Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent by e-mail to eisenbt@013.net. Solutions to previously stated problems can be seen at http://ssmj.tamu.edu

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Solutions to the problems stated in this issue should be posted before October 15, 2009

- 5068: *Proposed by Kenneth Korbin, New York, NY*

  Find the value of
  $$\sqrt{1+2009\sqrt{1+2010\sqrt{1+2011\sqrt{1+\cdots}}}}.$$

- 5069: *Proposed by Kenneth Korbin, New York, NY*

  Four circles having radii $\dfrac{1}{14}$, $\dfrac{1}{15}$, $\dfrac{1}{x}$ and $\dfrac{1}{y}$ respectively, are placed so that each of the circles is tangent to the other three circles. Find positive integers x and y with $15 < x < y < 300$.

- 5070: *Proposed by Isabel Iriberri Díaz and José Luis Díaz-Barrero, Barcelona, Spain*

  Find all real solutions to the system
  $$\left.\begin{array}{c} 9(x_1^2+x_2^2-x_3^2)=6x_3-1, \\ 9(x_2^2+x_3^2-x_4^2)=6x_4-1, \\ \cdots\cdots\cdots \\ 9(x_n^2+x_1^2-x_2^2)=6x_2-1. \end{array}\right\}$$

- 5071: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let h_a, h_b, h_c be the altitudes of $\triangle ABC$ with semi-perimeter s, in-radius r and circum-radius R, respectively. Prove that
  $$\frac{1}{4}\left(\frac{s(2s-a)}{h_a}+\frac{s(2s-b)}{h_b}+\frac{s(2s-c)}{h_c}\right) \le \frac{R^2}{r}\left(\sin^2 A+\sin^2 B+\sin^2 C\right).$$

- 5072: *Proposed by Panagiote Ligouras, Alberobello, Italy*

  Let a, b and c be the sides, l_a, l_b, l_c the bisectors, m_a, m_b, m_c the medians, and h_a, h_b, h_c the heights of $\triangle ABC$. Prove or disprove that

  a) $$\frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c} \ge \frac{4}{3}\left(m_a\cdot l_a+l_b\cdot h_b+h_c\cdot m_c\right)$$

b) $3\sum_{cyc}\dfrac{(-a+b+c)^3}{a} \geq 2\sum_{cyc}[m_a(l_a+h_a)].$

- 5073: *Proposed by Ovidiu Furdui, Campia-Turzii, Cluj, Romania*

  Let $m > -1$ be a real number. Evaluate

  $$\int_0^1 \{\ln x\} x^m dx,$$

  where $\{a\} = a - [a]$ denotes the fractional part of a.

Solutions

- 5050: *Proposed by Kenneth Korbin, New York, NY*

  Given $\triangle ABC$ with integer-length sides, and with $\angle A = 120^o$, and with $(a,b,c) = 1$.
  Find the lengths of b and c if side $a = 19$, and if $a = 19^2$, and if $a = 19^4$.

Solution 1 by Paul M. Harms, North Newton, KS

Using the law of cosines we have $a^2 = b^2 + c^2 - 2bc\cos 120^o = b^2 + c^2 + bc.$

When $a = 19$ we have $19^2 = 361 = b^2 + c^2 + bc$. The result $b = 5, c = 16$ with $a = 19$ satisfies the problem.

Some books indicate that the Diophantine equation $a^2 = b^2 + c^2 + bc$ has solutions of the form

$$b = u^2 - v^2,\ c = 2uv + v^2,\ \text{and}\ a = u^2 + v^2 + uv.$$

For the above $u = 3, v = 2$ and $a = 19 = 3^2 + 2^2 + 2(3)$.

Let $a_1^2 = b_1^2 + c_1^2 + b_1c_1$ be another Diophantine equation which has solutions of the form $b_1 = u_1^2 - v_1^2$, $c_1 = 2u_1v_1 + v^2$, and $a_1 = u_1 + v_1^2 + u_1v_1$. Let u$_1$ be the largest and v_1 be the smallest of the numbers $\{b, c\}$. If $b = c$, the Diophantine equation becomes $a_1^2 = 3b_1^2$ which has no integer solutions. Suppose $c > b$. (If $b > c$, a procedure similar to that below can be used).

