Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before October 15, 2011

• 5164: Proposed by Kenneth Korbin, New York, NY
A triangle has integer length sides \((a, b, c)\) such that \(a - b = b - c\). Find the dimensions of the triangle if the inradius \(r = \sqrt{13}\).

• 5165: Proposed by Thomas Moore, Bridgewater, MA
“Dedicated to Dr. Thomas Koshy, friend, colleague and fellow Fibonacci enthusiast.”
Let \(\sigma(n)\) denote the sum of all the different divisors of the positive integer \(n\). Then \(n\) is perfect, deficient, or abundant according as \(\sigma(n) = 2n, \sigma(n) < 2n, \) or \(\sigma(n) > 2n\). For example, 1 and all primes are deficient; 6 is perfect, and 12 is abundant. Find infinitely many integers that are not the product of two deficient numbers.

• 5166: Proposed by José Luis Díaz-Barrero, Barcelona, Spain
Let \(a, b, c\) be lengths of the sides of a triangle \(ABC\). Prove that
\[
\left(3^a + \frac{a - b}{b} 3^b\right) \left(3^b + \frac{b - c}{c} 3^c\right) \left(3^c + \frac{c - a}{a} 3^a\right) \geq 8.
\]

• 5167: Paolo Perfetti, Department of Mathematics, University “Tor Vergata,” Rome, Italy
Find the maximum of the real valued function
\[
f(x, y) = x^4 - 2x^3 - 6x^2 y^2 + 6xy^2 + y^4
\]
defined on the set \(D = \{(x, y) \in \mathbb{R}^2 : x^2 + 3y^2 \leq 1\}\).

• 5168: Proposed by G. C. Greubel, Newport News, VA
Find the value of \(a_n\) in the series
\[
\frac{7t + 2t^2}{1 - 36t^2 + 4t^2} = a_0 + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots + \frac{a_n}{t^n} + \cdots.
\]

• 5169: Proposed by Ovidiu Furdui, Cluj, Romania
Let \(n \geq 1\) be an integer and let \(i\) be such that \(1 \leq i \leq n\). Calculate:
\[
\int_0^1 \cdots \int_0^1 \frac{x_i}{x_1 + x_2 + \cdots + x_n} dx_1 \cdots dx_n.
\]
Solutions

• 5146: Proposed by Kenneth Korbin, New York, NY

Find the maximum possible value of the perimeter of an integer sided triangle with in-radius \( r = \sqrt{13} \).

Solution 1 by Kee-Wai Lau, Hong Kong, China

Let the lengths of the sides of the triangle be \( a, b, \) and \( c \) with \( c \leq b \leq a \).

Let \( x = b + c - a, \) \( y = c + a - b, \) \( z = a + b - c \) so that \( x, y, z \) are integers and \( 0 < x \leq y \leq z \).

It is well known that \( \frac{1}{2} \sqrt{\frac{xyz}{x+y+z}} \) or \( \frac{xyz}{x+y+z} = 52 \).

From \( xyz < xy(x+y+z) \), we see that \( xy > 52 \) and from \( xy < \frac{3xyz}{x+y+z} \), we have \( xy \leq 156 \).

Since \( a = \frac{y+z}{2}, \) \( b = \frac{z+x}{2}, \) \( c = \frac{x+y}{2} \), so we have to find positive integers \( x, y \) satisfying

\[
\begin{align*}
1 \leq x & \leq 12 \\
52 < xy & \leq 156
\end{align*}
\]

such that \( z = \frac{52(x+y)}{xy-52} \) is a positive integer greater than or equal to \( y \) and that \( x, y, z \) are of the same parity. With the help of a computer we find that

\((x, y, z) = (2, 28, 390), (2, 30, 208), (2, 40, 78), (2, 52, 54), (4, 14, 234), (4, 26, 30), (6, 10, 104), (6, 16, 26)\)

are the only solutions. Since \( a + b + c = x + y + z \), so the maximum possible value of the perimeter of an integer sided triangle with in-radius \( r = \sqrt{13} \) is 420.

Solution 2 by Brian D. Beasley, Clinton, SC

We designate the integer side lengths of the triangle by \( a, b, \) and \( c \). We also let \( x = a + b - c, \) \( y = c + a - b, \) and \( z = b + c - a \) and note that \( x + y + z = a + b + c \). Then the formula for the in-radius \( r \) of a triangle becomes

\[
r = \frac{1}{2} \sqrt{\frac{(a+b-c)(c+a-b)(b+c-a)}{a+b+c}} = \frac{1}{2} \sqrt{\frac{xyz}{x+y+z}}.
\]

For the given triangle, we thus have \( 52(x+y+z) = xyz \). Then \( xyz \) is even; combined with the fact that \( x, y, \) and \( z \) have the same parity, this implies that all three are even. Writing \( x = 2u, y = 2v, \) and \( z = 2w \), we obtain \( 13(u+v+w) = uvw \). Then 13 divides \( uvw \), so without loss of generality, we assume \( w = 13k \) for some natural number \( k \). This produces \( v = (u+13k)/(uk - 1) \). Using this equation, a computer search reveals eight solutions for \((u, v, w) \) (with \( u \leq v \)) and hence for \((a, b, c) \):

