Problems Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before October 15, 2012

- **5212:** *Proposed by Kenneth Korbin, New York, NY*

  Solve the equation
  $$2x + y - \sqrt{3x^2 + 3xy + y^2} = 2 + \sqrt{2}$$
  if x and y are of the form $a + b\sqrt{2}$ where a and b are positive integers.

- **5213:** *Proposed by Tom Moore, Bridgewater, MA*

  The triangular numbers T_n begin $1, 3, 6, 10, \ldots$ and, in general, $T_n = \dfrac{n(n+1)}{2},\ n = 1, 2, 3, \ldots$.

  For every positive integer $n > 1$, prove that n^4 is a sum of four triangular numbers.

- **5214:** *Proposed by Pedro H. O. Pantoja, Natal-RN, Brazil*

  Let a, b, c be positive real numbers. Prove that
  $$\frac{a^3(b+c)^2+1}{1+a+2b} + \frac{b^3(c+a)^2+1}{1+b+2c} + \frac{c^3(a+b)^2+1}{1+c+2a} \geq \frac{4abc(ab+bc+ca)+3}{a+b+c+1}.$$

- **5215:** *Proposed by Neculai Stanciu, Buzău, Romania*

  Evaluate the integral
  $$\int_{-1}^{1} \frac{2x^{1004} + x^{3014} + x^{2008}\sin x^{2007}}{1 + x^{2010}} dx.$$

- **5216:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let $f : \Re \to \Re^+$ be a function such that for all $a, b \in \Re$
  $$f(ab) = f(a)^b f(b)^{a^2}$$
  and $f(3) = 64$. Find all real solutions to the equation
  $$f(x) + f(x+1) - 3x - 2 = 0.$$

- **5217:** *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

  Find the value of:

$$\lim_{n\to\infty}\int_0^1\int_0^1 \sqrt[n]{(x^n+y^n)^k}\,dxdy,$$

  where k is a positive real number.

Solutions

- **5194:** *Proposed by Kenneth Korbin, New York, NY*

  Find two pairs of positive integers (a,b) such that,

$$\frac{14}{a}+\frac{a}{b}+\frac{b}{14}=41.$$

Solution 1 by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX

Multiplying $14/a+a/b+b/14=41$ by the LCM of the denominators, it follows that $14a^2+b(b-574)a+196b=0$.

To get positive integer solutions, $b-574<0$. Using MATLAB, we obtain the solutions $(252,567)$ and $(980,25)$. It is easily checked that these are solutions.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

If one intends to make the search amenable to a manual search then the search space needs to be narrowed down by exploiting divisibility properties of the numbers a and b. Equation (1) $\left(\frac{14}{a}+\frac{a}{b}+\frac{b}{14}=41\right)$ is equivalent to

$$14a^2+196b=ab(574-b). \qquad (2)$$

By (2), $14|ab^2$, which implies firstly that ($2|a$ or $2|b$) and secondly that ($7|a$ or $7|b$).

If $2|b$, then, again by (2), $4|14a^2$, which implies that $2|a$. So $2|a$.

If $7|b$, then, again by (2), $49|14a^2$, which implies that $7|a$. So $7|a$.

So a is a multiple of 14 and we write $a=14c$. (2) then reads as

$$196c^2+14b=bc(574-b). \qquad (3)$$

Let p be a prime different from 2 and 7. Let $p^\beta||b$, $p^\gamma||c$. $\Big(p^f||n$ means that $p^f|n$ and $p^{f+1}\not|n$ or in words: f is the exact exponent of p in the prime factorization of n.)

We claim that $\beta=2\gamma$.

If $p^\beta||b$, then by (3), $p^\beta|c^2$. So, $\gamma\geq\lceil\beta/2\rceil$.

If $p^\gamma || c$, then by (3), $p^\gamma | b$. Then, again by (3), $p^{2\gamma}|b$. So $\beta \geq 2\gamma$.

So $\gamma \geq \lceil \beta/2 \rceil \geq \gamma$ which indeed implies that $\beta = 2\gamma$.

