(a_2) = \cdots = p(a_n)$.

Since n different complex number have a polynomial value of zero for the $n - 1$ degree polynomial, the polynomial must be identically zero.

If $p(x)$ (given above is expanded, then all coefficients of the different powers of x must be zero. Consider the coefficient of x^{n-2}. From the first fraction of $p(x)$ the coefficient of x^{n-2} is

$$\frac{-(a_2 + a_3 + \cdots + a_n)}{(a_1 - a_2)(a_1 - a_3) \cdots (a_1 - a_n)} = \frac{-s_1}{(a_1 - a_2)(a_1 - a_3) \cdots (a_1 - a_n)}.$$

We see that the coefficient of x^{n-2} for $p(x)$ is

$$\frac{-s_1}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_n)}+\frac{-s_2}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_n)}+\cdots$$

$$+\quad\frac{-s_n}{(a_n-a_1)(a_n-a_2)\cdots(a_n-a_{n-1})}=0.$$

The left side of the last equality is equal to or the negative of the summation in the problem. Thus the summation in the problem is zero.

Also solved by the proposer.

- **5247:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate

$$\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\int_0^1 \ln(1+e^x)\ln(1+e^{2x})\cdots\ln(1+e^{nx})\ dx}.$$

Solution 1 by Anastasios Konronis, Athens, Greece

For $n\in N,\ x\in(0,1]$ we have

$$\begin{aligned}\ln(1+e^x)\cdot\ln(1+e^{2x})\cdots\ln(1+e^{nx}) &= n!x^n\prod_{k=1}^{n}\left(1+\frac{\ln(1+e^{-kx})}{kx}\right)=n!x^n\prod_{k=1}^{n}\left(1+\mathcal{O}\left(\frac{e^{-kx}}{kx}\right)\right)\\ &= n!x^n\left(1+\mathcal{O}\left(\frac{e^{-x}}{x^n}\right)\right)\\ &= n!\left(x^n+\mathcal{O}\left(e^{-x}\right)\right)\end{aligned}$$

so

$$\int_0^1 \ln(1+e^x)\cdot\ln(1+e^{2x})\cdots\ln(1+e^{nx})=\frac{n!}{n+1}\left(1+\mathcal{O}(n)\right).$$

Now from the above and taking into account that, from Stirling's formula,

$$\ln n! = n\ln n - n+\mathcal{O}(\ln n)$$

we get that

$$\begin{aligned}\frac{1}{n}\sqrt[n]{\int_0^1 \ln(1+e^x)\cdot\ln(1+e^{2x})\cdots\ln(1+e^{nx})\,dx} &= \frac{1}{n}\exp\left(\frac{1}{n}\ln\left(\frac{n!}{n+1}\left(1+\mathcal{O}(n)\right)\right)\right)\\ &= \frac{1}{n}\exp\left(\ln n-1+\mathcal{O}\left(\frac{\ln n}{n}\right)\right)=e^{-1}+\mathcal{O}\left(\frac{\ln n}{n}\right)\to e^{-1}\end{aligned}$$

Solution 2 by Arkady Alt, San Jose, California, USA.

Let $f_n(x) = \prod_{k=1}^{n} \ln\left(1 + e^{kx}\right)$. Since $f_n(x) > \prod_{k=1}^{n} \ln\left(e^{kx}\right) = x^n n!$ then

$$\int_0^1 f_n(x)\,dx > n! \int_0^1 x^n dx = \frac{n!}{n+1}.$$

On the other hand, since $f_n(x) f_n(1) \le 1$ we have

$$\int_0^1 f_n(x)\,dx \le f_n(1) \int_0^1 dx = f_n(1).$$

Thus,

$$\frac{1}{n}\sqrt[n]{\frac{n!}{n+1}} < \frac{1}{n}\sqrt[n]{\int_0^1 f_n(x)\,dx} \le \frac{1}{n}\sqrt[n]{f_n(1)}.$$