\[
\begin{align*}
(u, v, w) &= (2, 15, 13) & \Rightarrow (a, b, c) &= (17, 15, 28) & \Rightarrow \text{perimeter} &= 60 \\
(u, v, w) &= (3, 8, 13) & \Rightarrow (a, b, c) &= (11, 16, 21) & \Rightarrow \text{perimeter} &= 48 \\
(u, v, w) &= (1, 27, 26) & \Rightarrow (a, b, c) &= (28, 27, 53) & \Rightarrow \text{perimeter} &= 108 \\
(u, v, w) &= (1, 20, 39) & \Rightarrow (a, b, c) &= (21, 40, 59) & \Rightarrow \text{perimeter} &= 120 \\
(u, v, w) &= (3, 5, 52) & \Rightarrow (a, b, c) &= (8, 55, 57) & \Rightarrow \text{perimeter} &= 120 \\
(u, v, w) &= (1, 15, 104) & \Rightarrow (a, b, c) &= (16, 105, 119) & \Rightarrow \text{perimeter} &= 240 \\
(u, v, w) &= (2, 7, 117) & \Rightarrow (a, b, c) &= (9, 119, 124) & \Rightarrow \text{perimeter} &= 252 \\
(u, v, w) &= (1, 14, 195) & \Rightarrow (a, b, c) &= (15, 196, 209) & \Rightarrow \text{perimeter} &= 420
\end{align*}
\]
Thus the maximum value of the perimeter is 420.

Also solved by Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

• 5147: Proposed by Kenneth Korbin, New York, NY

Let

\[
\begin{align*}
x &= 5N^2 + 14N + 23 \\
y &= 5(N + 1)^2 + 14(N + 1) + 23
\end{align*}
\]

where N is a positive integer. Find integers \(a_i\) such that

\[
a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6 = 0.
\]

Solution 1 by G. C. Greubel, Newport News, VA

The equations for \(x\) and \(y\) are given by \(x = 5n^2 + 14n + 23\) and \(y = 5n^2 + 24n + 42\). We are asked to find the values of \(a_i\) such that the equation

\[
a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6 = 0
\]

is valid. In order to do so we need to calculate the values of \(x^2\), \(y^2\), and \(xy\). For this we have

\[
\begin{align*}
x^2 &= 25n^4 + 140n^3 + 426n^2 + 644n + 529 \\
y^2 &= 25n^4 + 240n^3 + 996n^2 + 2016n + 1764 \\
xy &= 25n^4 + 190n^3 + 661n^2 + 1140n + 966.
\end{align*}
\]

Using the above results we then have the equation

\[
0 = 25(a_1 + a_2 + a_3)n^4 + 10(14a_1 + 24a_2 + 19a_3)n^3
\]

\[
\quad + (426a_1 + 996a_2 + 661a_3 + 5a_4 + 5a_5)n^2
\]

\[
\quad + 2(322a_1 + 1008a_2 + 570a_3 + 7a_4 + 12a_5)n
\]

\[
\quad + (529a_1 + 1764a_2 + 966a_3 + 23a_4 + 42a_5 + a_6).
\]

From this we have five equations for the coefficients \(a_i\) given by

\[
\begin{align*}
0 &= a_1 + a_2 + a_3 \\
0 &= 14a_1 + 24a_2 + 19a_3 \\
0 &= 426a_1 + 996a_2 + 661a_3 + 5a_4 + 5a_5 \\
0 &= 322a_1 + 1008a_2 + 570a_3 + 7a_4 + 12a_5 \\
0 &= 529a_1 + 1764a_2 + 966a_3 + 23a_4 + 42a_5 + a_6.
\end{align*}
\]

From \(0 = 14a_1 + 24a_2 + 19a_3\) we have \(0 = 14(a_1 + a_2 + a_3) + 10a_2 + 5a_3 = 5(2a_2 + a_3)\), where we used the fact that \(0 = a_1 + a_2 + a_3\). This yields \(a_3 = -2a_2\). Using this result in \(0 = a_1 + a_2 + a_3\) yields \(a_2 = a_1\). The three remaining equations can be reduced to

\[
\begin{align*}
0 &= 20a_1 + a_4 + a_5 \\
0 &= 190a_1 + 7a_4 + 12a_5 \\
0 &= 361a_1 + 23a_4 + 42a_5 + a_6.
\end{align*}
\]
Solving this system we see that

\[ a_1 = a_1, \ a_2 = a_1, \ a_3 = -2a_1, \ a_4 = -10a_1, \ a_5 = -10a_1, \ a_6 = 289a_1. \]

We now verify that the above coefficients work.

\[ a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6 = 0, \]

becomes

\[ a_1 \left( x^2 + y^2 - 2xy - 10x - 10y + 289 \right) = 0, \]

and since \( a_1 \neq 0 \)

\[ x^2 + y^2 - 2xy - 10x - 10y + 289 = 0, \]

and

\[ (x - y)^2 - 10(x + y) + 289 = 0. \]

From the values of \( x \) and \( y \) presented to us in terms of \( n \) at the start of the problem, we see that \( x - y = -(10n + 19) \) and \( x + y = 10n^2 + 38n + 65 \).

Substituting these values into the above equations we obtain:

\[ 0 = (x - y)^2 - 10(x + y) + 289 \]
\[ = (10n + 19)^2 - 10(10n^2 + 38n + 65) + 289 \]
\[ = (100n^2 + 380n + 361) - (100n^2 + 380n + 650) + 289 \]
\[ = 361 - 650 + 289 \]
\[ = 0. \]

We have thus verified that for the coefficients we have obtained, and for the values of \( x \) and \( y \) that are given, \( a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6 = 0 \).

