Problems   Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before September 15, 2010

- **5116:** *Proposed by Kenneth Korbin, New York, NY*

  Given square $ABCD$ with point P on side AB, and with point Q on side BC such that

  $$\frac{AP}{PB} = \frac{BQ}{QC} > 5.$$

  The cevians DP and DQ divide diagonal AC into three segments with each having integer length. Find those three lengths, if $AC = 84$.

- **5117:** *Proposed by Kenneth Korbin, New York, NY*

  Find positive acute angles A and B such that

  $$\sin A + \sin B = 2\sin A \cdot \cos B.$$

- **5118:** *Proposed by David E. Manes, Oneonta, NY*

  Find the value of

  $$\sqrt{2011 + 2007\sqrt{2012 + 2008\sqrt{2013 + 2009\sqrt{2014 + \cdots}}}}$$

- **5119:** *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain*

  Let n be a non-negative integer. Prove that

  $$2 + \frac{1}{2^{n+1}} \prod_{k=0}^{n} \csc\left(\frac{1}{F_k}\right) < F_{n+1}$$

  where F_n is the n^{th} Fermat number defined by $F_n = 2^{2^n} + 1$ for all $n \ge 0$.

- **5120:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Calculate

  $$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \log\left(\frac{2n-k}{2n+k}\right).$$

- **5121:** *Proposed by Tom Leong, Scotrun, PA*

  Let n, k and r be positive integers. It is easy to show that

  $$\sum_{n_1+n_2+\cdots+n_r=n} \binom{n_1}{k}\binom{n_2}{k}\cdots\binom{n_r}{k} = \binom{n+r-1}{kr+r-1}, \quad n_1, n_2, \cdots n_r \in N$$

  using generating functions. Give a combinatorial argument that proves this identity.

Solutions

- **5098:** *Proposed by Kenneth Korbin, New York, NY*

  Given integer-sided triangle ABC with $\angle B = 60^\circ$ and with $a < b < c$. The perimeter of the triangle is $3N^2 + 9N + 6$, where N is a positive integer. Find the sides of a triangle satisfying the above conditions.

  Solution 1 by Michael N. Fried, Kibbutz Revivim, Israel

  Since $3n^2 + 9n + 6 = 3(n+1)(n+2) = 3K$, we can rephrase the problem as follows: Find an integer-sided triangle ABC with angle $B = 60^\circ$ and $a < b < c$ whose perimeter is the same as an equilateral triangle PBQ whose side is K.

  Let us then consider a triangle derived from PBQ by lengthening PB by an integer x and shortening BQ by an integer y such that the resulting triangle still has perimeter $3K$.

  Thus, we can write the following expression:

  $$\text{Perimeter } (\triangle ABC) = (K+x)+(K-y)+\Big[(K+x)^2+(K-y)^2-(K+x)(K-y)\Big]^{1/2} = 3K \qquad (1)$$

  Also we must make sure that,

  $$(K+x)^2 + (K-y)^2 - (K+x)(K-y) = M^2, \text{ for some interger } M \qquad (2)$$

  Note also that $BC < CA < AB$ since $\angle BAC < 60^\circ < \angle BCA$.

  Equation (1) can be transformed into the much simpler equation,

  $$xy = K(y-x) = (n+1)(n+2)(y-x) \qquad (3)$$

  The most obvious solution of (3) is $x = n+1$ and $y = n+2$.

  Substituting these expressions into the left hand side of (2) and simplifying, we get

  $$(K+x)^2+(K-y)^2-(K+x)(K-y) = (n+1)^4+2(n+1)^3+3(n+1)^2+2(n+1)+1 \qquad (4)$$

  But the right hand side of (4) is just $\big[(n+1)^2+(n+1)+1\big]^2$, so that (2) is satisfied when $x = n+1$ and $y = n+2$.

  Hence, we have at least one solution:

  $$\begin{aligned} AB &= K+x = (n+1)(n+2)+(n+1) = (n+1)(n+3) \\ BC &= K-y = (n+1)(n+2)-(n+2) = n(n+2) \\ CA &= (n+1)^2+(n+1)+1 \end{aligned}$$

Solution 2 by David Stone and John Hawkins, Statesboro, GA

We show that the following triangle satisfies the conditions posed in the problem:

$$\begin{aligned} a &= N2 + 2N = N(N+2) \\ b &= N^2 + 3N + 3 = (N+1)(N+2) + 1 \\ c &= N^2 + 4N + 3 = (N+1)(N+3). \end{aligned}$$

But by no means does this give all acceptable triangles and we exhibit some others (and methods to produce them).