Let $u_1 = c$ and $v_1 = b$. Then $b_1 = c^2 - b^2$ and $c_1 = 2cb + b^2$. The expression $b_1^2 + c_1^2 + b_1c_1 = (c^2 - b^2)^2 + (2cb + b^2)^2 + (c^2 - b^2)(2cb + b^2) = (c^2 + b^2 + bc)^2 = (a^2)^2 = a^4 = a_1^2$. In this case $a_1 = a^2$.

Now start with the above solution where $a = 19, u = 3, v = 2, b = 5$, and $c = 16$. For $a = 19^2$, let $u = 16$ and $v = 5$. Then we have the solution $b = 231^2, c = 185$ where a$^2 = 19^4 = 231 + 185^2 + 231(185)$.

For $a = 19^4$, let $u = 231$ and $v = 185$. Then $b = 19136, c = 119695$ and $a^2 = 19^8 = 19136^2 + 119695^2 + 19136(119695)$. Since 19 is not a factor of the b and c solutions above, $(a, b, c) = 1$.

The solutions I have found are $(19, 5, 16)$, $(19^2, 231, 185)$, and $(19^4, 19136, 119695)$.

Solution 2 by Bruno Salguerio Fanego, Viveiro, Spain

If $\triangle ABC$ is such a triangle, by the cosine theorem $a^2 = b^2 + c^2 - 2bc\cos A$, that is

$$c^2 + bc + b^2 - a^2 = 0,\ c = \frac{-b \pm \sqrt{4a^2 - 3b^2}}{2}\ \text{and}\ 4a^2 - 3b^2$$

must be positive integers and the latter a perfect square, with $(a, b, c) = 1$.

When $a = 19$, $0 < b \leq 2 \cdot 19/\sqrt{3} \Rightarrow 0 < b \leq 21$; $4 \cdot 19^2 - 3b^2$ is a positive perfect square for $b \in \{2^4, 5\}$ so $c \in \{5, 2^4\}$, and $(a, b, c) = 1$.

When $a = 19^2$, $0 < b \leq 2 \cdot 19^2/\sqrt{3} \Rightarrow 0 < b \leq 416$; $4 \cdot 19^4 - 3b^2$ is a positive perfect square that is not a multiple of 19 for $b \in \{3 \cdot 7 \cdot 11,\ 5 \cdot 37\}$, so $c \in \{5 \cdot 37,\ 3 \cdot 7 \cdot 11\}$, and $(a.b.c) = 1$.

When $a = 19^4$, $0 < b \leq 2 \cdot 19^4/\sqrt{3} \Rightarrow 0 < b \leq 150481$; $4 \cdot 19^8 - 3b^2$ is a positive perfect square that is not a multiple of 19 for $b \in \{5 \cdot 37 \cdot 647,\ 2^6 \cdot 13 \cdot 23\}$. So $c \in \{2^6 \cdot 13 \cdot 23,\ 5 \cdot 37 \cdot 647\}$, and $(a, b, c) = 1$.

And reciprocally, the triangular inequalities are verified by $a = 19, 16, 5$, by $a = 19^2, 231, 185$, and by $a = 19^4$, $119695, 19136$, so there is a $\triangle ABC$ with sides a, b and c with these integer lengths, and with $\angle A = 120^o$, and $(a, b, c) = 1$.

Thus, if $a = 19$, then $\{b, c\} = \{5, 16\}$; if $a = 19^2$, then $\{b, c\} = \{185, 231\}$, and if $a = 19^4$, then $\{b, c\} = \{19136, 119695\}$.

Note: When $a = 19^2, 4 \cdot 19^4 - 3b^2$ is a perfect square for $b \in \{2^4 \cdot 19,\ \ 3 \cdot 7 \cdot 11,\ \ 5 \cdot 37,\ \ 5 \cdot 19\}$.