**Solution 2 by Kee-Wai Lau, Hong Kong, China**

By putting \( N = 1, 2, 3, 4, 5 \), we obtain the system of equations

\[
\begin{align*}
1764a_1 + 5041a_2 + 2982a_3 + 42a_4 + 71a_5 + a_6 &= 0 \\
5041a_1 + 12100a_2 + 7810a_3 + 71a_4 + 110a_5 + a_6 &= 0 \\
12100a_1 + 25281a_2 + 17490a_3 + 110a_4 + 159a_5 + a_6 &= 0 \\
25281a_1 + 47524a_2 + 34662a_3 + 159a_4 + 218a_5 + a_6 &= 0 \\
47524a_1 + 82369a_2 + 62566a_3 + 218a_4 + 287a_5 + a_6 &= 0.
\end{align*}
\]

If \( a_1 = 0 \), then (1) reduces to

\[
\begin{align*}
5041a_2 + 2982a_3 + 42a_4 + 71a_5 + a_6 &= 0 \\
12100a_2 + 7810a_3 + 71a_4 + 110a_5 + a_6 &= 0 \\
25281a_2 + 17490a_3 + 110a_4 + 159a_5 + a_6 &= 0 \\
47524a_2 + 34662a_3 + 159a_4 + 218a_5 + a_6 &= 0 \\
82369a_2 + 62566a_3 + 218a_4 + 287a_5 + a_6 &= 0.
\end{align*}
\]

If \( a_1 = 0 \), then (1) reduces to

\[
\begin{align*}
5041a_2 + 2982a_3 + 42a_4 + 71a_5 + a_6 &= 0 \\
12100a_2 + 7810a_3 + 71a_4 + 110a_5 + a_6 &= 0 \\
25281a_2 + 17490a_3 + 110a_4 + 159a_5 + a_6 &= 0 \\
47524a_2 + 34662a_3 + 159a_4 + 218a_5 + a_6 &= 0 \\
82369a_2 + 62566a_3 + 218a_4 + 287a_5 + a_6 &= 0.
\end{align*}
\]

Since the determinant

\[
\begin{vmatrix}
5041 & 2982 & 42 & 71 & 1 \\
12100 & 7810 & 71 & 110 & 1 \\
25281 & 17490 & 110 & 159 & 1 \\
47524 & 34662 & 159 & 218 & 1 \\
82369 & 62566 & 218 & 287 & 1
\end{vmatrix} = -18000000, \]

so (2) has the unique solution \( a_2 = a_3 = a_4 = a_5 = a_6 = 0 \).
If \(a_1 \neq 0\), we write \(a_2 = a_1b_2, a_3 = a_1b_3, a_4 = a_1b_4, a_5 = a_1b_5, a_6 = a_1b_6\), so that (1) becomes

\[
\begin{align*}
1764 + 5041b_2 + 2982b_3 + 42b_4 + 71b_5 + b_6 &= 0 \\
5041 + 12100b_2 + 7810b_3 + 159b_4 + 110b_5 + b_6 &= 0 \\
12100 + 25281b_2 + 17490b_3 + 159b_4 + 218b_5 + b_6 &= 0 \\
25281 + 47524b_2 + 34662b_3 + 159b_4 + 218b_5 + b_6 &= 0 \\
47524 + 82369b_2 + 62566b_3 + 159b_4 + 287b_5 + b_6 &= 0.
\end{align*}
\]  

(3)

By Cramer’s rule, we find the unique solution of (3) to be

\[b_2 = 1, \quad b_3 = -2, \quad b_4 = -10, \quad b_5 = -10, \quad b_6 = 289.\]

It follows that the general solution to (1) is

\[a_1 = k, \quad a_2 = k, \quad a_3 = -2k, \quad a_4 = -10k, \quad a_5 = -10k, \quad a_6 = 289k,\]

(4)

where \(k\) is any integer. It can be checked readily by direct expansion that

\[kx^2 + ky^2 - 2kxy - 10kx - 10ky + 289k = 0\]

for any positive integer \(N\), and so the general solution to the equation of the problem is given by (4).

**Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain**

As in the published solutions to SSMJ problem 5144, we first compute

\[y - x = 5 \left[ (N + 1)^2 - N^2 \right] + 14 (N + 1 - N) + 23 - 23 = 5 (2N + 1) + 14 = 10N + 19\]

(1)

From \(x = 5N^2 + 14N + 23\) that is \(5N^2 + 14N + 23 - x = 0\), one obtains

\[N_{1,2} = \frac{-14 \pm \sqrt{14^2 - 20(23 - x)}}{10} = \frac{-7 \pm \sqrt{5x - 66}}{5}\]

and since \(N\) is a positive integer, we choose \(N = \frac{-7 + \sqrt{5x - 66}}{5}\)  

(2).

Substituting (2) into (1) gives

\[y - x = 2 \left( -7 + \sqrt{5x - 66} \right) + 19 = 5 + 2\sqrt{5x - 66}.\]

(3)

From (3) one obtains

\[(y - x - 5)^2 = \left( 2\sqrt{5x - 66} \right)^2, \text{ that is}\]

\[x^2 + y^2 - 2xy - 10x - 10y + 289 = 0\]

(4)

Relation (4) shows that it suffices to take the following integers for \(a_i\)

\[a_1 = a_2 = 1; \quad a_3 = -2; \quad a_4 = a_5 = -10; \quad a_6 = 289\]

**Comment:** Relation (4) shows that for any positive integer \(N\), all of the points with coordinates \((x, y) = (u_N, u_{N+1})\) for \(u_N = 5N^2 + 14N + 23\), are points situated on the parabola (*) with equation

\[
\begin{pmatrix}
X \\ Y
\end{pmatrix}
\begin{pmatrix}
289 & -5 & -5 \\
-5 & 1 & -1 \\
-5 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\ X \\ Y
\end{pmatrix} = 0.
\]
Because \( \det \begin{pmatrix} 289 & -5 & -5 \\ -5 & 1 & -1 \\ -5 & -1 & 1 \end{pmatrix} = 100 \neq 0 \) and \( \det \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 0 \).