The given sum for the perimeter does have a connection to triangles: $3N^2 + 9N + 6$ is $6T_{N+1}$, the $N+1st$ triangular number!

Since $a < b < c$ are all integers, we let m and n be positive integers such that $b = a + m$ and $c = a + m + n$.

By the Law of Cosines, $b^2 = a^2 + c^2 - 2ac\cos 60^\circ = a^2 - ac + c^2$. Replacing $b = a + m$ and $c = a + m + n$ we get

$$\begin{aligned} &(a+m)^2 = a^2 - a(a+m+n) + (a+m+n)^2 \text{ or} \\ &-am + an + n^2 + 2mn = 0. \qquad\qquad (1) \end{aligned}$$

Likewise, substituting $b = a + m$ and $c = a + m + n$ into the proscribed perimeter conditions produces

$$3a + 2m + n = 3(N+1)(N+2). \qquad (2)$$

From equation (1), we have $am = n(a + 2m + n)$; and from this we see that n must be a factor of am. There are many ways for this to happen, but the simplest possible is that $n|s$ or $n|m$.

Case I: $a = nA$.

Then

$$\begin{aligned} nAm &= n(nA + 2m + n), \\ Am &= nA + 2m + n, \text{ or} \\ (A-2)m &= n(A+1). \qquad\qquad (1a) \end{aligned}$$

The simplest possible solution to Equation (1a) is

$$\begin{aligned} n &= A - 2 \\ m &= A + 1 \end{aligned}$$

In this case, equation (2) becomes

$$\begin{aligned} 3nA + 2m + n &= 3\left(N+1\right)\left(N+2\right), \\ 3(A-2)A + 2(A+1) + (A-2) &= 3(N+1)(N+2), \\ 3A^2 - 3A &= 3(N+1)(N+2), \text{ or} \\ (A-1)A &= (N+1)(N+2). \end{aligned}$$

Because $A - 1$ and A are consecutive integers, as are $N + 1$ and $N + 2$, we must have $A = N + 2$ (so $n = N$ and $m = N + 3$). It then follows that

$$a = nA = N(N+2)$$

$$
\begin{aligned}
b &= a+m = N(N+2)+(N+3) = N^2+3N+3 \\
c &= a+m+n = N(N+2)+(N+3)+N = N^2+4N+3.
\end{aligned}
$$

It is straightforward to check that such a, b, c satisfy equations (1) and (2). Here are the first few solutions:

N	a	b	c
1	3	7	8
2	8	13	15
3	15	21	24
4	24	31	35
5	35	43	48
6	48	57	63
7	63	73	80
8	80	91	99
9	99	111	120
10	120	133	143
11	143	157	168
12	168	183	195
13	195	211	224
14	224	241	255
15	225	273	288

There are more solutions to the equation $(1a) : (A-2)m = n(A+1)$. For instance, we could look for solutions with $m = d(A+1)$ and $n = d(A-2)$, with $d > 1$. In this case, equation (2) becomes $dA(A-1) = (N+1)(N+2)$ which is quadratic in A. By varying d (and using Excel) we find more solutions:

d	N	a	b	c
2	2	6	14	16
2	19	390	422	448
3	8	72	93	105
3	34	1197	1263	1320
5	4	15	35	40
5	13	175	215	240
5	98	9675	9905	10120

Note that these solutions are scalar multiples of the (fundamental?) solutions found above. Many more solutions are possible.

Case II: $n|m$, or $m = nC$.

In this case, $am = n(a+2m+n)$ becomes

$$
\begin{aligned}
anC &= n(a+2nC+n) \text{ or} \\
aC &= a+2nC+n \text{ or} \\
(C-1)a &= n(2C+1) \qquad (1b)
\end{aligned}
$$

Once again, there are many ways to find solutions to this, but no general solution valid for all values of N. We stop by giving one more: with $N = 54$ we find
$a = 231, b = 4449, c = 4560$.