So b and c are of the form $b = 2^r 7^s k^2, c = 2^u 7^v k$ (4),
where r, s, u, v are nonnegative integers $0 \leq r \leq 9, 0 \leq u \leq 8, 0 \leq s, v \leq 3$,
$k \in \{1, 3, 5, 9, 11, 13, 15, 17, 19, 23\}$, because $b < 573, c < 421$.

We plug (4) into (3) and get

$$14\left(2^{1+2u}7^{1+2v} + 2^r 7^s\right) = 2^{r+u}7^{s+v}k\left(574 - 2^r 7^s k^2\right). \qquad (5).$$

A manual check reveals that (5) can hold only for $k = 5$ and $k = 9$. They give rise to the two pairs $(b, c) = (25, 70)$ and $(b, c) = (567, 18)$ which in turn yield the two solutions $(a, b) \in \{(980, 25), (252, 567)\}$.

Yet another approach to solve (1) consists in solving (3) for b. We find

$$b = \frac{7\left(-1 + 41c \pm \sqrt{(1-41c)^2 - 4c^3}\right)}{c}.$$

Obviously $4c^3 \leq (41c - 1)^2 < (41c)^2$. So $c < 420$ (as above). The term under the root sign equals the square of an integer. We are left with a finite set of values of c for which we need to check this condition. We find that the only values of c are $c = 18$ and $c = 70$. They give rise to the solutions already mentioned.

Comment by editor: When Ken submitted this problem he accompanied it with the following explanation.

Let K be a factor of 14, and let $a = K^2 y$ and let $b = y^2$. Then

$$\frac{K}{a} + \frac{a}{b} + \frac{b}{K} = \frac{1 + K^3 + y^3}{Ky}$$

which is equal to an integer if K is a factor of $y^3 + 1$ and if y is a factor of $K^3 + 1$.

Also solved by Brian D. Beasley, Clinton, SC; Pat Costello, Richmond, KY; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Titu Zvonaru, Comănesti, Romania jointly with Neculai Stanciu, Buzău, Romania; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5195:** *Proposed by Kenneth Korbin, New York, NY*

If N is a prime number or a power of primes congruent to 1 (mod 6), then there are positive integers a and b such that $3a^2 + 3ab + b^2 = N$ with $(a, b) = 1$.

Find a and b if $N = 2011$, and if $N = 2011^2$, and if $N = 2011^3$.

Solution 1 by Kee-Wai Lau, Hong Kong, China

From $2a^2 + 3ab + b^2 = N$, we obtain $b = \dfrac{\sqrt{4N - 3a^2} - 3a}{2}$, so that $a < \sqrt{\dfrac{N}{3}}$.

A computer search yields the following results.

$$\text{For } N = 2011, \text{ we have } (a, b) = (10, 29)$$

$$\begin{aligned} \text{For } N &= 2011^2, \text{ we have } (a,b) = (880, 541) \\ \text{For } N &= 2011^3, \text{ we have } (a,b) = (46619, 10711) \end{aligned}$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

This problem is best put in the context of Eisenstein integers. Let $\omega = \dfrac{-1+i\sqrt{3}}{2}$. The set of Eisenstein integers $Z[\omega] = \{a + b\omega | a, b \in Z\}$ has the following properties:

- (i) $Z[\omega]$ forms a commutative ring of algebraic integers in the real number field $Q(\omega)$
- (ii) $Z[\omega]$ is an Euclidean domain whose norm N is given by $N(a+b\omega) = a^2 - ab + b^2$. As a result of this $Z[\omega]$ is a factorial ring.
- (iii) The group of units in $Z[\omega]$ is the cyclic group formed by the sixth root of unity in the complex plane. Specifically, they are $\{\pm 1, \pm\omega, \pm\omega^2\}$. These are just the Eisenstein integers of norm one.
- (iv) An ordinary prime number (or rational prime) which is 3 or congruent to 1 (mod 3) is of the form $x^2 - xy + y^2$ for some integers x, y and may therefore be factored into $(x + y\omega)(x + y\omega^2)$ and because of that it is not prime in the Eisenstein integers. Ordinary primes congruent to 2 (mod 3) cannot be factored in this way and they are primes in the Eisenstein integers as well.