Let $a_n = \dfrac{f_n(1)}{n^n}$.
Since

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{a_{n-1}} &= \lim_{n\to\infty} \left(\frac{f_n(1)}{n^n} \cdot \frac{(n-1)^{n-1}}{f_{n-1}(1)} \right) \\
&= \lim_{n\to\infty} \left(\left(1 - \frac{1}{n}\right)^{n-1} \cdot \frac{\ln(1+e^n)}{n} \right) \\
&= \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^{n-1} \cdot \lim_{n\to\infty} \frac{\ln(1+e^{-n}) + n}{n} \\
&= e^{-1} \cdot 1 = e^{-1}
\end{aligned}$$

then by *, the Multiplicative Stolz Theorem $\lim_{n\to\infty} \frac{1}{n}\sqrt[n]{f_n(1)} = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_n}{a_{n-1}} = e^{-1}$.

Also we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n}\sqrt[n]{\frac{n!}{n+1}} &= \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} \cdot \frac{1}{\sqrt[n]{n+1}} \\
&= \lim_{n\to\infty} \sqrt[n]{n!}n \cdot \lim_{n\to\infty} \frac{1}{\sqrt[n]{n+1}} \\
&= e^{-1} \cdot 1 = e^{-1}.
\end{aligned}$$

(Note: $\lim_{n\to\infty} \dfrac{\sqrt[n]{n!}}{n} = e^{-1}$. Indeed,

$$\left(\frac{n}{e}\right)^n < n! < \left(\frac{n+1}{e}\right)^n (n+1) \Rightarrow$$

$$\frac{1}{e} < \frac{\sqrt[n]{n!}}{n} < \frac{1}{e} \cdot \frac{n+1}{n} \cdot \sqrt[n]{n+1},$$

or again, applying the Multiplicative Stolz Theorem to $\sqrt[n]{\frac{n!}{n^n}}$).

Then by the squeeze principle,

$$\lim_{n\to\infty} \frac{1}{n} \sqrt[n]{f_n(1)} = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = e^{-1} = e^{-1} \text{ yields}$$

$$\lim_{n\to\infty} \frac{1}{n} \sqrt[n]{\int_0^1 f_n(x)\,dx} = e^{-1}.$$

* We use the Multiplicative Stolz Theorem in the following form:
If the sequence $\left(\frac{a_{n+1}}{a_n}\right)_{n\geq 1}$ has a limit then the sequence $\left(\sqrt[n]{a_n}\right)_{n\geq 1}$ has a limit and

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Solution 3 by Kee-Wai, Hong Kong, China

We show that the limit equals $\frac{1}{e}$.

Denote the integrand by $f(x)$. Since $f(x) > (x)(2x)\cdots(nx) = (n!)x^n$, so

$$\int_0^1 f(x)dx > \frac{n!}{n+1}. \qquad (1)$$

For $0 \leq x \leq 1$ and $k = 1, 2, \cdots, n$, we have

$$1 + e^{kx} \leq 1 + e^k < 2e^k < e^{1+k}, \text{so that}$$

$$f(x) < (n+1)! \text{ and}$$

$$\int_0^1 f(x)dx < (n+1)!. \qquad (2)$$

By Stirling's formula for $n!$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sqrt[n]{\frac{n!}{n+1}} = \lim_{n\to\infty} \frac{1}{n} \sqrt[n]{(n+1)!} = \frac{1}{e}.$$

Now by $(1), (2)$ and the squeezing principle, we obtain the result we claimed.

Also solved by Paul M. Harms, North Newton, KS; Adrian Naco, Polytechnic University, Tirana, Albania and the proposer.

Mea Culpa (yet again)

Featured solution 5241(3) that appeared in the April 2013 issue of the column was submitted jointly by **Anastasios Kotronis and Konstantinos Tsouvalas, University of Athens, Athens, Greece.** I inadvertently forgot to list Konstantinos' name. Sorry.