Also solved by Brian D. Beasley, Clinton, SC; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; David C. Wilson, Winston-Salem, NC, and the proposer.

- **5099:** *Proposed by Kenneth Korbin, New York, NY*

  An equilateral triangle is inscribed in a circle with diameter d. Find the perimeter of the triangle if a chord with length $d-1$ bisects two of its sides.

  Solution 1 by Boris Rays, Brooklyn, NY

  Let O be the center of the inscribed equilateral triangle ABC. Let the intersection of the altitude from vertex A with side BC be F; from vertex B with side AC be H, and from vertex C with side AB be E . Since $\triangle ABC$ is equilateral, AF, BH, and CE are also the respective angle bisectors, perpendicular bisectors, and medians of the equilateral triangle, and $AH = HC = CE = FB = BE = EA$.

  Let line segment EF be extended in each direction, intersecting BH at K, and the circumscribing circle of $\triangle ABC$ at points D and G, where D is on the minor arc$\widehat{AB}$ and G is on the minor arc $\widehat{BC}$. Note that points D, E, K, F, and G lie on line segment $\overline{DG}$ and that $AO = OG$. Also note, by the givens of the problem, that

$$\begin{aligned} DG &= d-1 \text{ and} \\ AO &= BO = CO = r = \frac{d}{2}, \qquad (1) \end{aligned}$$

  where r and d are correspondingly the radius and diameter of the circumscribed circle.

$$\begin{aligned} BH &\perp AC, AH = HC, \angle BAO = \angle OAH = 30^\circ. \\ OH &= \frac{1}{2}AO = \frac{d}{4}. \\ AH &= \sqrt{\left(\frac{d}{2}\right)^2 - \left(\frac{d}{4}\right)^2} = \frac{d}{4}\sqrt{3}. \\ AC &= 2AH = \frac{d}{2}\sqrt{3} \qquad (2) \end{aligned}$$

  The perimeter P of triangle $\triangle ABC$ will be

$$\begin{aligned} P &= 3 \cdot AC = \frac{3}{2}\sqrt{3}d. \qquad (3) \\ BK &= \frac{1}{2}BH = \frac{1}{2} \cdot 3 \cdot OH = \frac{3}{8}d. \\ KO &= BO - BK = \frac{d}{2} - \frac{3}{8}d = \frac{d}{8}. \\ GK &= \frac{1}{2}DG = \frac{d-1}{2}. \end{aligned}$$

  Triangle $\triangle GKO$ is a right triangle with $DG \perp BH$ and $GK \perp BO$. Therefore,

$$GO^2 = GK^2 + KO^2 \qquad (4)$$

  Substituting the values of the component parts of $\triangle GKO$ into (4),

$$GO = r = \frac{d}{2},\ GK = \frac{d-1}{2},\ KO = \frac{d}{8},$$

we obtain
$$\left(\frac{d-1}{2}\right)^2 - \left(\frac{d}{8}\right)^2 = \left(\frac{d}{2}\right)^2. \qquad (5)$$

Simplifying the last equation (5) we find that $d = 4 \cdot \left(4 + \sqrt{15}\right)$. Therefore,

$$\begin{aligned} AC &= \frac{4(4+\sqrt{15})}{2}\sqrt{3} = 2\left(4\sqrt{3} + 3\sqrt{5}\right), \text{and} \\ P &= 3 \cdot 2\left(4\sqrt{3} + 3\sqrt{5}\right) = 24\sqrt{3} + 18\sqrt{5}. \end{aligned}$$

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Solution 2 by Brian D. Beasley, Clinton, NC