**Solution 4 by David Stone and John Hawkins, Statesboro, GA**

We will show that the proscribed points \((x, y)\) lie on the conic

\[
x^2 + y^2 - 2xy - 10xy - 10y + 289 = 0.
\]

This is a parabola. In fact, it is the parabola \( x = \frac{1}{2\sqrt{2}} y^2 \) rotated counterclockwise \( \frac{\pi}{4} \) and translated “up the diagonal \( y = x \)” by a distance \( \frac{289}{20}\sqrt{2} \), having its vertex at \( \left( \frac{289}{20}, \frac{289}{20} \right) \).

We will actually consider the more general problem

\[
\begin{aligned}
x &= aN^2 + bN + c \\
y &= a(N + 1)^2 + b(N + 1) + c
\end{aligned}
\]

with the restrictions on \( N \) removed.

Treating these as parametric equations, we can eliminate the parameter \( N \) (without getting bogged down in the quadratic formula).

Expanding the expression for \( y \) gives

\[
y = aN^2 + 2aN + a + bN + b + c = \left(aN^2 + bN + c\right) + 2aN + a + b = x + 2aN + a + b.
\]

Solving for \( N \) gives \( N = \frac{y - x - (a + b)}{2a} \).

Substituting back into the expression for \( x \):

\[
x = a\left(\frac{y - x - a - b}{2a}\right)^2 + b\left(\frac{y - x - a - b}{2a}\right) + c,
\]

which simplifies to

\[
\begin{aligned}
(1) \quad x^2 + y^2 - 2xy - 2ax - 2ay + \left(a^2 - b^2 + 4ac\right) &= 0.
\end{aligned}
\]

This is our solution for the general problem. So we do indeed have a quadratic equation for our figure; the discriminate equals zero.

From calculus, we know that a \( 45^\circ \) rotation will remove the \( xy \) term. The transformation equations are

\[
x = \frac{1}{\sqrt{2}} (x' - y') \quad \text{and} \quad y = \frac{1}{\sqrt{2}} (x' + y')
\]

Substituting these into Equation (1), we get

\[
\frac{(x' - y')^2}{2} + \frac{(x' + y')^2}{2} - 2\frac{(x' - y')(x' + y')}{2} - 2a\frac{(x' - y')}{\sqrt{2}} - 2a\frac{(x' + y')}{\sqrt{2}} + \left(a^2 - b^2 + 4ac\right) = 0.
\]
This simplifies to
\[ 2 (y')^2 - 4a \frac{\sqrt{2}}{\sqrt{2}} x' + \left( a^2 - b^2 + 4ac \right) = 0. \]

This becomes more familiar as
\[ x' - \frac{a^2 - b^2 + 4ac}{2a\sqrt{2}} = \frac{1}{a\sqrt{2}} (y')^2. \]

We recognize a nice parabola in the \( x', y' \) plane. In fact, if we translate to the new origin, \( \left( \frac{a^2 - b^2 + 4ac}{2a\sqrt{2}}, 0 \right) \) (in the \( x', y' \) plane) and let
\[ x'' = x' - \frac{a^2 - b^2 + 4ac}{2a\sqrt{2}} \quad \text{and} \quad y'' = y' \]
our equation becomes
\[ x'' = \frac{1}{a\sqrt{2}} (y'')^2. \]

Substituting the values \( a = 5, b = 14, c = 23 \) produces the solution to the given problem.

Comment 1: We see that \( x \) and \( y \) are interchangeable in Equation (1), reflecting the fact that the line \( y = x \) is the axis of symmetry of our parabola. Therefore, more lattice points than originally mandated fall on the parabola.

For convenience, let \( u_n = aN^2 + bN + c \). By the given condition, for any integer \( N \), the point \( (u_N, u_{N+1}) \) lies on the parabola. By symmetry, \( (u_{N+1}, u_N) \) also lies on the parabola.

Comment 2: We see that this sequence satisfies the first order non-linear recurrence:
\[ u_{N+1} = u_N + (2N + 1) a + b. \]
We have shown that the points \( (u_N, u_{N+1}) \), \( N \in \mathbb{Z} \), lie on the parabola given by Equation (1) (as do the points \( (u_{N+1}, u_N) \)). This is reminiscent of the result that pairs of Fibonacci numbers \( (F_N, F_{N+1}) \) lie on the hyperbolas \( y^2 - xy - x^2 = \pm 1 \). In fact, such pairs are the only lattice points on these hyperbolas.

So we wonder if the points \( (u_N, u_{N+1}) \) and \( (u_{N+1}, u_N) \) are the only lattice points on the parabola given by Equation (1).

Also solved by Brian D. Beasley, Clinton, SC; Edwin Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; Raúl A. Simón, Santiago, Chile, and the proposer.