So based on this, if p is a prime number congruent to 1 (mod 6) then p factors as $p = (c + d\omega)(c + d\omega^2)$ where $c + d\omega$ and $c + d\omega^2$ are two Eisenstein primes that are complex conjugates to each other. Of course $(c, d) = 1$, since $c + d\omega$ and $c + d\omega^2$ are both Eisenstein primes. By assumption $N = p^k$ for some natural number k. Then $N = p^k = (c + d\omega)^k(c + d\omega^2)^k$. Let $(c + d\omega)^k = e + f\omega$. We claim that e and f are coprime.

Assume that there is a prime q that divides both e and f. Then $q|(c + d\omega)^k|(c + d\omega)^k(c + d\omega^2)^k = p^k$. So $q = p$ and therefore $q = (c + d\omega)(c + d\omega^2)$. Then $(c + d\omega^2)|(c + d\omega)^{k-1}$ which implies firstly that $k > 1$, (since an Eisenstein prime cannot divide 1) and secondly that $(c + d\omega^2)|(c + d\omega)$, (since $(c + d\omega^2)$ is an Eisenstein prime). Because $|c + d\omega^2| = |c + d\omega|$ we conclude that there is a unit u such that $(c + d\omega^2) = u(c + d\omega)$. So $c, d \in \{0, \pm 1\}$ which cannot be, since $N(c + d\omega) = p \equiv 1 \pmod 6$.

So there is a factorization $N = p^k = (e + f\omega)(e + f\omega^2)$, where e and f are coprime integers. We claim that we can assume in addition that either (i) $0 < e < f < 2e$ or (ii) $0 < e < -f$.

Indeed, since

$$(+1)(+1) = (-1)(-1) = (+\omega)(+\omega^2) = (+\omega^2)(+\omega) = (-\omega)(-\omega^2) = (-\omega^2)(-\omega)$$

we conclude that

$$\begin{aligned} N(e + f\omega) &= N(-e - f\omega) = N(f + e\omega) = N(-f - e\omega) \\ &= N(f + (f - e)\omega) = N(-f + (e - f)\omega) = N(f - e + f\omega) = N(e - f - f\omega). \end{aligned}$$

So if we consider the eight Eisenstein integers

$$e+f\omega,\ -e-f\omega,\ f+e\omega,\ -f-e\omega,\ f+(f-e)\omega,\ -f+(e-f)\omega,\ f-e+f\omega,\ e-f-f\omega$$

there is one among these of the form $g+h\omega$ such that either $0<g<h<2g$ or $0<g<-h$, for if g and h have the same sign we can first assume that $g>0$ and $h>0$ (by replacing, if necessary g by $-g$ and h by $-h$). Next we can assume that $h>g$ (by replacing, if necessary g by h and h by g). Next we can assume that $h<2g$ (by replacing, if necessary, g by $h-g$). If g and h have different signs then we can first assume that $g>0,\ h<0$ (by replacing , if necessary, g by $-g$ and h by $-h$). Next we can assume that $g<-h$ (by replacing, if necessary g by $-h$ and h by $-g$).

In case (i) we define: $a:=f-e>0,\ b:=2e-f>0$. Then

$$\begin{aligned} N=p^k &= (e+f\omega)(e+f\omega^2)=e^2-ef+f^2=(a+b)^2-(a+b)(2a+b)+(2a+b)^2 \\ &= a^2+2ab+b^2-(2a^2+3ab+b^2)+4a^2+4ab+b^2=3a^2+3ab+b^2. \end{aligned}$$

In case (ii) we define: $a=e>0,\ b:=-e-f>0$. Then

$$\begin{aligned} N=p^k &= (e+f\omega)(e+f\omega^2)=e^2-ef+f^2=a^2+a(a+b)+(a+b)^2 \\ &= a^2+a^2+ab+a^2+2ab+b^2=3a^2+3ab+b^2. \end{aligned}$$