We model the circle using $x^2 + y^2 = r^2$, where $r = d/2$, and place the triangle with one vertex at $(0, r)$, leaving the other two vertices in the third and fourth quadrants. Labeling the fourth quadrant vertex as (a, b), we have $b = r - \sqrt{3}a$ and thus $a = \sqrt{3}r/2$, $b = -r/2$. Then two of the midpoints of the triangle's sides are $\left(\frac{\sqrt{3}}{4}r, \frac{1}{4}r\right)$ and $\left(0, -\frac{1}{2}r\right)$. We find the endpoints of the chord through these two midpoints by substituting its equation, $y = \sqrt{3}x - r/2$, into the equation of the circle; the two x-coordinates of these endpoints are $x = sr$ and $x = tr$, where
$$s = \frac{\sqrt{3} + \sqrt{15}}{8} \quad \text{and} \quad t = \frac{\sqrt{3} - \sqrt{15}}{8}.$$
Hence the length of the chord is
$$\sqrt{(s-t)^2r^2 + (\sqrt{3}(s-t))^2r^2} = d(s-t).$$
If the chord length is $d - k$, where $0 < k < d$, then $d = k/(1 - s + t) = 4k(4 + \sqrt{15})$. Thus the perimeter of the triangle is $P = 3\sqrt{3}r = k(24\sqrt{3} + 18\sqrt{5})$. For the given problem, since $k = 1$, we obtain $P = 24\sqrt{3} + 18\sqrt{5}$.

Also solved by Michael Brozinsky, Central Islip, NY; Paul M. Harms, North Newton, KS; John Nord, Spokane, WA; Raúl A. Simón, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5100:** *Proposed by Mihály Bencze, Brasov, Romania*

  Prove that
  $$\sum_{k=1}^{n} \sqrt{\frac{k}{k+1}} \binom{n}{k} \le \sqrt{\frac{n(2^{n+1} - n)2^{n-1}}{n+1}}$$

  Solution 1 by Kee-Wai Lau, Hong Kong, China

  We need the identities
  $$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n \qquad (1)$$
  $$\sum_{k=0}^{n} k\binom{n}{k} x^{k-1} = n(1+x)^{n-1} \qquad (2)$$

$$\text{and} \quad \sum_{k=0}^{n} \binom{n}{k} \frac{x^{k+1}}{k+1} \quad = \quad \frac{(1+x)^{n+1}-1}{n+1} \qquad (3)$$

Identity (1) is the well known binomial expansion, whilst identities (2) and (3) follow respectively by differentiating and integrating (1). By the Cauchy-Schwarz inequality and putting $x = 1$ in (2) and (3) we obtain

$$\begin{aligned}
\sum_{k=1}^{n} \sqrt{\frac{k}{k+1}\binom{n}{k}} &= \sum_{k=1}^{n} \left(\sqrt{k\binom{n}{k}}\right)\left(\sqrt{\binom{n}{k}\frac{1}{k+1}}\right) \\
&\le \sqrt{\left(\sum_{k=1}^{n} k\binom{n}{k}\right)\left(\sum_{k=1}^{n}\binom{n}{k}\frac{1}{k+1}\right)} \\
&= \sqrt{(n2^{n-1})\left(\frac{2^{n+1}-1}{n+1}-1\right)} \\
&= \sqrt{\frac{n\left(2^{n+1}-n-2\right)2^{n-1}}{n+1}},
\end{aligned}$$

and the inequality of the problem follows.

Solution 2 by Shai Covo, Kiryat-Ono, Israel

We shall prove a substantially better upper bound than the one stated in the problem. Namely, we show that

$$\sum_{k=1}^{n} \sqrt{\frac{k}{k+1}\binom{n}{k}} < \frac{n}{n+1}\left(2^n - \frac{1}{2}\right).$$

It is readily checked that our bound is less than the bound of $\sqrt{\dfrac{n(2^{n+1}-n)2^{n-1}}{n+1}}$ that the problem suggests; moreover, we have verified numerically that it is much tighter.

Now to the proof. The key observation is that

$$\sqrt{\frac{k}{k+1}} < 1 - \frac{1}{2(k+1)}$$

for all $k \in N$ (actually, for any real $k > 0$; its origin lies in the *mean value theorem* applied to the function $f(x) = \sqrt{x}$ and points $a = k/(k+1), b = 1$.