When $a = 19^4, 4 \cdot 19^8 - 3b^2$ is a perfect square for
$b \in \{5 \cdot 37 \cdot 647,\ 2^4 \cdot 19^3,\ 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 19,\ 3 \cdot 7 \cdot 11 \cdot 19^2,\ 5 \cdot 19^2 \cdot 37,\ 17 \cdot 19 \cdot 163,\ 5 \cdot 19^3,\ 2^6 \cdot 13 \cdot 23\}$.

Also solved by John Hawkins and David Stone (jointly), Statesboro, GA; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; David C.Wilson, Winston-Salem, NC, and the proposer.

- 5051: *Proposed by Kenneth Korbin, New York, NY*

Find four pairs of positive integers (x, y) such that $\dfrac{(x-y)^2}{x+y} = 8$ with $x < y$.

Find a formula for obtaining additional pairs of these integers.

Solution 1 by Charles McCracken, Dayton, OH

The given equation can be solved for y in term of x by expanding the numerator and multiplying by the denominator to get

$$x^2 - 2xy + y^2 = 8((x + y) \Longrightarrow y^2 - (2x + 8)y + (x^2 - 8x) = 0.$$

Solving this by the quadratic formula yields $y = x + 4 + 4\sqrt{x+1}$.

Since the problem calls for integers we choose values of x that will make $x + 1$ a square. Specifically

$$\begin{aligned} x &= 3, 8, 15, 24, 35, \cdots \text{or} \\ x &= k^2 + 2k,\ k \geq 1 \end{aligned}$$

The first four pairs are $(3, 15)$, $(8, 24)$, $(15, 35)$, $(24, 48)$.

In general, $x = k^2 + 2k$ and $y = k^2 + 6k + 8$, $k \geq 1$.

Solution 2 by Armend Sh. Shabani, Republic of Kosova

The pairs are $(3, 15), (8, 24), (15, 35), (24, 48)$. In order to find a formula for additional pairs we write the given relation $(y - x)^2 = 8(x + y)$ in its equivalent form $y - x = 2\sqrt{2(x + y)}$.

From this it is clear that $x+y$ should be of the form $2s^2$, and this gives the system of equations:

$$\begin{cases} x+y=2s^2 \\ y-x=4s \end{cases}$$

The solutions to this system are $x=s^2-2s,\ y=s^2+2s$, and since the solutions should be positive, we choose $s \geq 3$.

Solution 3 by Boris Rays, Brooklyn, NY

Let

$$\begin{cases} x+y=a \\ y-x=b \end{cases}$$

Since $x<y$ and a and b are positive integers, it follows that $b^2=8a$ and that b=$2\sqrt{2a}$. Since b is a positive integer we may choose values of a so that $2a$ is a perfect square. Specifically, let $a=2^{2n-1}$, where $n=1,2,3,\cdots$. Therefore, $2a=2\cdot 2^{2n-1}=2^{2n}=(2^n)^2$, where $n=1,2,3,\cdots$. Similarly, $b=2^{n+1}\ n=1,2,3,\cdots$.

Substituting these values of a and of b into the original system gives:

$$\begin{aligned} x &= \frac{2^{2n-1}-2^{n+1}}{2}=2^n(2^{n-2}-1) \\ y &= \frac{2^{2n-1}+2^{n+1}}{2}=2^n(2^{n-2}+1) \end{aligned}$$

and since we want $x,y>0$ we choose $n=3,4,5,\cdots$. The ordered triplets

$$(n,x,y):(3,8,24),(4,48,80),(5,224,288),(6,960,1088).$$

satisfy the problem. It can also be easily shown that our general values of x and y also satisfy the original equation.

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Pat Costello, Richmond, KY; Michael C. Faleski, University Center, MI; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Jahangeer Kholdi (with John Viands and Tyler Winn (students),Western Branch High School, Chesapeake, VA), Portsmouth, VA; Tuan Le (student, Fairmont, High School), Anaheim, CA; David E. Manes, Oneonta, NY; Melfried Olson, Honolulu, HI; Jaquan Outlaw (student, Heritage High School) Newport News, VA and Robert H. Anderson (jointly), Chesapeake, VA; Boris Rays, Brooklyn, NY; Vicki Schell, Pensacola, FL; David Stone and John Hawkins (jointly), Statesboro, GA; David C. Wilson, Winston-Salem, NC, and the proposer.