We find (upon using that $\omega^3=1,\omega^2+\omega+1=0$),

$$\begin{aligned} 2011 &= (10+49\omega)(10+49\omega^2), && (1) \\ 2011^2 &= (10+49\omega)^2(10+49\omega^2)^2=(2301+1421\omega)(2301+1421\omega^2), && (2) \\ 2011^3 &= (10+49\omega)^3(10+49\omega^2)^3=(46619-57330\omega)(46619-57330\omega^2). && (3) \end{aligned}$$

We note that $39+49\omega^2$ is an associate of $10+49\omega$ since $-(39+49\omega^2)=-(39-49-49\omega)=10+49\omega$.

So, $2011=(39+49\omega)(39+49\omega^2)$, and $0<39<49<78$. We are in case (i) and find $a=10,\ b=29$. Indeed, if we define $f(a,b)=3a^2+3ab+b^2$, then $f(10,29)=2011$.

We note that $2301+1421\omega^2$ is an associate of $1421+2301\omega$ since $\omega(2301+1421\omega^2)=1421+2301\omega$. So, $2011^2=(1421+2301\omega)(1421+2301\omega^2)$, and $0<1421<2301<2842$. We are in case (i) and find $a=880,\ b=541$. Indeed, $f(880,541)=2011^2$.

We note that $2011^3=(46619-57330\omega)(46619-57330\omega^2)$, and $0<46619<57330$. We are in case (ii) and find $a=46619,\ b=10711$. Indeed, $f(46619,10711)=2011^3$.

Also solved by Brian D. Beasley, Clinton, SC; Pat Costello, Richmond, KY; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5196:** *Proposed by Neculai Stanciu, Buzău, Romania*

Determine the last six digits of the product $(2010)\ (5^{2014})$.

Solution 1 by Robert Howard Anderson, Chesapeake, VA

To determine the last six digits of a product you must know the last six digits of each number you plan to multiply.

To do this the last six digits of 5^{2014} we need to look at the patterns of the solutions to lower powers.

All power of 5 end in 5, and all even powers of five end in 25, then all even powers greater than 2 end in 625. The 4th digit is either 5 or 0; the digit can be determined by using 2008 (mod 4) as 5.

The 5th digit is either 1,9,6, or 4; the digit can be determined by using 2008 (mod 8) as 1.

The 6th digit is either 3,7,1,5,8,2,6,or 0; the digit can be determined by using 2006 (mod 16) as 5.

The last six digits of 5^{2014} are 515625.

The last six digits of (2010)(515625) are 406250; so the last six digits of $(2010)5^{2014}$ are 406250.

Solution 2 by Ercole Suppa, Teramo, Italy

Clearly the last digit of $N = (2010)\left(5^{2014}\right)$ is 0. Therefore in order to find the last six digits of N it is enough to calculate the last five digits of $(201)\left(5^{2014}\right)$.

Let us first calculate a few powers of 5, and to do it we need to know just the last five digits of the previous power of 5:

$5^1 = 5$	$5^2 = 25$	$5^3 = 25$	$5^4 = 625$
$5^5 = 3125$	$5^6 = 15625$	$5^7 = 78125$	$5^8 = \cdots 90625$
$5^9 = \cdots 53125$	$5^{10} = \cdots 65625$	$5^{11} = \cdots 28125$	$5^{12} = \cdots 40625$
$5^{13} = \cdots 03125$	$5^{14} = \cdots 15625$		

Observe that the last five digits of 5^{14} are the same as those of 5^6. Therefore, starting with 5^6 the last five digits of powers of 5 will repeat periodically:

$$15625, 78125, 90625, 53125, 65625, 28125, 40625, 0325, 15625, \cdots.$$

This means that increasing the exponent of eight does not change the last five digits of powers of 5. Since $2014 = 6 + 8 \cdot 251$, it follows that 5^6 and 5^{2014} have the same last five digits, so

$$201 \cdot 5^{2014} \equiv 201 \cdot 5^6 \equiv 201 \cdot 15625 \equiv 40625 \pmod{10^5}$$

and this implies that the last six digits of $(2010)\left(5^{2014}\right)$ are 406250.