Thus, using the elementary identity $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ (twice), we get

$$\sum_{k=1}^{n} \sqrt{\frac{k}{k+1}}\binom{n}{k} \quad < \quad \sum_{k=1}^{n}\binom{n}{k} - \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k+1}\binom{n}{k}$$

$$
\begin{aligned}
&= 2^n - 1 - \frac{1}{2(n+1)} \sum_{k=1}^{n} \binom{n+1}{k+1} \\
&= 2^n - 1 - \frac{2^{n+1} - (n+1) - 1}{2(n+1)} \\
&= \frac{n}{n+1}\left(2^n - \frac{1}{2}\right).
\end{aligned}
$$

Solution 3 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

We prove the slightly more general statement

$$\sum_{k=1}^{n} \sqrt{\frac{k}{k+1}\binom{n}{k}} \leq \sqrt{\frac{(2^n - 1)\left[(n-1)\,2^n + 1\right]}{n+1}}. \qquad (1)$$

To show that this implies the desired inequality, we begin by letting $P(n)$ be the statement: $2^{n+1} > (n-1)^2 + 3$. $P(1)$ is obvious and if we assume $P(n)$ is true for some $n \geq 1$, then

$$
\begin{aligned}
2^{n+2} &= 2 \cdot 2^{n+1} > 2\left[(n-1)^2 + 3\right] = \left(n^2 + 3\right) + \left(n^2 - 4n + 5\right) \\
&= \left(n^2 + 3\right) + \left[(n-2)^2 + 1\right] > n^2 + 3,
\end{aligned}
$$

and $P(n+1)$ is also true. By Mathematical Induction, $P(n)$ is true for all $n \geq 1$.

Then, for $n \geq 1$,

$$
\begin{aligned}
& n\left(2^{n+1} - n\right) 2^{n-1} - (2^n - 1)\left[(n-1)\,2^n + 1\right] \\
=\; & 2^{n-1}\left[2^{n+1} - (n-1)^2 - 3\right] + 1 \\
>\; & 0
\end{aligned}
$$

and we have

$$(2^n - 1)\left[(n-1)\,2^n + 1\right] < n\left(2^{n+1} - n\right) 2^{n-1}. \qquad (2)$$

It follows that statement (1) implies the given inequality.

To prove statement (1), we note that since $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$, we get $\displaystyle\sum_{k=1}^{n} \frac{\binom{n}{k}}{2^n - 1} = 1$.
Because $f(x) = \sqrt{x}$ is concave down on $[0, \infty)$, Jensen's Theorem implies that

$$\sum_{k=1}^{n} \binom{n}{k} \frac{1}{2^n - 1} \sqrt{\frac{k}{k+1}} \leq \sqrt{\sum_{k=1}^{n} \binom{n}{k} \frac{1}{2^n - 1} \frac{k}{k+1}} = \sqrt{\frac{1}{2^n - 1} \sum_{k=1}^{n} \binom{n}{k} \frac{k}{k+1}},$$

and hence,

$$\sum_{k=1}^{n} \binom{n}{k} \sqrt{\frac{k}{k+1}} \leq \sqrt{(2^n - 1) \sum_{k=1}^{n} \binom{n}{k} \frac{k}{k+1}}. \qquad (3)$$

For $k = 1, 2, \ldots, n$,

$$\binom{n}{k}\frac{k}{k+1} = \frac{k}{n+1}\binom{n+1}{k+1}$$

and we get

$$\sum_{k=1}^{n}\binom{n}{k}\frac{k}{k+1} = \frac{1}{n+1}\sum_{k=1}^{n}k\binom{n+1}{k+1} = \frac{1}{n+1}\sum_{k=2}^{n+1}(k-1)\binom{n+1}{k}. \qquad (4)$$

Finally, the Binomial Theorem yields

$$\sum_{k=0}^{n+1}\binom{n+1}{k}x^k = (1+x)^{n+1}\,.$$

It follows that when $x \neq 0$,

$$\sum_{k=1}^{n+1}\binom{n+1}{k}x^{k-1} = \frac{(1+x)^{n+1}-1}{x}$$

and, by differentiating,

$$\sum_{k=2}^{n+1}(k-1)\binom{n+1}{k}x^{k-2} = \frac{x\,(n+1)\,(1+x)^n - \left[(1+x)^{n+1}-1\right]}{x^2}.$$

In particular, when $x = 1$,

$$\sum_{k=2}^{n+1}(k-1)\binom{n+1}{k} = (n+1)\,2^n - 2^{n+1} + 1 = (n-1)\,2^n + 1. \qquad (5)$$

Then, statements (3), (4), and (5) imply statement (1), which (by statement (2)) yields the desired inequality.