- 5052: *Proposed by Juan-Bosco Romero Márquez, Valladolid, Spain*

  If $a \geq 0$, evaluate:

$$\int_0^{+\infty} arctg\frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2}\ \frac{dx}{1+x^2}.$$

Solution by Kee-Wai Lau, Hong Kong, China

Denote the integral by I. We show that

$$I = \begin{cases} \dfrac{\pi}{4} arctg \dfrac{2a}{1-a^2}, & 0 \le a < 1; \\ \\ \dfrac{\pi^2}{8}, & a = 1; \\ \\ \dfrac{\pi}{4}\Big(\pi - arctg\dfrac{2a}{a^2-1} - 4 arctg \dfrac{\sqrt{a^4+a^2-1}-a}{1+a^2}\Big), & a > 1. \end{cases} \tag{1}$$

Let $J = \displaystyle\int_0^{+\infty} \frac{2a(ax^2+2x+a)arctg(x)}{(1+x^2)\Big((a^2+1)x^2+4ax+a^2+1\Big)}dx$. Integrating by parts, we see that for $0 \le a < 1$,

$$\begin{aligned} I &= \int_0^{+\infty} arctg\frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2} d(arctg(x)) \\ &= \Big[arctg\frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2} arctg(x)\Big]_0^{+\infty} \\ &- \int_0^{+\infty} arctg(x) d\Big(arctg\frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2}\Big) \\ &= J. \end{aligned}$$

For $a \ge 1$, let $r_a = \dfrac{\sqrt{a^4+a^2-1}-a}{1+a^2}$ be the non-negative root of the quadratic equation $(1+a^2)x^2+2ax+1-a^2=0$ so that

$$\begin{aligned} I &= \Big[arctg\frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2} arctg(x)\Big]_0^{r_a} \\ &+ \Big[arctg\frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2} arctg(x)\Big]_{r_a}^{+\infty} + J \\ &= -\pi arctg(r_a) + J. \end{aligned}$$

By substituting $x = \dfrac{1}{y}$ and making use of the fact that $arctg(1/y) = \dfrac{\pi}{2} - arctg(y)$ we obtain

$$J = 2a \int_0^{+\infty} \frac{(ay^2+2y+a)arctg(1/y)}{(1+y^2)\Big((a^2+1)y^2+4ay+a^2+1\Big)}dy$$

$$= \quad 2a\Big(\frac{\pi}{2}\int_0^{+\infty}\frac{(ay^2+2y+a)}{(1+y^2)\Big((a^2+1)y^2+4ay+a^2+1\Big)}dy\Big)-J$$

so that $J=\dfrac{\pi a}{2}\displaystyle\int_0^{+\infty}\frac{(ay^2+2y+a)}{(1+y^2)\Big((a^2+1)y^2+4ay+a^2+1\Big)}dy$. Resolving into partial fractions we obtain

$$J=\frac{\pi}{4}\Big(\int_0^{+\infty}\frac{dy}{1+y^2}+(a^2-1)\int_0^{+\infty}\frac{dy}{(1+a^2)y^2+4ay+1+a^2}\Big).$$

Clearly, $J=\dfrac{\pi^2}{8}$ for $a=1$. For $p>0,\ pr>q^2$, we have the well know result

$$\int_0^{+\infty}\frac{dy}{py^2+2qy+r}=\frac{1}{\sqrt{pr-q^2}}arctg\frac{q}{\sqrt{pr-q^2}},$$

so that for $a\geq 0,\ a\neq 1$

$$J=\frac{\pi}{4}\Big(\frac{\pi}{2}+\frac{a^2-1}{|a^2-1|}arctg\frac{2a}{|a^2-1|}\Big).$$

Hence (1) follows and this completes the solution.

Also solved by Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy, and the proposer.