Solution 3 by Albert Stadler, Herrliberg, Switzerland

By Fermat's Little Theorem, $5^{\varphi(32)} = 5^{16} \equiv 1 (\text{mod } 32)$. So,

$$5^{2009} \equiv 5^{2009-16\cdot 125} \equiv 5^9 \equiv (-3)^3 \equiv 5 (\text{mod } 32),$$

which means that there is an integer k such that

$$5^{2009} - 5 = 32k.$$

We multiply this equation by $2010 \cdot 5^5$ and get

$$2010 \cdot 5^{2014} - 2010 \cdot 5^6 = 201 \cdot 10^6 k.$$

But

$$2010 \cdot 5^6 = 31406250.$$

So the last six digits are 406250.

Solution 4 by Kee-Wai Lau, Hong Kong, China

We show that $(2010)\left(5^{2014}\right) = \cdots 406250$.

It is easy to check that

$$(2010)\left(5^{2014}\right) = 406250 + \left(2^4\right)\left(5^6\right)\left(5^{2011}-1\right) + (2)\left(5^7\right)\left(5^{2008}-1\right).$$

Hence to prove our result, we need only show that $5^{2011}-1$ is a multiple of 4 and $5^{2008}-1$ is a multiple of 32.

In fact,

$$\begin{aligned} 5^{2011}-1 \equiv 1^{2011}-1 &\equiv 0 \pmod{4}, \text{ and} \\ 5^{2008}-1 = 390625^{251}-1 &\equiv 1^{251}-1 \equiv 0 \pmod{32}, \end{aligned}$$

and this completes the solution.

Also solved by Daniel Lopez Aguayo, UNAM Morelia, Mexico; Brian D. Beasley, Clinton, SC; Pat Costello, Richmond, KY; Bruno Salgueiro Fanego, Viveiro Spain; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5197:** *Proposed by Pedro H. O. Pantoja, UFRN, Brazil*

Let x, y, z be positive real numbers such that $x^2+y^2+z^2=4$. Prove that,

$$\frac{1}{6-x^2}+\frac{1}{6-y^2}+\frac{1}{6-z^2} \le \frac{1}{xyz}.$$

Solution 1 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

The inequality is evidently

$$\sum_{\text{cyc}} \frac{1}{2+x^2+y^2} \le \frac{1}{xyz}.$$

$a^2+1 \ge 2|a|$ yields

$$\sum_{\text{cyc}} \frac{1}{2+x^2+y^2} \le \sum_{\text{cyc}} \frac{1}{2x+2y} \le \frac{1}{xyz}$$

and $(\sqrt{x}-\sqrt{y})^2 \ge 0$ yields

$$\sum_{\text{cyc}} \frac{1}{2x+2y} \le \sum_{\text{cyc}} \frac{1}{4\sqrt{xy}} \le \frac{1}{xyz} \iff \sum_{\text{cyc}} z\sqrt{xy} \le 4$$

which is implied by

$$\sum_{\text{cyc}} z\frac{1}{2}(x+y) \leq 4 \iff xy+yz+zx \leq 4.$$

But this follows by the well known $xy+yz+zx \leq x^2+y^2+z^2$, thus concluding the proof.