Also solved by G. C. Greubel, Newport News, VA, and the proposer

- **5101:** *Proposed by K. S. Bhanu and M. N. Deshpande, Nagpur, India*

An unbiased coin is tossed repeatedly until r heads are obtained. The outcomes of the tosses are written sequentially. Let R denote the total number of runs (of heads and tails) in the above experiment. Find the distribution of R.

Illustration: if we decide to toss a coin until we get 4 heads, then one of the possibilities could be the sequence $T\ T\ H\ H\ T\ H\ T\ H$ resulting in 6 runs.

Solution by Shai Covo, Kiryat-Ono, Israel

It is readily seen that R can be represented as

$$R = 1 + Y_1 + 2\sum_{i=2}^{r}Y_i, \qquad (1)$$

where $Y_i,\ i = 1, \ldots, r$, is a random variable equal to 1 if the i-th head follows a tail and equal to 0 otherwise. The $Y_{i's}$ are thus *independent* Bernoulli (1/2) variables, that is

$P(Y_i = 1) = P(Y_i = 0) = 1/2$. Noting that R is odd if and only if $Y_1 = 0$, and even if and only if $Y_1 = 1$, it follows straightforwardly from (1) that

$$\begin{aligned} P(R=n) &= \frac{1}{2} P\left(\sum_{i=2}^{r} Y_i = \frac{n-1}{2}\right) \text{ for } n = 1, 3, \ldots, (2r-1) \text{ and} \\ P(R=n) &= \frac{1}{2} P\left(\sum_{i=2}^{r} Y_i = \frac{n-2}{2}\right) \text{ for } n = 2, 4, \ldots, 2r. \end{aligned} \tag{2}$$

Finally, since $\sum_{i=2}^{r} Y_i$ has a binomial distribution with parameters $r-1$ and $\frac{1}{2}$ (defined as 0 if $r = 1$), we conclude that

$$P(R=n) = \binom{r-1}{(n-1)/2} \frac{1}{2^r} \text{ for } n = 1, 3, \ldots, (2r-1)$$

and

$$P(R=n) = \binom{r-1}{(n-2)/2} \frac{1}{2^r} \text{ for } n = 2, 4, \ldots, 2r.$$

Remark 1. More generally, if the probability of getting a head on each throw is $p \in (0,1)$, then $P(R=n)$ is given, in a shorter form, by

$$P(R=n) = \binom{r-1}{\lfloor \frac{n-1}{2} \rfloor} (1-p)^{\lfloor n/2 \rfloor} p^{r - \lfloor n/2 \rfloor}, \ n = 1, 2, \ldots, 2r,$$

where $\lfloor \cdot \rfloor$ denotes the floor function. This is proved in the same way as in the unbiased case, only that now the Y_i are Bernoulli $(1-p)$ variables.

Remark 2. From (1) and the fact that $E(Y_i) = 1/2$ and $Var(Y_i) = 1/4$, we find that the expectation and variance of R are given by
$E(R) = 1 + \frac{1}{2} + 2(r-1)\frac{1}{2} = r + \frac{1}{2}$ and $Var(R) = \frac{1}{4} + 4(r-1)\frac{1}{4} = r - \frac{3}{4}$.
In the more general case of Remark 1, where $E(Y_i) = 1-p$ and $Var(Y_i) = (1-p)p$, the expectation and variance of R are given by

$$E(R) = 2(1-p)r + p \quad \text{and} \quad Var(R) = 4(1-p)pr - 3(1-p)p.$$

Also solved by David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- **5102:** *Proposed by Miquel Grau-Sánchez and José Luis Díaz-Barrero, Barcelona, Spain*

Let n be a positive integer and let $a_1, a_2, \cdots, a_n$ be any real numbers. Prove that

$$\frac{1}{1 + a_1^2 + \ldots + a_n^2} + \frac{1}{F_n F_{n+1}} \left(\sum_{k=1}^{n} \frac{a_k F_k}{1 + a_1^2 + \ldots + a_k^2} \right)^2 \leq 1,$$

where F_k represents the k^{th} Fibonacci number defined by $F_1 = F_2 = 1$ and for $n \geq 3, F_n = F_{n-1} + F_{n-2}$.