- 5053: *Proposed by Panagiote Ligouras, Alberobello, Italy*

Let a, b and c be the sides, r the in-radius, and R the circum-radius of $\triangle ABC$. Prove or disprove that

$$\frac{(a+b-c)(b+c-a)(c+a-b)}{a+b+c}\leq 2rR.$$

Solution by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Roger Zarnowski (jointly), San Angelo, TX

The given inequality is essentially the same as Padoa's Inequality which states that

$$abc\geq(a+b-c)(b+c-a)(c+a-b),$$

with equality if and only if $a=b=c$. We will prove this using the approach presented in [1]. Let $x=\dfrac{a+b-c}{2}$, $y=\dfrac{b+c-a}{2}$, and $z=\dfrac{c+a-b}{2}$. Then, $x,y,z>0$ by the Triangle Inequality and $a=x+z$, $b=x+y$, $c=y+z$. By the Arithmetic - Geometric Mean Inequality,

$$\begin{aligned} abc &= (x+z)(x+y)(y+z)\\ &\geq (2\sqrt{xz})(2\sqrt{xy})(2\sqrt{yz})\\ &= (2x)(2y)(2z)\\ &= (a+b-c)(b+c-a)(c+a-b), \end{aligned}$$

with equality if and only if $x=y=z$, i.e., if and only if $a=b=c$.

If $A = Area(\triangle ABC)$ and $s = \dfrac{a+b+c}{2}$, then

$$R = \frac{abc}{4A} \quad \text{and} \quad A = rs = r\left(\frac{a+b+c}{2}\right),$$

which imply that $2rR = \dfrac{abc}{a+b+c}$. Hence, the problem reduces to Padoa's Inequality.

Reference:

[1] R. B. Nelsen, Proof Without Words: Padoa's Inequality, **Mathematics Magazine** 79 (2006) 53.

Also solved by Scott H. Brown, Montgomery, AL; Michael Brozinsky, Central Islip, NY; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Tuan Le (student, Fairmont High School), Anaheim, CA; David E. Manes, Oneonta, NY; Manh Dung Nguyen (student, Special High School for Gifted Students), HUS, Vietnam; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.)

- 5054: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let x, y, z be positive numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{x^2+xy+y^2} + \frac{y^3}{y^2+yz+z^2} + \frac{z^3}{z^2+zx+x^2} \geq 1.$$

Solution 1 by Ovidiu Furdui, Campia Turzii, Cluj, Romania

First we note that if a and b are two positive numbers then the following inequality holds

$$\frac{a^2-ab+b^2}{a^2+ab+b^2} \geq \frac{1}{3} \qquad (1).$$

Let

$$S = \frac{x^3}{x^2+xy+y^2} + \frac{y^3}{y^2+yz+z^2} + \frac{z^3}{z^2+zx+x^2}.$$

We have,

$$\begin{aligned} S &= \frac{x^3-y^3+y^3}{x^2+xy+y^2} + \frac{y^3-z^3+z^3}{y^2+yz+z^2} + \frac{z^3-x^3+x^3}{z^2+zx+x^2} \\ &= (x-y) + \frac{y^3}{x^2+xy+y^2} + (y-z) + \frac{z^3}{y^2+yz+z^2} + (z-x) + \frac{x^3}{z^2+zx+x^2} \\ &= \frac{y^3}{x^2+xy+y^2} + \frac{z^3}{y^2+yz+z^2} + \frac{x^3}{z^2+zx+x^2}. \end{aligned}$$

It follows, based on (1), that

$$\begin{aligned} S &= \frac{1}{2}(S+S) \\ &= \frac{1}{2}\left(\frac{x^3+y^3}{x^2+xy+y^2} + \frac{y^3+z^3}{y^2+yz+z^2} + \frac{z^3+x^3}{z^2+zx+x^2}\right) \\ &= \frac{1}{2}\left((x+y)\frac{x^2-xy+y^2}{x^2+xy+y^2} + (y+z)\frac{y^2-yz+z^2}{y^2+yz+z^2} + (z+x)\frac{z^2-xz+x^2}{z^2+zx+x^2}\right) \end{aligned}$$

$$\geq \frac{1}{2}\left(\frac{x+y}{3}+\frac{y+z}{3}+\frac{z+x}{3}\right)$$
$$= \frac{x+y+z}{3} \geq \sqrt[3]{xyz} = 1, \text{ and the problem is solved.}$$