Soluiton 2 by David E. Manes, Oneonta, NY

Let $L = \dfrac{1}{6-x^2} + \dfrac{1}{6-y^2} + \dfrac{1}{6-z^2}$. Since $x^2+y^2+z^2=4$, it follows that

$$6-x^2 = 2+y^2+z^2,\ 6-y^2 = 2+x^2+z^2,\ 6-z^2 = 2+x^2+y^2.$$

Therefore,

$$L = \frac{1}{6-x^2} + \frac{1}{6-y^2} + \frac{1}{6-z^2} = \frac{1}{2+y^2+z^2} + \frac{1}{2+x^2+z^2} + \frac{1}{2+x^2+y^2}.$$

Using the Arithmetic Mean-Geometric Mean Inequality twice, one obtains

$$\begin{aligned}
L &= \frac{1}{2+(y^2+z^2)} + \frac{1}{2+(x^2+z^2)} + \frac{1}{2+(x^2+y^2)} \\
&\leq \frac{1}{2+(2yz)} + \frac{1}{2+(2xz)} + \frac{1}{2+(2xy)} \\
&= \frac{1}{2}\left(\frac{1}{1+yz} + \frac{1}{1+xz} + \frac{1}{1+xy}\right) \\
&\leq \frac{1}{2}\left(\frac{1}{2\sqrt{yz}} + \frac{1}{2\sqrt{xz}} + \frac{1}{2\sqrt{xy}}\right) \\
&= \frac{1}{4}\left(\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{\sqrt{xyz}}\right).
\end{aligned}$$

As a result, to show that $L \leq \dfrac{1}{xyz}$ it suffices to show that

$$\begin{aligned}
\frac{1}{4}\left(\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{\sqrt{xyz}}\right) &\leq \frac{1}{xyz}, \text{ if and only if} \\
\frac{1}{4}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right) &\leq \frac{1}{\sqrt{xyz}}, \text{ if and only if} \\
\frac{1}{4}\left(x\sqrt{yz}+y\sqrt{xz}+z\sqrt{xy}\right) &\leq 1.
\end{aligned}$$

However, the Cauchy-Schwarz inequality, and the inequality $xy+yz+zx \leq x^2+y^2+z^2$ (*which also follows from the C-S inequality; editor's comment*) imply that

$$\frac{1}{4}\left(x\sqrt{yz}+y\sqrt{xz}+z\sqrt{xy}\right) \leq \frac{1}{4}\sqrt{x^2+y^2+z^2}\sqrt{yz+xz+xy}$$

$$\leq \quad \frac{1}{4}\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2+z^2} = 1.$$

Accordingly, if $x, y, z > 0$, and $x^2+y^2+z^2 = 4$, then

$$\frac{1}{6-x^2} + \frac{1}{6-y^2} + \frac{1}{6-z^2} \leq \frac{1}{xyz}.$$

Solution 3 by Arkady Alt, San Jose, CA

Let $a := \dfrac{x^2}{4}$ $b := \dfrac{y^2}{4}$, $c := \dfrac{z^2}{4}$ then inequality becomes

$$\frac{1}{6-4a} + \frac{1}{6-4b} + \frac{1}{6-4c} \leq \frac{1}{8\sqrt{abc}},$$

where $a+b+c = 1$.

Let $E = E(a,b,c) := \ \sqrt{abc}\sum\limits_{cyc}\dfrac{1}{3-2a}$, $p := ab+bc+ca, q := abc$.

Since $\sum\limits_{cyc}(3-2b)(3-2c) = \sum\limits_{cyc}(9-6(b+c)+4bc) = 15+4p$,
$(3-2a)(3-2b)(3-2c) = 9+12p-8q$ then $E = \dfrac{(15+4p)\sqrt{q}}{9+12p-8q}$.

Since $q \leq \dfrac{p^2}{3}^*$ and E is increasing in q then

$$\begin{aligned} \frac{E}{\sqrt{3}} &\leq \frac{(15+4p)\,p}{27+36p-8p^2} \\ &\leq \frac{\left(15+4\cdot\frac{1}{3}\right)\cdot\frac{1}{3}}{27+36\cdot\frac{1}{3}-8\cdot\frac{1}{9}} = \frac{1}{7} \end{aligned}$$

because $\dfrac{(15+4p)\,p}{27+36p-8p^2}$ is increasing in positive p and
$p \leq \dfrac{1}{3} \iff ab+bc+ca \leq \dfrac{(a+b+c)^2}{3}$.