• 5148: Proposed by Pedro Pantoja (student, UFRN), Natal, Brazil

Let \( a, b, c \) be positive real numbers such that \( ab + bc + ac = 1 \). Prove that
\[ \frac{a^2}{\sqrt{b(b+2c)}} + \frac{b^2}{\sqrt{c(c+2a)}} + \frac{c^2}{\sqrt{a(a+2b)}} \geq 1. \]

Solution 1 by David E. Manes, Oneonta, NY

Let \( L = \frac{a^2}{\sqrt{b(b+2c)}} + \frac{b^2}{\sqrt{c(c+2a)}} + \frac{c^2}{\sqrt{a(a+2b)}} \). To prove that \( L \geq 1 \), we will use
Jensen’s inequality that states if \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are positive numbers satisfying
\[
\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1,
\]
and \( x_1, x_2, \ldots, x_n \) are any \( n \) points in an interval where \( f \) is continuous and convex, then
\[
\lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots + \lambda_n f(x_n) \geq f\left(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n\right).
\]
The function \( f(x) = \frac{1}{\sqrt[3]{x}} \) is continuous and convex on the interval \((0, \infty)\). Let
\[
\alpha = a^2 + b^2 + c^2 \\
\lambda_1 = \frac{a^2}{\alpha} \\
\lambda_2 = \frac{b^2}{\alpha} \\
\lambda_3 = \frac{c^2}{\alpha} \\
x_1 = b^2 + 2bc \\
x_2 = c^2 + 2ac \\
x_3 = a^2 + 2ab
\]
Then \( \lambda_1 + \lambda_2 + \lambda_3 = 1 \) and Jensen’s inequality implies
\[
\frac{1}{\alpha} L = \frac{a^2}{\alpha} f\left(b^2 + 2bc\right) + \frac{b^2}{\alpha} f\left(c^2 + 2ac\right) + \frac{c^2}{\alpha} f\left(a^2 + 2ab\right)
\geq f\left(\frac{a^2 (b^2 + 2bc) + b^2 (c^2 + 2ac) + c^2 (a^2 + 2ab)}{\alpha}\right)
= \frac{\alpha}{\sqrt[3]{a^2b^2 + b^2c^2 + c^2a^2 + 2a^2bc + 2ab^2c + 2abc^2}}
= \frac{\alpha}{\sqrt[3]{(ab + bc + ac)^2}} = \sqrt[3]{\alpha}.
\]
Hence, \( L \geq \alpha^{4/3} = (a^2 + b^2 + c^2)^{4/3} \geq 1 \) since the inequality
\((a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0\) with the constraint \( ab + bc + ac = 1 \) implies
\(a^2 + b^2 + c^2 \geq 1\). Note that equality occurs if and only if \( a = b = c = \frac{1}{\sqrt[3]{3}}\).

**Solution 2** by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhii, Kriftel, Germany

Using Cauchy-Schwarz inequality we have,
\[
\left(\frac{\sqrt[3]{b(b + 2c)}}{\sqrt[3]{b(b + 2c)}} + \frac{\sqrt[3]{c(c + 2a)}}{\sqrt[3]{b(b + 2c)}} + \frac{\sqrt[3]{a(a + 2b)}}{\sqrt[3]{b(b + 2c)}}\right) \left(\frac{a^2}{\sqrt[3]{b(b + 2c)}} + \frac{b^2}{\sqrt[3]{c(c + 2a)}} + \frac{c^2}{\sqrt[3]{a(a + 2b)}}\right) \geq (a + b + c)^2,
\]
which implies that,
\[
\frac{a^2}{\sqrt[3]{b(b + 2c)}} + \frac{b^2}{\sqrt[3]{c(c + 2a)}} + \frac{c^2}{\sqrt[3]{a(a + 2b)}} \geq \frac{(a + b + c)^2}{\sqrt[3]{b(b + 2c)}} + \frac{(a + b + c)^2}{\sqrt[3]{c(c + 2a)}} + \frac{(a + b + c)^2}{\sqrt[3]{a(a + 2b)}}.
\]
Using the fact that the function \( f(x) = \sqrt[3]{x} \) is a concave function, since the second derivative is negative, we have that any three numbers \( x, y, z \), according to Jensen’s
inequality, satisfy the inequality $f(x) + f(y) + f(z) \leq 3f\left(\frac{x + y + z}{3}\right)$ and applying this we have

$$\frac{a^2}{\sqrt[3]{b(b + 2c)}} + \frac{b^2}{\sqrt[3]{c(c + 2a)}} + \frac{c^2}{\sqrt[3]{a(a + 2b)}} \geq \frac{(a + b + c)^2}{\sqrt[3]{b(b + 2c)} + \sqrt[3]{c(c + 2a)} + \sqrt[3]{a(a + 2b)}}$$

$$\geq \frac{(a + b + c)^2}{3 \sqrt[3]{\left(\frac{b(b + 2c) + c(c + 2a) + a(a + 2b)}{3}\right)}}$$

$$= \frac{(a + b + c)^2}{3 \sqrt[3]{(a + b + c)^2}}$$

So it is enough to prove that

$$\frac{(a + b + c)^2}{3 \sqrt[3]{(a + b + c)^2}} \geq 1,$$

which implies

$$\frac{(a + b + c)^2}{3} \geq \frac{3}{\sqrt[3]{3}}$$

$$\frac{(a + b + c)^2}{3 \sqrt[3]{(a + b + c)^2}} \geq \frac{3}{\sqrt[3]{3}}$$

$$\frac{(a + b + c)^2}{3 \sqrt[3]{(a + b + c)^2}} \geq \frac{3}{\sqrt[3]{3}}$$

Using the given condition and the AM-GM inequality we have

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$$

$$\geq 3ab + 3bc + 3ac$$

$$= 3(ab + bc + ac)$$

$$= 3$$

and this is the end of the proof.