Solution 2 by Manh Dung Nguyen (student, Special High School for Gifted Students) HUS, Vietnam

Firstly, we have,

$$\sum \frac{x^3-y^3}{(x^2+xy+y^2)} = \sum \frac{(x-y)(x^2+xy+y^2)}{(x^2+xy+y^2)} = \sum (x-y) = 0.$$

Hence,

$$\sum \frac{x^3}{x^2+xy+y^2} = \sum \frac{y^3}{x^2+xy+y^2}.$$

So it suffices to show that,

$$\sum \frac{x^3+y^3}{x^2+xy+y^2} \geq 2.$$

On the other hand,

$$3(x^2-xy+y^2)-(x^2+xy+y^2) = 2(x-y)^2 \geq 0.$$

Thus,

$$\sum \frac{x^3+y^3}{x^2+xy+y^2} = \sum \frac{(x+y)(x^2-xy+y^2)}{x^2+xy+y^2} = \sum \frac{x+y}{3} = \frac{2(x+y+z)}{3}.$$

By the AM-GM Inequality, we have,

$$x+y+z \geq 3\sqrt[3]{xyz} = 3,$$

so we are done.
Equality hold if and only if $x=y=z=1$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

It can be checked readily that,

$$\frac{x^3}{x^2+xy+y^2} = \frac{(2x-y)}{3} + \frac{(x+y)(x-y)^2}{3(x^2+xy+y^2)} \geq \frac{(2x-y)}{3}.$$

Similarly, $\dfrac{y^3}{y^2+yz+z^2} \geq \dfrac{(2y-z)}{3}$, $\dfrac{z^3}{z^2+zx+x^2} \geq \dfrac{(2z-x)}{3}$.

Hence by the arithmetic mean-geometric mean inequality, we have:

$$\frac{x^3}{x^2+xy+y^2} + \frac{y^3}{y^2+yz+z^2} + \frac{z^3}{z^2+zx+x^2}$$

$$\geq \frac{x+y+z}{3}$$

$$\geq \sqrt[3]{xyz}$$

$$= 1.$$

Also solved by Dionne Bailey, Elsie Campbell and Charles Diminnie (jointly), San Angelo, TX; Scott H. Brown, Montgomery, AL; Michael Brozinsky, Central Islip, NY; Bruno Salgueiro Fanego, Viveiro, Spain; Tuan Le (student, Fairmont High School), Anaheim, CA; Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy; Boris Rays, Brooklyn, NY; Armend Sh. Shabani, Republic of Kosova, and the proposer.

- 5055: *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania*

Let α be a positive real number. Find the limit

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k^{\alpha}}.$$

Solution 1 by Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy

Answer:

$$\text{The limit is} \begin{cases} 0, & \text{if } \alpha > 1; \\ 1, & \text{if } 0 < \alpha < 1; \\ \ln 2, & \text{if } \alpha = 1. \end{cases}$$

Proof: Let $\alpha > 1$.

Writing $k^{\alpha} = \sum_{i=1}^{N}\frac{k^{\alpha}}{N}$, by the AGM we have

$$\frac{1}{n+k^{\alpha}} = \frac{1}{\frac{n}{2}+\frac{n}{2}+\frac{k^{\alpha}}{N}+\ldots+\frac{k^{\alpha}}{N}} \leq \frac{1}{\frac{n}{2}+\left(\frac{n}{2}\frac{k^{\alpha N}}{N^N}\right)^{\frac{1}{N+1}}}$$

$$= \frac{1}{\frac{n}{2}+\frac{n^{\frac{1}{N+1}}k^{\frac{\alpha N}{N+1}}}{2^{\frac{1}{N+1}}N^{\frac{N}{N+1}}}} \leq \frac{1}{n^{\frac{1}{N+1}}\left(\frac{1}{2}+\frac{k^{\frac{\alpha N}{N+1}}}{2^{\frac{1}{N+1}}N^{\frac{N}{N+1}}}\right)}$$

and we observe that $\alpha N/(N+1) > 1$ if $N > 1/(\alpha-1)$. Thus we write

$$0 < \sum_{k=1}^{n}\frac{1}{n+k^{\alpha}} \leq n^{-1/(N+1)}\sum_{k=1}^{\infty}\frac{1}{\left(\frac{1}{2}+\frac{k^{\frac{\alpha N}{N+1}}}{2^{\frac{1}{N+1}}N^{\frac{N}{N+1}}}\right)}$$

The series converges and the limit is zero.