Thus,

$$\begin{aligned} E \leq \frac{\sqrt{3}}{7} \quad &\iff \quad 4E \leq \frac{4\sqrt{3}}{7} \\ &\iff \quad 8\sqrt{abc}\sum_{cyc}\frac{1}{6-4a} \leq \frac{4\sqrt{3}}{7} \\ &\iff \quad xyz\sum_{cyc}\frac{1}{6-x^2} \leq \frac{4\sqrt{3}}{7} \end{aligned}$$

$$\iff \sum_{cyc} \frac{1}{6-x^2} \le \frac{4\sqrt{3}}{7xyz}.$$

$$\frac{1}{6-x^2}+\frac{1}{6-y^2}+\frac{1}{6-z^2} \quad \le \quad \frac{4\sqrt{3}}{7xyz}.$$

Remark: Since $\dfrac{4\sqrt{3}}{7} < 1$ we have $\dfrac{1}{6-x^2}+\dfrac{1}{6-y^2}+\dfrac{1}{6-z^2} \le \dfrac{4\sqrt{3}}{7xyz} < \dfrac{1}{xyz}$.

So, the inequality in the formulation of problem could have been stated with the stronger statement

$$\frac{1}{6-x^2}+\frac{1}{6-y^2}+\frac{1}{6-z^2} \quad \le \quad \frac{4\sqrt{3}}{7xyz}, \text{instead of with the weaker one of}$$

$$\frac{1}{6-x^2}+\frac{1}{6-y^2}+\frac{1}{6-z^2} \quad \le \quad \frac{1}{xyz}.$$

* *Editor's comment:* The inequality $q \le \dfrac{p^2}{3}$ is equivalent to $3abc(a+b+c) \le (ab+bc+ca)^2$ which is equivalent to $abc(a+b+c) \le a^2b^2+b^2c^2+c^2a^2$ which is implied by adding up $a^2bc \le 0.5a^2(b^2+c^2)$ and its cyclic variants.

Also solved by Kee-Wai Lau*, Hong Kong, China; Ecole Suppa, Teramo, Italy; Albert Stadler*, Herrliberg, Switzerland; Titu Zvonaru, Comănesti, Romania jointly with Neculai Stanciu, Buzău, Romania, and the proposer. (* Observed, specifically stated and proved the stricter inequality.)

- **5198:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let m, n be positive integers. Calculate,

$$\sum_{k=1}^{2n}\prod_{i=0}^{m}\left(\lfloor\frac{k+1}{2}\rfloor + a + i\right)^{-1},$$

where a is a nonnegative number and $\lfloor x\rfloor$ represents the greatest integer less than or equal to x.

Solution 1 by Arkady Alt, San Jose, CA

$$\sum_{k=1}^{2n}\prod_{i=0}^{m}\left(\left\lfloor\frac{k+1}{2}\right\rfloor + a + i\right)^{-1}$$

$$= \sum_{k=1}^{n}\prod_{i=0}^{m}\left(\left\lfloor\frac{2k-1+1}{2}\right\rfloor + a + i\right)^{-1} + \sum_{k=1}^{n}\prod_{i=0}^{m}\left(\left\lfloor\frac{2k+1}{2}\right\rfloor + a + i\right)^{-1}$$

$$= 2\sum_{k=1}^{n}\prod_{i=0}^{m}(k+a+i)^{-1}$$

$$
\begin{aligned}
&= 2\sum_{k=1}^{n} \frac{1}{(k+a)(k+1+a)\dots(k+m+a)} \\
&= \frac{2}{m}\sum_{k=1}^{n}\left(\frac{1}{(k+a)(k+1+a)\dots(k+m-1+a)} - \frac{1}{(k+1+a)(k+2+a)\dots(k+m+a)}\right) \\
&= \frac{2}{m}\left(\frac{1}{(1+a)(2+a)\dots(m+a)} - \frac{1}{(n+1+a)(n+2+a)\dots(n+m+a)}\right).
\end{aligned}
$$