**Solution 3 by Andrea Fanchini, Cantù, Italy**

Recall Holder’s inequality that states that if $a_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$ are positive real numbers, then:

$$\prod_{i=1}^{m} \left( \sum_{j=1}^{n} a_{ij} \right) \geq \left( \sum_{j=1}^{n} \prod_{i=1}^{m} a_{ij} \right)^{m}.$$

Setting $n = 3$ and $m = 4$ and using this inequality we have

$$\left( \sum_{cyc} \frac{a^2}{\sqrt[3]{b^2 + 2bc}} \right) \left( \sum_{cyc} \frac{a^2}{\sqrt[3]{b^2 + 2bc}} \right) \left( \sum_{cyc} \frac{a^2}{\sqrt[3]{b^2 + 2bc}} \right) \left( \sum_{cyc} a^2 \left( b^2 + 2bc \right) \right) \geq \left( a^2 + b^2 + c^2 \right)^4,$$
and being that \(a^2 + b^2 + c^2 \geq ab + bc + ca\),

\[
\left(\sum_{\text{cyc}} \frac{a^2}{\sqrt{b^2 + 2bc}}\right) \left(\sum_{\text{cyc}} \frac{a^2}{\sqrt{b^2 + 2bc}}\right) \left(\sum_{\text{cyc}} \frac{a^2}{\sqrt{b^2 + 2bc}}\right) \left(\sum_{\text{cyc}} \frac{a^2(b^2 + 2bc)}{2bc}\right) \geq (ab + bc + ca)^4 = 1
\]
because

\[
\left(\sum_{\text{cyc}} a^2(b^2 + 2bc)\right) = (ab + bc + ca)^2 = 1,
\]
and so the proposed inequality holds.

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, University “Tor Vergata,” Rome, Italy, and the proposer.

• 5149: Proposed by José Luis Díaz-Barrero, Barcelona, Spain

A regular \(n\)-gon \(A_1, A_2, \cdots, A_n\) \((n \geq 3)\) has center \(F\), the focus of the parabola \(y^2 = 2px\), and no one of its vertices lies on the \(x\) axis. The rays \(FA_1, FA_2, \cdots, FA_n\) cut the parabola at points \(B_1, B_2, \cdots, B_n\).

Prove that

\[
\frac{1}{n} \sum_{k=1}^{n} FB_k^2 > p^2.
\]

Solution by Ángel Plaza (University of Las Palmas de Gran Canaria) and Javier Sánchez-Reyes (University of Castilla-La Mancha), Spain

In polar coordinates \((r, \theta)\) centered at the focus the parabola is given by \(r = p/(1 + \cos \theta)\). Defining the arguments \(\theta_k = \theta_n + 2k\pi/n\) for \(k = 1, 2, \ldots, n\) corresponding to the vertices \(A_k\) of the polygon, we have to prove that

\[
\frac{1}{n} \sum_{k=1}^{n} \frac{p^2}{(1 + \cos \theta_k)^2} > p^2,
\]

\[
\frac{1}{n} \sum_{k=1}^{n} \frac{1}{(1 + \cos \theta_k)^2} > 1,
\]

where \(\theta_k \neq 0\) and \(\theta_k \neq \pi\). Since the function \(f(x) = 1/x^2\) is strictly convex and \(\sum_{k=1}^{n} \cos \theta_k = 0\), for example because the sum of all the \(n\)th complex roots of unity is zero, it follows that

\[
\frac{1}{n} \sum_{k=1}^{n} \frac{1}{(1 + \cos \theta_k)^2} > \left(\frac{\sum_{k=1}^{n} \cos \theta_k}{n}\right)^{-2} = 1.
\]

Also solved by Raúl A. Simón, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA and the proposer.
Let \( \{A_n\}_{n=1}^{\infty} \) be a sequence of matrices such that \( \det(A_n) \neq 0,1 \) for all \( n \in \mathbb{N} \). Calculate:

\[
\lim_{n \to \infty} \frac{n^n \ln(|\det(A_n)|)}{\ln(|\det(\text{adj}^{n^m}(A_n))|)}
\]

where \( \text{adj}^{n^m} \) refers to \( \text{adj} \circ \text{adj} \circ \cdots \circ \text{adj}, n \) times, the \( n \)th iterate of the classical adjoint.

**Solved 1 by the proposer**

A simple calculation of \( \text{adj}^{n^m}(A) \), \( m = 1, 2, \cdots, 5 \) using equalities:

\[
\begin{align*}
(i) & \quad \text{adj}(A) \cdot A = A \cdot \text{adj}(A) = \det(A) \cdot I_{n \times n}. \\
(ii) & \quad \det(A^{-1}) = (\det(A))^{-1} \\
(iii) & \quad \det(kA) = k^n \det(A)
\end{align*}
\]

suggests the following conjecture:

\[
\text{adj}^{n^{m+1}}(A) = \det(A)^{P_m(n)} A^{(-1)^m}; \quad P_m(n) = \frac{(n-1)^m + (-1)^{m-1}}{n}, \quad m, n \in \mathbb{N} \quad (**) \]