Let $\alpha < 1$. Trivially we have $\sum_{k=1}^{n}\frac{1}{n+k^{\alpha}} \leq \sum_{k=1}^{n}\frac{1}{n} = 1.$

Moreover,

$$\sum_{k=1}^{n}\frac{1}{n+k^{\alpha}} \geq \sum_{k=1}^{n}\frac{1}{n}\frac{1}{1+\frac{k^{\alpha}}{n}} \geq \sum_{k=1}^{n}\frac{1}{n}(1-\frac{k^{\alpha}}{n}) = 1-\sum_{k=1}^{n}\frac{k^{\alpha}}{n^2} \geq 1-\frac{n^{1+\alpha}}{n^2},$$

$1 \geq (1-x^2)$ has been used. By comparison the limit equals one since

$$1 \leq \sum_{k=1}^{n} \frac{1}{n+k^\alpha} \leq 1 - \frac{n^{1+\alpha}}{n^2}$$

The last step is $\alpha = 1$. We need the well known equality $H_n \approx \sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + o(1)$ and then

$$\sum_{k=1}^{n} \frac{1}{n+k} = \sum_{k=n+1}^{2n} (H_{2n} - H_n) = \ln(2n) - \ln n + o(1) \to \ln 2$$

The proof is complete.

Solution 2 by David Stone and John Hawkins, Statesboro, GA

Below we show that for $0 < \alpha < 1$, the limit is 1; for $\alpha = 1$, the limit is $\ln 2$; and for $\alpha > 1$, the limit is 0.

For $\alpha = 1$ we get

$$\int_0^1 \frac{1}{1+u} du \geq \sum_{k=1}^{n} \frac{1}{n+k} \geq \int_{1/n}^{(n+1)/n} \frac{1}{1+u} du.$$

Since $\frac{1}{2} \leq \frac{1}{1+u} \leq 1$, we know that the limit exists as n approaches infinity and is given by

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^\alpha} = \int_0^1 \frac{1}{1+u} du = \ln(1+u)\Big|_0^1 = \ln 2 - \ln 1 = \ln 2.$$

Next suppose $\alpha < 1$. Then

$$0 < k^\alpha \leq n^\alpha \text{ for } 1 \leq k \leq n, \text{ so}$$

$$\begin{aligned}
n < n + k^\alpha &\leq n + n^\alpha \quad \text{and} \\
\frac{1}{n+n^n} &\leq \frac{1}{n+k^\alpha} < \frac{1}{n}. \quad \text{Thus,} \\
\sum_{k=1}^{n} \frac{1}{n+n^\alpha} &\leq \sum_{k=1}^{n} \frac{1}{n+k^\alpha} < \sum_{k=1}^{n} \frac{1}{n} = 1, \quad \text{or} \\
\frac{n}{n+n^\alpha} &\leq \sum_{k=1}^{n} \frac{1}{n+k^\alpha} < 1. \quad \text{Hence,} \\
\lim_{n\to\infty} \frac{n}{n+n^\alpha} &\leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^\alpha} \leq 1 \\
\lim_{n\to\infty} \frac{1}{1+\alpha n^{\alpha-1}} &\leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^\alpha} \leq 1. \quad \text{But,} \\
\lim_{n\to\infty} \frac{1}{1+\alpha n^{\alpha-1}} &= 1, \text{ since } \alpha - 1 < 0. \text{ Therefore,} \\
\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^\alpha} &= 1.
\end{aligned}$$

Finally, suppose $\alpha > 1$.