Solution 2 by Anastasios Kotronis, Athens, Greece

By a direct calculation, using the identity $\Gamma(x+1) = x\Gamma(x), \quad x > 0$ for the Γ function, we can see that

$$\prod_{i=0}^{m} \frac{1}{b+i} = \frac{\Gamma(b)}{\Gamma(b+m+1)} = \frac{1}{m}\left(\frac{\Gamma(b)}{\Gamma(b+m)} - \frac{\Gamma(b+1)}{\Gamma(b+m+1)}\right) \qquad b > 0. \tag{1}$$

Now

$$
\begin{aligned}
&\sum_{k=1}^{2n}\prod_{i=0}^{m}\left(\left[\frac{k+1}{2}\right]+a+i\right)^{-1} \\
&= \sum_{k=1,3,\dots,2n-1}\prod_{i=0}^{m}\left(\frac{k+1}{2}+a+i\right)^{-1} + \sum_{k=2,4,\dots,2n}\prod_{i=0}^{m}\left(\frac{k}{2}+a+i\right)^{-1} \\
&= 2\sum_{k=1}^{n}\prod_{i=0}^{m}(k+a+i)^{-1} \\
&\overset{(1)}{=} \frac{2}{m}\sum_{k=1}^{n}\left(\frac{\Gamma(a+k)}{\Gamma(a+k+m)} - \frac{\Gamma(a+k+1)}{\Gamma(a+k+m+1)}\right) \\
&= \frac{2}{m}\left(\frac{\Gamma(a+1)}{\Gamma(a+1+m)} - \frac{\Gamma(a+n+1)}{\Gamma(a+n+m+1)}\right).
\end{aligned}
$$

Also solved by Albert Stadler, Herrliberg, Switzerland and the proposer.

- **5199:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $k > 0$ and $n \geq 0$ be real numbers. Calculate,

$$\int_0^1 x^n \ln\left(\sqrt{1+x^k} - \sqrt{1-x^k}\right) dx.$$

Solution by Anastasios Kotronis, Athens, Greece

$$
\begin{aligned}
I &= \left.\frac{x^{n+1}\ln\left(\sqrt{1+x^k}-\sqrt{1-x^k}\right)}{n+1}\right|_0^1 - \frac{k}{2(n+1)}\int_0^1 \frac{x^{n+k}\left(\frac{1}{\sqrt{1+x^k}}+\frac{1}{\sqrt{1-x^k}}\right)}{\sqrt{1+x^k}-\sqrt{1-x^k}}\,dx \\
&= \frac{\ln 2}{2(n+1)} - \frac{k}{2(n+1)}\int_0^1 \frac{x^{n+k}\left(\sqrt{1-x^k}+\sqrt{1+x^k}\right)}{\left(\sqrt{1+x^k}-\sqrt{1-x^k}\right)\sqrt{1-x^{2k}}}\,dx \\
&= \frac{\ln 2}{2(n+1)} - \frac{k}{4(n+1)}\int_0^1 \frac{x^{n}\left(\sqrt{1-x^k}+\sqrt{1+x^k}\right)^2}{\sqrt{1-x^{2k}}}\,dx \\
&= \frac{\ln 2}{2(n+1)} - \frac{k}{2(n+1)}\int_0^1 \left(\frac{x^n}{\sqrt{1-x^{2k}}}+x^n\right)dx \\
&= \frac{\ln 2}{2(n+1)} - \frac{k}{2(n+1)^2} - \frac{k}{2(n+1)}\int_0^1 \frac{x^n}{\sqrt{1-x^{2k}}}\,dx \\
&\overset{x^{2k}=u}{=\!=} \frac{\ln 2}{2(n+1)} - \frac{k}{2(n+1)^2} - \frac{1}{4(n+1)}B\left(\frac{n+1}{2k},\frac{1}{2}\right) \quad (B(u,v) \text{ denotes the Euler beta function}) \\
&= \frac{\ln 2}{2(n+1)} - \frac{k}{2(n+1)^2} - \frac{1}{4(n+1)}\frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2k}\}\right)}{\Gamma\left(\frac{n+k+1}{2k}\right)}.
\end{aligned}
$$

Also solved by Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.