Solution by Kee-Wai Lau, Hong Kong, China

By Cauchy-Schwarz's inequality and the well known identity $\displaystyle\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$ we have

$$
\begin{aligned}
& \frac{1}{F_n F_{n+1}} \left(\sum_{k=1}^{n} \frac{a_k F_k}{1 + a_1^2 + \ldots + a_k^2} \right)^2 \\
= \ & \frac{1}{F_n F_{n+1}} \left(\sum_{k=1}^{n} \left(\frac{a_k}{1 + a_1^2 + \ldots + a_k^2} \right) F_k \right)^2 \\
\leq \ & \frac{1}{F_n F_{n+1}} \left(\sum_{k=1}^{n} \frac{a_k^2}{(1 + a_1^2 + \ldots + a_k^2)^2} \right) \left(\sum_{k=1}^{n} F_k^2 \right) \\
= \ & \sum_{k=1}^{n} \frac{a_k^2}{(1 + a_1^2 + \ldots + a_k^2)^2}
\end{aligned}
$$

Hence it remains for us to show that

$$\frac{1}{1 = a_1^2 + \ldots + a_n^2} + \sum_{k=1}^{n} \frac{a_k^2}{(1 + a_1^2 + \ldots + a_k^2)^2} \leq 1. \qquad (1)$$

Denote the left hand side of (1) by $f(n)$. Since $f(1) = \dfrac{1 + 2a_1^2}{1 + 2a_1^2 + a_1^4}$, so $f(1) \leq 1$.

Now

$$
\begin{aligned}
& f(m+1) - f(m) \\
= \ & \frac{1}{1 + a_1^2 + \ldots + a_{m+1}^2} + \frac{a_{m+1}^2}{(1 + a_1^2 + \ldots + a_{m+1}^2)^2} - \frac{1}{1 + a_1^2 + \ldots + a_m^2} \\
= \ & \frac{(1 + a_1^2 + \ldots + a_m^2)(1 + a_1^2 + \ldots + a_m^2 + 2a_{m+1}^2) - (1 + a_1^2 + \ldots + a_{m+1}^2)^2}{(1 + a_1^2 + \ldots + a_{m+1}^2)^2 (1 + a_1^2 + \ldots + a_m^2)} \\
= \ & - \frac{a_{m+1}^4}{(1 + a_1^2 + \ldots + a_{m+1}^2)^2 (1 + a_1^2 + \ldots + a_m^2)} \\
\leq \ & 0,
\end{aligned}
$$

so in fact $f(n) \leq 1$ for all positive integers n. Thus (1) holds and this completes the solution.

Also solved by the proposers.

- **5103:** *Proposed by Roger Izard, Dallas, TX*

A number of circles of equal radius surround and are tangent to another circle. Each of the outer circles is tangent to two of the other outer circles. No two outer circles intersect in two points. The radius of the inner circle is a and the radius of each outer circle is b. If

$$a^4 + 4a^3b - 10a^2b^2 - 28ab^3 + b^4 = 0,$$

determine the number of outer circles.

Solution by Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie, San Angelo, TX

Let $\mathcal{C_A}$ be the inner circle centered at point A with radius a. Similarly, let $\mathcal{C_B}$ be a fixed outer circle centered at point B with radius b. Circle $\mathcal{C_B}$ is tangent to two other outer circles; let T_1 and T_2 be these points of tangency. Then,

$$\overline{BT_1} \perp \overline{AT_1} \quad \text{and} \quad \overline{BT_2} \perp \overline{AT_2}.$$

If θ is the measure of $\angle T_1AT_2$, then $0^\circ < \theta < 180^\circ$. Further, triangle T_1AB is a right triangle where

$$m\angle T_1AB = \frac{\theta}{2}, \quad T_1B = b, \quad \text{and} \quad AB = a + b$$

which yields

$$\sin\left(\frac{\theta}{2}\right) = \frac{b}{a+b}. \qquad (1)$$

The given condition $a^4 + 4a^3b - 10a^2b^2 - 28ab^3 + b^4 = 0$ implies that

$$\begin{aligned}
a^4 + 4a^3b + b^4 &= 10a^2b^2 + 28ab^3 \\
a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 &= 16a^2b^2 + 32ab^3 \\
(a+b)^4 &= 16b^2(a^2 + 2ab) \\
(a+b)^4 &= 16b^2(a^2 + 2ab + b^2) - 16b^4 \\
(a+b)^4 &= 16b^2(a+b)^2 - 16b^4 \\
1 &= \frac{16b^2(a+b)^2 - 16b^4}{(a+b)^4} \\
1 &= 16\left(\frac{b}{a+b}\right)^2 - 16\left(\frac{b}{a+b}\right)^4.
\end{aligned}$$