We note that $\dfrac{1}{n+k^\alpha}$ is a decreasing function of k and as a result we can write

$$0 \leq \sum_{k=1}^{\infty} \frac{1}{n+k^\alpha} \leq \int_0^n \frac{1}{n+k^\alpha} dk = \frac{1}{n}\int_0^1 \frac{1}{1+\dfrac{k^n}{n^{\alpha/\alpha}}} dk.$$

Using the substitution $u = \dfrac{k}{u^{1/\alpha}}$ with $du = \dfrac{1}{n^{1/\alpha}} dk$, the above becomes,

$$\begin{aligned}
0 \leq \sum_{k=1}^{n} \frac{1}{n+k^\alpha} &\leq \frac{n^{1/\alpha}}{n}\int_0^{n^{(n-1)/n}} \frac{1}{1+u^\alpha} du = \frac{1}{n^{(\alpha-1)/\alpha}}\int_0^{n^{(n-1)/\alpha}} \frac{1}{1+u^\alpha} du \\
&\leq \frac{1}{n^{(\alpha-1)/\alpha}}\int_0^{n} \frac{1}{1+u^\alpha} du \\
&\leq \frac{1}{n^{(\alpha-1)}\alpha}\int_0^{1} \frac{1}{1+u^\alpha} du + \frac{1}{n^{(\alpha-1)/\alpha}}\int_1^{n} \frac{1}{1+u^\alpha} du \\
&\leq \frac{1}{n^{(\alpha-1)/\alpha}}(1) + \frac{1}{n^{(\alpha-1)/\alpha}}\int_1^{n} \frac{1}{1+u} du \\
&= \frac{1}{n^{(\alpha-1)/\alpha}}(1) + \frac{1}{n^{(\alpha-1)/\alpha}}(1)\Big[\ln(1+n) - \ln 2\Big].
\end{aligned}$$

That is,

$$0 \leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^\alpha} \leq \lim_{n\to\infty} \frac{1}{n^{(\alpha-1)/\alpha}} + \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{2}\right)}{n^{(\alpha-1)/\alpha}}.$$

Using L'Hospital's rule repeatedly we get,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n^{(\alpha-1)/\alpha}} + \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{2}\right)}{n^{(\alpha-1)/\alpha}} &= 0 + \lim_{n\to\infty} \frac{\frac{2}{n+1}}{\left(\frac{\alpha-1}{\alpha}\right)n^{-1/\alpha}} \\
&= \lim_{n\to\infty} \frac{2\alpha n^{1/\alpha}}{(\alpha-1)(n+1)} \\
&= \lim_{n\to\infty} \frac{2}{(\alpha-1)(n)^{1-1/\alpha}} \\
&= 0.
\end{aligned}$$

Thus, $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^\alpha} = 0$ for $\alpha > 1$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^\alpha} = \begin{cases} 1, & 0<\alpha<1; \\ \ln 2, & \alpha = 1; \\ 0, & \alpha > 1. \end{cases}$

For $0 < \alpha < 1$, we have

$$\frac{1}{1+n^{\alpha-1}} = \sum_{k=1}^{n} \frac{1}{n+n^{\alpha}} \leq \sum_{k=1}^{n} \frac{1}{n+k^{\alpha}} < \sum_{k=1}^{n} \frac{1}{n} = 1 \text{ and so } \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^{\alpha}} = 1.$$

For $\alpha = 1$ we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^{\alpha}} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \frac{1}{(1+k/n)} = \int_0^1 \frac{dx}{1+x} = \ln 2.$$

For $\alpha > 1$, let t be any real number satisfying $\frac{1}{\alpha} < t < 1$ and let $m = \lfloor n^t \rfloor$.

We have

$$0 < \sum_{k=1}^{n} \frac{1}{n+k^{\alpha}} = \sum_{k=1}^{m} \frac{1}{n+k^{\alpha}} + \sum_{k=m+1}^{n} \frac{1}{n+k^{\alpha}} < \frac{m}{n} + \frac{n-m}{(m+1)^{\alpha}} \leq \frac{1}{n^{1-t}} + \frac{1}{n^{\alpha t-1}},$$

which tends to 0 as n tends to infinity. It follows that $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^{\alpha}} = 0.$

This completes the solution.

Also solved by Valmir Krasniqi, Prishtina, Kosova, and the proposer.