We prove the conjecture by induction on the positive integer \( m \). The assertion trivially holds for the case \( m = 1 \). Let it hold for some positive integer \( m > 1 \). Then

\[
\begin{align*}
\text{adj}^{n^{m+1}}(A) &= \text{adj}(\text{adj}^{n^m}(A)) \\
&= \det(\text{adj}^{n^m}(A)) (\text{adj}^{n^m}(A))^{-1} \\
&= \det(\det(A)^{P_m(n)} A^{(-1)^m}) (\det(A)^{P_m(n)} A^{(-1)^m})^{-1} \\
&= \det(A)^{(n-1)P_m(n) + (-1)^m} (A)^{(-1)^{m+1}}.
\end{align*}
\]

Besides,

\[
P_{m+1}(n) = (n-1)P_m(n) + (-1)^m = \frac{(n-1)^{m+1} + (-1)^m}{n},
\]

proving the assertion for positive integer \( m + 1 \). Accordingly, using \( (**) \) we have

\[
\lim_{n \to \infty} \frac{n^n \ln(|\det(A_n)|)}{\ln(|\det(\text{adj}^{n^m}(A_n))|)} = \lim_{n \to \infty} \frac{n^n \ln(|\det(A_n)|)}{\ln(|\det(\det(A_n)^{P_m(n)} A^{(-1)^m})|)}
\]

\[
= \lim_{n \to \infty} \frac{n^n \ln(|\det(A_n)|)}{\ln(|(\det(A_n)^{nP_m(n)}) \det(A_n^{(-1)^n})|)}
\]

\[
= \lim_{n \to \infty} \frac{n^n \ln(|\det(A_n)|)}{\ln(|(\det(A_n)^{nP_m(n)} + (-1)^n)|)}
\]
\[
\lim_{n \to \infty} \frac{n^n}{nP_n(n) + (-1)^n} = \\
= \lim_{n \to \infty} \left( \frac{n}{n - 1} \right)^n = e.
\]

**Solution 2 by David Stone and John Hawkins, Statesboro, GA**

We shall find a formula for \( adj^n(nA) \) and then show the limit is \( e \).

First recall some properties of the inverse and the classical adjoint, where \( A \) is \( n \times n \) and invertible and \( c \) a non-zero scalar.

1. \( adj(A) = \text{det}(A)A^{-1} \)
2. \( adj(A)^{-1} = \frac{1}{\text{det}(A)}A = \text{adj}(A^{-1}) \)
3. \( \text{det}[adj(A)] = [\text{det}(A)]^{n-1} \)
4. \( \text{det}(cA) = c^n \text{det}(A) \)
5. \( (cA)^{-1} = \frac{1}{c}A^{-1} \)
6. \( adj(cA) = c^{n-1}adj(A) \)

Then we see

7. \( adj^2(A) = adj[adj(A)] = \text{det}[adj(A)][adj(A)]^{-1} \) by (1)
   \[ = [\text{det}(A)]^{n-1} \frac{1}{\text{det}(A)}A \text{ by (3) and (2)} \]
   \[ = [\text{det}(A)]^{n-2}A. \]

Continuing with our calculations, we have

8. \( adj^3(A) = adj[adj^2(A)] \) \[ = adj[\{[\text{det}(A)]^{n-2}A \}] \text{ by (7)} \]
   \[ = \left\{ [\text{det}(A)]^{n-2} \right\}^{n-1} adj(A) \text{ by (6)} \]
   \[ = [\text{det}(A)]^{(n-1)(n-2)} \text{det}(A)A^{-1} \text{ by (1)} \]
   \[ = [\text{det}(A)]^{n^2-3n+3}A^{-1} \]
We observe that repeated applications of $\text{adj}$ will produce terms of the form $[\det(A)]^{p_k(n)} A^{(-1)^k}$, where $p_i(n)$ is a polynomial of degree $k-1$ in $n$.

Specifically, for $k = 1, 2, 3, \ldots, n-1$, we have

$$
\text{(9) } \text{adj}^{(k+1)}(A) = \text{adj} \left[ \text{adj}^k(A) \right]
$$

$$
= \text{adj} \left[ [\det(A)]^{p_k(n)} A^{(-1)^k} \right] \text{ by induction}
$$

$$
= \left\{ [\det(A)]^{p_k(n)} \right\}^{n-1} \text{adj} \left( A^{(-1)^k} \right) \text{ by (6)}
$$

$$
= [\det(A)]^{(n-1)p_k(n)} \text{det} \left( A^{(-1)^k} \right) A^{(-1)^k(n-1)} \text{ by (1)}
$$

$$
= [\det(A)]^{(n-1)p_k(n)+(n-1)^k} A^{(-1)^{k+1}}
$$

Therefore, we can recursively compute the polynomials which give the exponent on $\det(A)$ and obtain a concrete formula for $\text{adj}(A)$:

$$
\text{adj}^{k+1}(A) = [\det(A)]^{(n-1)p_k(n)} A^{\left( (-1)^k \right) A^{(n-1)p_k(n)+(n-1)^k}}
$$

By (1) $\text{adj}(A) = \det(A) A^{-1}$, so $p_1(n) = 1$.

By (7) $\text{adj}^2(A) = [\det(A)]^{n-2} A$, so $p_2(n) = n - 2$.

Then $p_3(n) = (n-1)p_2(n) + (-1)^2 = (n-1)(n-2) + 1 = n^2 - 3n + 3$, agreeing with (8).

Continuing, we find that

$p_4(n) = n^3 - 4n^2 + 6n - 4$ and

$p_5(n) = n^4 - 5n^3 + 10n^2 - 10n + 5$.