By equation (1) and the half-angle formula, $\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$, it follows that:

$$\begin{aligned}
1 &= 16\left(\frac{1-\cos\theta}{2}\right) - 16\left(\frac{1-\cos\theta}{2}\right)^2 \\
1 &= 8(1-\cos\theta) - 4(1-\cos\theta)^2 \\
1 &= 4 - 4\cos^2\theta
\end{aligned}$$

$$\begin{aligned}\cos^2\theta &= \frac{3}{4}\\ \cos\theta &= \pm\frac{\sqrt{3}}{2}\\ \theta &= 30^\circ \text{ or } 150^\circ.\end{aligned}$$

Since the number of outer circles is $\dfrac{360^\circ}{\theta}$, then $\theta = 30^\circ$ and there must be 12 outer circles.

Comment by editor: **David Stone and John Hawkins of Statesboro, GA** observed in their solution that "the circle passing through the centers of the outer bracelet of circles has circumference almost equal, but slightly larger than, the perimeter of the regular polygon determined by these centers: $2\pi(a+b) \approx n(2b)$. Thus $n \approx \dfrac{a+b}{b}\pi$ (in fact, n must be slightly smaller than $\dfrac{a+b}{b}\pi$)."

They went on to say that since

$$\begin{aligned}a^4 + 4a^3b - 10a^2b^2 - 28ab^3 + b^4 &= 0,\\ \frac{a^4}{b^4} + \frac{4a^3b}{b^4} - \frac{10a^2b^2}{b^4} - \frac{28ab^3}{b^4} + \frac{b^4}{b^4} &= 0, \text{ implies}\\ x^4 + 4x^3 - 10x^2 - 28x + 1 &= 0, \text{ where } x = \frac{a}{b}.\end{aligned}$$

Therefore, $\dfrac{a}{b} = \sqrt{6} \pm \sqrt{2} - 1$, and since $n \approx \dfrac{a+b}{b}\pi$, $n = 12$. But then they went further. The equation $\sin\left(\dfrac{\pi}{n}\right) = \dfrac{b}{a+b} = \dfrac{1}{1+\dfrac{a}{b}}$, provides the link between n and the ratio $\dfrac{a}{b}$; we can solve for either:

$$n = \frac{\pi}{\sin^{-1}\left(\dfrac{1}{1+a/b}\right)} \quad \text{and} \quad \frac{a}{b} = \frac{1}{\sin(\pi/n)} - 1.$$

The problem poser cleverly embedded a nice ratio for $\dfrac{a}{b}$ in the fourth degree polynomial; nice in the sense that the n turned out to be an integer. In fact, the graph of the increasing function $y = \dfrac{\pi}{\sin^{-1}\left(\dfrac{1}{1+r}\right)}$ is continuous and increasing for the positive ratio r. Thus **any** lager value of n is uniquely attainable (given the correct choice of $r = \dfrac{a}{b}$).

Or we can reverse the process: fix the number of surrounding circles and calculate $r = \dfrac{a}{b}$.

A nice example (by letting $b = 1$): if we want to surround a circle with a bracelet of 100 unit circles, how large should it be? Answer:

$$\text{radius } = a = \frac{a}{1} = \frac{1}{\sin\dfrac{\pi}{100}} - 1 = 30.836225.$$

Also solved by Michael Brozinsky, Central Islip, NY; Michael N. Fried, Kibbutz Revivim, Israel; Paul M. Harms, North Newton, KS; Kenneth

Korbin, New York, NY; Boris Rays, Brooklyn, NY; Raúl A. Simón, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA; The Taylor University Problem Solving Group, Upland, IN, and the proposer.