The appearance of the binomial coefficients is unmistakable. We deduce that, for $k = 1, 2, 3, \ldots, n$,

$p_k(n) = \frac{(n-1)^k + (-1)^{k-1}}{n}$, a polynomial of degree $k-1$.

The capstone of this sequence of polynomials: $p_n(n) = \frac{(n-1)^n + (-1)^n}{n}$, allows us to calculate $\text{adj}^{\infty}(A)$ as:

$$
\text{(10) } \text{adj}^{\infty}(A) = [\det(A)]^{\frac{(n-1)^n + (-1)^{n-1}}{n}} A^{(-1)^n}
$$

Therefore, $A_n \in M_{n \times n}(\mathbb{C})$,

$$
\det \left( \text{adj}^{\infty}(A_n) \right) = \det \left\{ [\det(A)]^{\frac{(n-1)^n + (-1)^{n-1}}{n}} A^{(-1)^n} \right\} \text{ by (10)}
$$

$$
= \left( [\det(A)]^{\frac{(n-1)^n + (-1)^{n-1}}{n}} \right)^n \det \left[ A^{(-1)^n} \right] \text{ by (4)}
$$

$$
= [\det(A)]^{(n-1)^n + (-1)^{n-1}} \det \left[ A^{(-1)^n} \right]
$$

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\[
\begin{align*}
\det(A) &= (n-1)^n + (-1)^{n-1} + (-1)^n \\
&= (n-1)^n.
\end{align*}
\]

Thus,
\[
\ln(\det(adj^n(A_n))) = \ln(\det(A_n)^{(n-1)n}) = (n-1)^n \ln(|\det(A_n)|),
\]

so, for \(n \geq 2\),
\[
\frac{n^n \ln(\det(A_n))}{\ln(|\det(adj^n(A_n))|)} = \frac{n^n \ln(\det(A_n))}{(n-1)^n \ln(|\det(A_n)|)} = \frac{n^n}{(n-1)^n} = \left(\frac{n}{n-1}\right)^n.
\]

That is, the individual \(A_n\) has disappeared and our complex fraction has become very simple.

Now it is easy to show by calculus that the limit is \(e\).

- **5151**: Proposed by Ovidiu Furdui, Cluj, Romania

Find the value of
\[
\prod_{n=1}^{\infty} \left(\sqrt{\frac{\pi}{2}} \cdot \frac{(2n-1)!\sqrt{2n+1}}{2^n n!}\right)^{(-1)^n}.
\]

More generally, if \(x \neq n\pi\) is a real number, find the value of
\[
\prod_{n=1}^{\infty} \left(\frac{x}{\sin x} \left(1 - \frac{x^2}{n^2 \pi^2}\right) \cdots \left(1 - \frac{x^2}{(n\pi)^2}\right)\right)^{(-1)^n}.
\]

**Solution by the proposer**

The first product equals \(\sqrt{\frac{2\sqrt{2}}{\pi}}\) and the second one equals \(\frac{2 \sin \frac{x}{2}}{x}\). Recall the infinite product representation for the sine function
\[
\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2 \pi^2}\right).
\]

Since the first product can be obtained from the second one, when \(x = \pi/2\), we concentrate on the calculation of the second product. Let
\[
\begin{align*}
S_{2n} &= \sum_{k=1}^{2n} (-1)^k \left(\ln \left(1 - \frac{x^2}{\pi^2}\right) + \cdots + \ln \left(1 - \frac{x^2}{k^2 \pi^2}\right) + \ln \frac{x}{\sin x}\right) \\
&= -\left(\ln \left(1 - \frac{x^2}{\pi^2}\right) + \ln \frac{x}{\sin x}\right) + \left(\ln \left(1 - \frac{x^2}{\pi^2}\right) + \ln \left(1 - \frac{x^2}{2^2 \pi^2}\right) + \ln \frac{x}{\sin x}\right) \\
&\cdots
\end{align*}
\]
\[
-\left(\ln \left(1 - \frac{x^2}{\pi^2}\right) + \ln \left(1 - \frac{x^2}{2^2 \pi^2}\right) + \cdots + \ln \left(1 - \frac{x^2}{(2n - 1)^2 \pi^2}\right) + \ln \frac{x}{\sin x}\right)
\]
\[
+ \left(\ln \left(1 - \frac{x^2}{\pi^2}\right) + \ln \left(1 - \frac{x^2}{2^2 \pi^2}\right) + \cdots + \ln \left(1 - \frac{x^2}{(2n - 1)^2 \pi^2}\right) + \ln \left(1 - \frac{x^2}{(2n)^2 \pi^2}\right) + \ln \frac{x}{\sin x}\right)
\]
\[
= \ln \left(\frac{1 - x^2}{(2\pi)^2} \left(1 - \frac{x^2}{(4\pi)^2}\right) \cdots \left(1 - \frac{x^2}{(2n\pi)^2}\right)\right)
\]
\[
= \ln \left(\frac{1 - \left(\frac{x}{2}\right)^2}{\pi^2} \left(1 - \frac{(x/2)^2}{(2\pi)^2}\right) \cdots \left(1 - \frac{(x/2)^2}{(n\pi)^2}\right)\right).
\]

Letting \(n\) tend to infinity in the preceding equality we get that \(\lim_{n \to \infty} S_{2n} = \ln \frac{2\sin(x/2)}{x}\), and the problem is solved.

Also solved by Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy.