Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before May 15, 2014

- **5295:** *Proposed by Kenneth Korbin, New York, NY*

  A convex cyclic hexagon has sides

  $$\left(5,\ 7\sqrt{17},\ 23\sqrt{13},\ 25\sqrt{13},\ 25\sqrt{17},\ 45\right).$$

  Find the diameter of the circumcircle and the area of the hexagon.

- **5296:** *Proposed by Roger Izard, Dallas, TX*

  Consider the "Star of David," a six pointed star made by overlapping the triangles ABC and FDE. Let

  $$\overline{AB} \cap \overline{DF} = G, \text{ and } \overline{AB} \cap \overline{DE} = H,$$
  $$\overline{AC} \cap \overline{DF} = L, \text{ and } \overline{AC} \cap \overline{FE} = K,$$
  $$\overline{BC} \cap \overline{DE} = I, \text{ and } \overline{BC} \cap \overline{FE} = J,$$

  in such a way that:

  $$\frac{CK}{AC} = \frac{EI}{DE} = \frac{BI}{BC} = \frac{GD}{DF} = \frac{AG}{AB} = \frac{FK}{EF} \text{ and}$$

  $$\frac{AL}{AC} = \frac{DH}{DE} = \frac{BH}{AB} = \frac{EJ}{EF} = \frac{FL}{DF} = \frac{CJ}{CB}.$$

  Let $r = \dfrac{CK}{AC}$ and let $p = \dfrac{AL}{AC}$. Prove that $r + p = \dfrac{3pr+1}{2}$.

- **5297:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  Let $s_n = n^2,\ t_n = \dfrac{n(n+1)}{2},\ p_n = \dfrac{n(3n-1)}{2}$, for positive integers n, be the square, triangular and pentagonal numbers respectively. Prove, independently of each other, that

$$\begin{aligned} i) \ \ t_a + p_b &= t_c \\ ii) \ \ t_a + s_b &= p_c \\ iii) \ \ p_a + s_b &= s_c, \end{aligned}$$

for infinitely many positive integers, a, b, and c.

- **5298:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

  Let $(a_n)_{n\geq 1}$ be an arithmetic progression and m a positive integer. Calculate:

$$\lim_{n\to\infty}\left(\left(\sum_{k=1}^{m}\left(1+\frac{1}{n}\right)^{n+a_k} - me\right)n\right).$$

- **5299:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Without the aid of a computer, show that

$$\ln^2 2\int_0^1 \frac{x^{3/2}2^x \sin x}{(1+x\ln 2)^2}\,dx \ \geq\ \frac{1-\ln 2}{1+\ln 2}\int_0^1 \sqrt{x}\ \sin x\ dx.$$

- **5300:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $n \geq 1$ be an integer. Prove that

$$\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^{2n} x} = \sum_{k=0}^{n-1}\binom{n-1}{k}\cdot\frac{1}{2n-2k-1}.$$

Solutions

- **5277:** *Proposed by Kenneth Korbin, New York, NY*

  Find x and y if a triangle with sides $(2013, 2013, x)$ has the same area and the same perimeter as a triangle with sides $(2015, 2015, y)$.

  Solution 1 by Carl Libis, Lane College, Jackson, TN

  The perimeter of $(2013, 2013, x)$ equals the perimeter of $(2015, 2015, y)$ implies that $x = y + 4$.

  Also, the altitude h_1 of $(2013, 2013, y + 4)$ bisects $y + 4$.

Use the Pythagorean Theorem on right triangle $(2013, h_1, (y+4)/2)$ to obtain $h_1 = \sqrt{2013^2 - (2 + y/2)^2}$. Similarly for altitude h_2 of $(2015, 2015, y)$ we obtain $h_2 = \sqrt{2015^2 - (y/2)^2}$.

Equal areas implies that

$$\left(2+\frac{y}{2}\right)\sqrt{2013^2-\left(2+\frac{y}{2}\right)^2} = \frac{y}{2}\sqrt{2015^2-\left(\frac{y}{2}\right)^2}.$$

Square both sides, simplify, and then factor to obtain

$$\begin{aligned} 0 &= y^3 + 2020y^2 - 81043224y - 16208660 \\ &= (y+4030)(y^2 - 2010y - 4022) \\ &= (y+4030)(y^2 - 2010y - 4022) \\ &= (y+4030)\left(y - 1005 - \sqrt{1014047}\right)\left(y - 1005 + \sqrt{1014047}\right). \end{aligned}$$

The only positive solution of the three solutions is $y = 1005 + \sqrt{1014047} \approx 2012$.

Thus the values are: $y \approx 2012$ and $x \approx 2016$.

Solution 2 by proposer

The method to obtain x and y is to solve the system of equations:

$$\begin{cases} \dfrac{2y^2+8y+12}{y+2} & = 2013+2015, \quad \text{and} \\ x = y+4. \end{cases}$$

If a triangle with sides (a, a, b) has the same area and the same perimeter as a triangle with sides (c, c, d), where a, b, c and d are positive integers, then the value of the area and the perimeter can be expressed in terms of b and d. Namely,

$$\begin{aligned} \text{Area} &= \frac{bd\sqrt{b^2+bd+d^2}}{2b+2d} \\ \text{Perimeter} &= \frac{2b^2+2bd+2d^2}{b+d}. \end{aligned}$$

Comment by **David Stone and John Hawkins, Georgia Southern University, Statesboro, GA.** More generally, if we let $k > 2$ be some positive constant and enforce the same "equal-area and equi-perimeter" condition on the two triangles (k, k, x) and $(k+2, k+2, y)$, we find the single solution

$$y = \frac{k-3+\sqrt{(k+1)^2-8}}{2} \text{ and } x = y+4 = \frac{k+5+\sqrt{(k+1)^2-8}}{2}.$$

Also solved by Dionne Bailey, Elsie Camjpbell, and Charles Diminnie, Angelo State University, TX; Brian D. Beasely, Presbyterian College, Clinton, SC; D. M. Batinetu-Giurgiu, Bucharest, Romania, Neculai Stanciu, Buza, Romania, and Titu Zvonaru, Comanesi, Romania; Bruno Salgueiro Fanego, Viveiro, Spain; Michael Fried, Ben-Gurion University, Beer-Sheva, Israel; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain, and the proposer.

- **5278:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

The triangular numbers $6 = (2)(3)$ and $10 = (2)(5)$ are each twice a prime number. Find all triangular numbers that are twice a prime.

Solution 1 by Ed Gray, Highland Beach, FL

The triangular numbers are given by: (1) $T_n = \dfrac{n(n+1)}{2}$, so if a triangular number is double a prime p, we must have the following equation: (2) $\dfrac{n(n+1)}{2} = 2p$.

First, suppose n is an even integer. Then $n = 2k$ for some integer k, and $\dfrac{n(n+1)}{2}$ becomes $\dfrac{2k(2k+1)}{2} = k(2k+1)$. If $k(2k+1) = 2p$, then k must be even, say $k = 2r$ and $k(2k+1) = 2r(4r+1) = 2p$. So, $r(4r+1) = p$. But p is prime and this implies that $r = 1, k = 2, n = 4$ and $\dfrac{(n)(n+1)}{2} = 10$.

Second, If n is odd, let $n = 2k+1$; then

$$\frac{n(n+1)}{2} = \frac{(2k+1)(2k+2)}{2} = (2k+1)(k+1) = 2p.$$

Here, $k+1$ must be even, say $k+1 = 2r$, and $(2k+1)(k+1) = 2r(4r-1) = 2p$. Since p is prime, $r = 1, k = 1, n = 3$ and $\dfrac{n(n+1)}{2} = 6$. Hence, all relevant triangular numbers were given in the statement of the problem.

Solution 2 by Paul M. Harms, North Newton, KS

Triangular numbers have the form $\dfrac{n(n+1)}{2}$ where n is a positive integer. For each positive integer n either n or $n+1$ has a factor of 2. When n is a positive integer greater then 4, the number $n, (n+1), \dfrac{n}{2}$, and $\dfrac{n+1}{2}$ are all greater than 2.

When $n > 4$, and an even integer, then $\dfrac{n}{2}$, is a prime number greater than 2 or a product of prime numbers, and $n+1$ is also a prime number greater than 2 or a product of prime numbers. In this case, $\dfrac{n}{2}(n+1)$ cannot be two times one prime number.

Similarly, when $n > 4$ and an odd number, n as well as $\dfrac{n+1}{2}$ are prime numbers greater than 2 or are a product of prime numbers. Then $n\dfrac{(n+1)}{2}$ cannot be two times one prime number.

The triangular numbers that are twice a prime must come from positive integers n which are not greater than 4. We see that the triangular numbers 6 when $n=3$ and 10 when $n=4$ are the only triangular numbers which are twice a prime number.

Also solved by Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo State University, San Angelo, TX; Brian D. Beasely, Presbyterian College, Clinton, SC; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Neculai Stanciu and Titu Zvonaru, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5279:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania*

Let $f : \Re_+ \longrightarrow \Re_+$ be a convex function on $\Re_+$, where $\Re_+$ stands for the positive real numbers. Prove that

$$3\left(f^2(x)+f^2(y)+f^2(z)\right)-9f^2\left(\frac{x+y+z}{3}\right) \geq (f(x)-f(y))^2+(f(y)-f(z))^2+(f(z)-f(x))^2.$$

Solution 1 by Arkady Alt, San Jose, CA

Since

$$\begin{aligned} & 3\left(f^2(x)+f^2(y)+f^2(z)\right)-(f(x)-f(y))^2+(f(y)-f(z))^2+(f(z)-f(x))^2 \\ = \ & (f(x)+f(y)+f(z))^2, \end{aligned}$$

the original inequality is equivalent to

$$(f(x)+f(y)+f(z))^2 \geq 9f^2\left(\frac{x+y+z}{3}\right) \iff \frac{f(x)+f(y)+f(z)}{3} \geq f\left(\frac{x+y+z}{3}\right),$$

where the latter inequality is Jensen's Inequality for the convex function $f : \Re_+ \longrightarrow \Re_+$.

Solution 2 by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

Since f is convex, then $f\left(\dfrac{x+y+z}{3}\right) \leq \dfrac{f(x)+f(y)+f(z)}{3}$ and the left-hand side of the given inequality is

$$\begin{aligned} LHS &\geq 3\left(f^2(x)+f^2(y)+f^2(z)\right)-(f(x)+f(y)+f(z))^2 \\ &= 2\left(f^2(x)+f^2(y)+f^2(z)\right)-(2f(x)f(y)+2f(y)f(z)+2f(z)f(x)) \\ &= (f(x)-f(y))^2+(f(y)-f(z))^2+(f(z)-f(x))^2. \end{aligned}$$

Solution 3 by Michael Brozinsky, Central Islip, NY

Since f is convex we know that if $a \leq b$ and $0 < t < 1$ that

$$f\left(t \cdot a + (1-t) \cdot b\right) \leq t \cdot f(a) + (1-t) \cdot f(b).$$

(See, for example, the Chord Theorem in Calculus with Analytic Geometry (1978) by Flanders and Price, pages 153-154.)

Without loss of generality, let $0 < x \leq y \leq z$ and since $x \leq \dfrac{y+z}{2}$, we have, using the above result twice that:

$$\begin{aligned} f\left(\frac{x+y+z}{3}\right) = f\left(\frac{1}{3} \cdot x + \frac{2}{3} \cdot \left(\frac{y+z}{2}\right)\right) &\leq \frac{1}{3} \cdot f(x) + \frac{2}{3} \cdot \left(\frac{y+z}{2}\right) \\ &\leq \frac{1}{3} \cdot f(x) + \frac{2}{3} \cdot \left(\frac{1}{2} \cdot f(z) + \frac{1}{2} \cdot f(z)\right) \\ &= \frac{f(x) + f(y) + f(z)}{3}. \end{aligned}$$

Hence, $f(x) + f(y) + f(z) \geq 3 \cdot f\left(\dfrac{x+y+z}{3}\right)$ where the right hand side is positive by definition of f.

Squaring both sides gives

$$f^2(x) + f^2(y) + f^2(z) + 2 \cdot f(x) \cdot f(y) + 2 \cdot f(x) \cdot f(z) + 2 \cdot f(y) \cdot f(z) - 9 \cdot f^2\left(\frac{x+y+z}{3}\right) \geq 0,$$

which is clearly equivalent to the inequality to be proved.

Also solved by Ed Gray, Highland Beach, FL; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kee-Wai Lau, Hong Kong, China; Adrian Naco, Polytechnic University, Tirana, Albania; Titu Zvonaru, Comănesti, Romania, and the proposers.

- **5280:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $a \geq b \geq c$ be nonnegative real numbers. Prove that

$$\frac{1}{3}\left(\frac{(a+b)(c+a)}{2+\sqrt{a+b}} + \frac{(c+a)(b+c)}{2+\sqrt{c+a}} + \frac{(b+c)(a+b)}{2+\sqrt{b+c}}\right) \leq \frac{(a+b)^2}{2+\sqrt{b+c}}.$$

Solution 1 by Greg Cook, Student, Angelo State University, San Angelo,TX

First, since $a \geq b \geq c \geq 0$, then $(a+b)(c+a) \leq (a+b)^2$ and $2 + \sqrt{a+b} \geq 2 + \sqrt{b+c}$. Then,

$$\frac{(a+b)(c+a)}{2+\sqrt{a+b}} \leq \frac{(a+b)^2}{2+\sqrt{b+c}}. \qquad (1)$$

Again since $a \geq b \geq c \geq 0$, then $(c+a)(b+c) \leq (a+b)^2$ and $2 + \sqrt{c+a} \geq 2 + \sqrt{b+c}$. Then,

$$\frac{(c+a)(b+c)}{2+\sqrt{c+a}} \le \frac{(a+b)^2}{2+\sqrt{b+c}}. \qquad (2)$$

Also, since $a \ge b \ge c \ge 0$, then $(b+c)(a+b) \le (a+b)^2$. Then,

$$\frac{(b+c)(a+b)}{2+\sqrt{b+c}} \le \frac{(a+b)^2}{2+\sqrt{b+c}}. \qquad (3)$$

Combining (1), (2), and (3),

$$\frac{(a+b)(c+a)}{2+\sqrt{a+b}} + \frac{(c+a)(b+c)}{2+\sqrt{c+a}} + \frac{(b+c)(a+b)}{2+\sqrt{b+c}} \le 3\left(\frac{(a+b)^2}{2+\sqrt{b+c}}\right).$$

Finally,

$$\frac{1}{3}\left(\frac{(a+b)(c+a)}{2+\sqrt{a+b}} + \frac{(c+a)(b+c)}{2+\sqrt{c+a}} + \frac{(b+c)(a+b)}{2+\sqrt{b+c}}\right) \le \frac{(a+b)^2}{2+\sqrt{b+c}}.$$

Solution 2 by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

The inequality is a consequence of the Chebyshev's sum inequality. Note that sequences $(a+b)(c+a),\ (c+a)(b+c),\ (b+c)(a+b)$ and $\frac{1}{2+\sqrt{a+b}},\ \frac{1}{2+\sqrt{c+a}},\ \frac{1}{2+\sqrt{b+c}}$ are oppositely sorted. Therefore, the left-hand side of the given inequality LHS is bounded as

$$\begin{aligned} LHS &\le \frac{1}{3}\left((a+b)(c+a)+(c+a)(b+c)+(b+c)(a+b)\right) \\ &\quad \frac{1}{3}\left(\frac{1}{2+\sqrt{a+b}}+\frac{1}{2+\sqrt{c+a}}+\frac{1}{2+\sqrt{b+c}}\right) \\ &\le (a+b)(c+a)\frac{1}{2+\sqrt{b+c}} \\ &\le \frac{(a+b)^2}{2+\sqrt{b+c}}. \end{aligned}$$

Solution 3 by Arkady Alt, San Jose, CA

Note that:

1. $c \le b \iff c+a \le a+b \iff \dfrac{(a+b)(c+a)}{2+\sqrt{a+b}} \le \dfrac{(a+b)^2}{2+\sqrt{a+b}}$ and
$c \le a \iff 2+\sqrt{b+c} \le 2+\sqrt{a+b} \iff \dfrac{(a+b)^2}{2+\sqrt{a+b}} \le \dfrac{(a+b)^2}{2+\sqrt{b+c}}$ yields
$\dfrac{(a+b)(c+a)}{2+\sqrt{a+b}} \le \dfrac{(a+b)^2}{2+\sqrt{b+c}}$;

2. $\begin{cases} a+b \ge c+a \\ a+b \ge b+c \end{cases}$ $\dfrac{(c+a)(b+c)}{2+\sqrt{c+a}} \le \dfrac{(a+b)^2}{2+\sqrt{c+a}}$ and $2+\sqrt{c+a} \ge 2+\sqrt{b+c}$
yields $\dfrac{(c+a)(b+c)}{2+\sqrt{c+a}} \le \dfrac{(a+b)^2}{2+\sqrt{b+c}}$;

3. $\dfrac{(b+c)\,(a+b)}{2+\sqrt{b+c}} \le \dfrac{(a+b)^2}{2+\sqrt{b+c}} \iff b+c \le a+b \iff c \le a.$

Then $\dfrac{1}{3}\left(\dfrac{(a+b)\,(c+a)}{2+\sqrt{a+b}} + \dfrac{(c+a)\,(b+c)}{2+\sqrt{c+a}} + \dfrac{(b+c)\,(a+b)}{2+\sqrt{b+c}}\right) \le$
$\dfrac{1}{3} \cdot 3\ frac(a+b)^2 2+\sqrt{b+c} = \dfrac{(a+b)^2}{2+\sqrt{b+c}}.$

Solution 4 by Michael Brozinsky, Central Islip, NY

Denote the left hand side and right hand side of the given inequality by L and R respectively. The inequality will be established if we can show the maximum value of L and the minimum value of R are equal to one another. Specifically, we will show that $\max L = \min R = \dfrac{4a^2}{2+2\sqrt{2a}}$, and that this occurs when $a=b=c$.

If we differentiate L, with respect to b we obtain

$$\frac{\partial}{\partial b}\left(\frac{1}{3}\left(\frac{(a+b)\cdot(c+a)}{2+\sqrt{a+b}} + \frac{(c+a)\cdot(b+c)}{2+\sqrt{c+a}} + \frac{(b+c)\cdot(a+b)}{2+\sqrt{b+c}}\right)\right) = \frac{1}{3}\cdot(A+B) \text{ where}$$

$$\begin{aligned} A &= \frac{c+a}{2+\sqrt{a+b}} - \frac{1}{2}\frac{\sqrt{a+b}\ (c+a)}{(2+\sqrt{a+b})^2} + \frac{c+a}{2+\sqrt{a+b}} \\ &= \frac{1}{2}\frac{(c+a)\left(16+4\sqrt{c+a}+10\sqrt{a+b}+\sqrt{a+b}\sqrt{c+a}+2a+2b\right)}{\left(2+\sqrt{a+b}\right)^2(2+\sqrt{c+a})} \end{aligned}$$

and

$$\begin{aligned} B &= \frac{a+b}{2+\sqrt{b+c}} + \frac{b+c}{2+\sqrt{b+c}} - \frac{1}{2}\frac{\sqrt{b+c}\ (a+b)}{(2+\sqrt{b+c})^2} \\ &= \frac{1}{2}\,\frac{4a+a\sqrt{b+c}+8b+3b\sqrt{b+c}+4c+2c\sqrt{b+c}}{\left(2+\sqrt{b+c}\right)^2}. \end{aligned}$$

Since A and B are clearly non-negative and since $a \ge b \ge c$ we have L increases with b and so has its maximum when $b=a$.

Replacing b by a in L (call this expression M) and differentiating with respect to c gives

$$\frac{\partial}{\partial c}(M) = \frac{\partial}{\partial c}\left(\frac{1}{3}\left(\frac{2a(c+a)}{2+\sqrt{2a}} + \frac{(c+a)^2}{2+\sqrt{c+a}} + \frac{2(c+a)a}{2+\sqrt{c+a}}\right)\right)$$

$$= \frac{2}{3}\left(\frac{a}{2+\sqrt{2a}}\right) + \frac{2}{3}\left(\frac{c+a}{2+\sqrt{c+a}}\right) - \frac{1}{6}\frac{(c+a)\sqrt{c+a}}{\left(2+\sqrt{c+a}\right)^2}$$

$$+\quad \frac{2}{3}\left(\frac{a}{2+\sqrt{c+a}}\right)-\frac{1}{3}\frac{\sqrt{c+a}\;a}{(2+\sqrt{c+a})^2}$$

which simplifies to

$$\frac{1}{6}\,\frac{1}{(2+\sqrt{2a})(2+\sqrt{c+a})^2}\Big(48a+26\sqrt{c+a}\;a+4ac+4a^2+16c+6c\sqrt{c+a}$$
$$+8c\sqrt{2a}+3c\sqrt{2a}\sqrt{c+a}+16a\sqrt{2a}+5a\sqrt{2a}\sqrt{c+a}\Big).$$

Since this derivative is clearly nonnegative, M increases with c and since $a \geq c$, M is maximized when $c = a$. So, L is maximized when b and c are both a. This value is $\dfrac{4a^2}{2+\sqrt{2a}}$.

Now if R is differentiated with respect to a we obtain.

$$\frac{\partial}{\partial a}\left(\frac{(a+b)^2}{2+\sqrt{b+c}}\right)=\frac{2(a+b)}{2+\sqrt{b+c}}$$

which is clearly nonnegative and so R increases with a and since $a \geq b$ is minimized when $a = b$.

Replacing a by b in R (call this expression N) we have

$$\frac{\partial}{\partial b}(N)=\frac{\partial}{\partial b}\left(\frac{(2b)^2}{2+\sqrt{b+c}}\right)=\frac{2b\left(8\sqrt{b+c}+3b+4c\right)}{\left(2+\sqrt{b+c}\right)^2\sqrt{b+c}}$$

which is clearly nonnegative. So, N increases with b, and since $b \geq c$ is minimized when $b = c$, R is minimized when $a = b = c$, and has value of $\dfrac{4a^2}{2+\sqrt{2a}}$.

Editor's Comment: **D. M. Bătinetu-Giurgiu, Neuclai Stanciu and Titu Zvonaru, all of Romania,** jointly constructed and proved a generalization of Problem 5280. Their generalization follows:

Let $n \in N,\ n \geq 3,\ a = x_1 \geq b = x_2 \geq x_3 \geq \ldots \geq c = x_{n-1} \geq d = x_n > 0$ and $u, v \in R_+ = (0, \infty)$.

If $x_{n+1} = x_1, x_{n+2} = x_2$, then

$$\sum_{k=1}^{n}\frac{(x_k+x_{k+1})(x_k+x_{k+2})}{u+v\sqrt{x_{k+1}+x_{k+2}}}\leq\frac{n(a+b)^2}{u+v\sqrt{c+d}}.$$

Letting $n = 3,\ x_1 = a,\ x_2 = b,\ x_3 = c$ and $u = 2,\ v = 1$, they showed that the inequality holds.

Also solved by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College Bucharest, Neuclai Stanciu, "George Emil Palade" School, Buzău, and Titu Zvonaru, Comănesti, Romania; Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo State University, San Angelo, TX; Ed Gray, Highland

Beach, FL; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Adrian Naco, Polytechnic University, Tirana, Albania; Perfetti Paolo, Department of Mathematics, "Tor Vergata" University, Rome, Italy, and the proposer.

- **5281:** *Proposed by Arkady Alt, San Jose, CA*

For the sequence $\{a_n\}_{n\geq 1}$ defined recursively by $a_{n+1} = \dfrac{a_n}{1+a_n^p}$ for $n \in \mathcal{N}, a_1 = a > 0$, determine all positive real p for which the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

Solution 1 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

Answer: $p < 1$.

Proof: Since $a_{n+1} < a_n$, $a_n \to 0$.
It follows that

$$a_{n+1} = a_n - a_n^{p+1} + a_n^{2p+1} + O(a_n^{3p+1})$$

We employ the standard result of the exercise num.174 at page 38 of the book by G. Pólya, G. Szegö, *Problems and Theorems in Analysis, I.*

Assume that $0 < f(x) < x$ and $f(x) = x - ax^k + bx^l + x^l \varepsilon(x)$, $\lim_{x\to 0} \varepsilon(x) = 0$, for $0 < x < x_0$ where $1 < k < l$ and a, b both positive. The sequence x_n defined by $x_{n+1} = f(x_n)$ satisfies

$$\lim_{n\to\infty} n^{1/(k-1)} x_n = (a(k-1))^{-1/(k-1)}.$$

In our case we have $a = 1$, $k = p+1$, $b = 1$, $l = 2p+1$. Thus the sequence satisfies

$$a_n = p^{-1/p} n^{-1/p} + o(n^{-1/p})$$

and then the series converges if and only if $p < 1$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

We show that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent if $0 < p < 1$ and divergent if ≥ 1.

We assume in what follows that $n \in N$. Clearly $a_n > 0$ and by the given recursive relation, we have $a_{n+1} < a_n$. Therefore $L = \displaystyle\lim_{n\to\infty} a_n$ exists and from $L = \dfrac{L}{1+L^p}$, we see that $L = 0$. Inductively, we have

$$a_{n+1} = \frac{a}{\displaystyle\prod_{k=1}^{n} (1 + a_k^p)}. \qquad (1)$$

By making use of the well-known inequality $1 + x < e^x$ for $x > 0$, we deduce from (1) that $a_{n+1} > ae^{-\sum_{k=1}^{n} a_k^p} > 0$. Since $\displaystyle\lim_{n\to\infty} a_{n+2} = 0$, so $\displaystyle\sum_{k=1}^{n} a_k^p$ is divergent. Now there exits $k_0 \in N$, depending at most on a and p, such that $a_k < 1$ whenever $k > k_0$. Hence if $p \geq 1$, then for any integer $M > k_0$, we have $\sum_{k=k_0+1}^{M} a_k \geq \sum_{k=k_0+1}^{M} a_k^p$. Thus $\sum_{k=+1}^{\infty} a_k$ is divergent.

We next consider the case $0 < p < 1$. Let $m = \left\lfloor \frac{1}{1-p} \right\rfloor + 1$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x. By (1), for any $n > m$, we have

$$0 < a_{n+1} \leq \frac{a}{(1+a_n^p)^n} < \frac{a}{(1+a_{n+1}^p)^n} < \frac{a}{\binom{n}{m} a_{n+1}^{mp}},$$

so that

$$0 < a_{n+1} < \left(\frac{am!}{\prod_{k=0}^{m-1}(n-k)} \right)^{1/(1+mp)} \leq \left(\frac{am!}{(n-m+1)^m} \right)^{1/(1+mp)}.$$

It is easy to check that $\frac{m}{1+mp} > 1$, and so $\sum_{n=1}^{\infty} a_n$ is convergent.

This completes the solution.

Also solved by Ed Gray, Highland Beach, FL, and the proposer.

- **5282:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate

$$\int_0^1 x \ln\left(\sqrt{1+x} - \sqrt{1-x}\right) \ln\left(\sqrt{1+x} + \sqrt{1-x}\right) dx.$$

Solution 1 by Anastasios Kotronis, Athens, Greece

Using the identity

$$ab = \frac{1}{4} \cdot a + b^2 - a - b^2,$$

with $a = \ln \sqrt{1+x} - \sqrt{1-x}$ and $b = \ln \sqrt{1+x} + \sqrt{1-x}$ we have

$$\begin{aligned}
I &= \int_0^1 x \ln \sqrt{1+x} - \sqrt{1-x} \ln \sqrt{1+x} + \sqrt{1-x}\, dx \\
&= \frac{1}{4} \int_0^1 x \ln^2(2x) - \ln^2 \frac{1 - \sqrt{\frac{1-x}{1+x}}}{1 + \sqrt{\frac{1-x}{1+x}}}\, dx \\
&= \frac{1}{4} \int_0^1 x \ln^2(2x)\, dx - \frac{1}{4} \int_0^1 x \ln^2 \frac{1 - \sqrt{\frac{1-x}{1+x}}}{1 + \sqrt{\frac{1-x}{1+x}}}\, dx \\
&= I_1 - I_2.
\end{aligned}$$

Integrating by parts twice we easily get that

$$I_1 = \frac{\ln^2 2}{8} - \frac{\ln 2}{8} + \frac{1}{16}. \qquad (1)$$

In order to calculate I_2, we first note that

$$\begin{aligned}
\int \frac{u(1-u^2)}{(1+u^2)^3}\,du \quad u^2 = y \quad & \frac{1}{2}\int \frac{1-y}{(1+y)^3}\,dy \\
= \quad & \int \frac{1}{(1+y)^3}\,dy - \frac{1}{2}\int \frac{1}{(1+y)^2} \\
= \quad & \frac{u^2}{2(1+u^2)^2} + c,
\end{aligned}$$

so, letting $\sqrt{\dfrac{1-x}{1+x}} = y$ and letting $\dfrac{1-y}{1+y} = u$ we have

$$\begin{aligned}
\frac{1}{4}\int x \ln^2 \frac{1-\sqrt{\frac{1-x}{1+x}}}{1+\sqrt{\frac{1-x}{1+x}}}\,dx &= \int \frac{y(1-y^2)}{(1+y^2)^3} \ln^2 \frac{1-y}{1+y}\,dy \\
&= \int \frac{u(1-u^2)}{(1+u^2)^3} \ln^2 u\,du \\
&= \frac{u^2 \ln^2 u}{2(1+u^2)^2} - \int \frac{u}{2(1+u^2)^2} \ln u\,du \\
&= \frac{u^2 \ln^2 u}{2(1+u^2)^2} - \int -\frac{1}{2(1+u^2)}' \ln u\,du \\
&= \frac{u^2 \ln^2 u}{2(1+u^2)^2} + \frac{\ln u}{2(1+u^2)} - \frac{1}{2}\int \frac{1}{u} - \frac{u}{1+u^2}\,du \\
&= \frac{u^2 \ln^2 u}{2(1+u^2)^2} + \frac{\ln u}{2(1+u^2)} - \frac{\ln u}{2} + \frac{\ln(1+u^2)}{4} + \\
&= A(x) + c
\end{aligned}$$

which yields

$$I_2 = A(x)\Big|_0^1 = \lim_{x\to 0+} A(x) - \lim_{x\to 1-} A(x) = \frac{\ln 2}{4}, \qquad (2)$$

and hence, from (1) and (2), $I = \dfrac{\ln^2 2}{8} - \dfrac{\ln 8}{8} + \dfrac{1}{16}$.

Solution 2 by Arkady Alt, San Jose, CA

Solution A.

Let $I = \int\limits_0^1 x \ln\left(\sqrt{1+x}+\sqrt{1-x}\right) \ln\left(\sqrt{1+x}-\sqrt{1-x}\right) dx.$

Then $4I = \int\limits_0^1 x \ln\left(\sqrt{1+x}+\sqrt{1-x}\right)^2 \ln\left(\sqrt{1+x}-\sqrt{1-x}\right)^2 dx = \int\limits_0^1 xu(x)\,v(x)\,dx,$

where $u(x) = \ln\left(2+2\sqrt{1-x^2}\right), v(x) = \ln\left(2-2\sqrt{1-x^2}\right).$

Since $u(x)+v(x) = \ln(4x^2) = 2\ln(2x)$ then

$u^2(x)+v^2(x)+2u(x)\,v(x) = 4\ln^2(2x) \iff u(x)\,v(x) = 2\ln^2(2x) - \dfrac{u^2(x)+v^2(x)}{2}$

and, therefore, $4I = 2\int\limits_0^1 x\ln^2(2x)\,dx - \dfrac{1}{2}\left(\int\limits_0^1 xu^2(x)\,dx + \int\limits_0^1 xv^2(x)\,dx\right).$

1. Using substitution and integration by parts we obtain

$2\int\limits_0^1 x\ln^2(2x)\,dx = [t=2x; dt=2dx] = \dfrac{1}{2}\int\limits_0^2 t\ln^2(t)\,dt = \ln^2 2 - \dfrac{1}{2}\int\limits_0^2 t\ln t\,dt =$

$\ln^2 2 - \ln 2 + \dfrac{1}{2}.$

2. Let $t = 2+2\sqrt{1-x^2}$. Since $xdx = -\dfrac{(t-2)\,dt}{4}$ then

$\int\limits_0^1 xu^2(x)\,dx = \dfrac{1}{4}\int\limits_4^2 -(t-2)\ln^2 t dt = \dfrac{1}{4}\int\limits_2^4 (t-2)\ln^2 t dt.$

3. Let $t = 2-2\sqrt{1-x^2}$. Since $xdx = \dfrac{(2-t)\,dt}{4}$ then

$\int\limits_0^1 xv^2(x)\,dx = \dfrac{1}{4}\int\limits_0^2 (2-t)\ln^2 t dt = -\dfrac{1}{4}\int\limits_0^2 (t-2)\ln^2 t dt.$

Hence $\dfrac{1}{2}\left(\int\limits_0^1 xu^2(x)\,dx + \int\limits_0^1 xv^2(x)\,dx\right) = \dfrac{1}{8}\left(\int\limits_2^4 (t-2)\ln^2 t dt - \int\limits_0^2 (t-2)\ln^2 t dt\right) =$

$\dfrac{1}{8}\left(\int\limits_0^4 (t-2)\ln^2 t dt - 2\int\limits_0^2 (t-2)\ln^2 t dt\right).$

Using integration by parts twice we obtain

$\int (t-2)\ln^2 t dt = \begin{bmatrix} p' = t-2; p = \dfrac{t^2}{2} - 2t \\ q = \ln^2 t; q' = \dfrac{2\ln t}{t} \end{bmatrix} = \left(\dfrac{t^2}{2} - 2t\right)\ln^2 t - \int (t-4)\ln t dt =$

$\left(\dfrac{t^2}{2} - 2t\right)\ln^2 t - \left(\dfrac{t^2}{2} - 4t\right)\ln t + \dfrac{t^2}{4} - 4t.$

Since $\int_0^4 (t-2)\ln^2 t dt = \left(\left(\frac{t^2}{2}-2t\right)\ln^2 t - \left(\frac{t^2}{2}-4t\right)\ln t + \frac{t^2}{4} - 4t\right)_0^4 = 8\ln 4 - 12$
and
$\int_0^2 (t-2)\ln^2 t dt = \left(\left(\frac{t^2}{2}-2t\right)\ln^2 t - \left(\frac{t^2}{2}-4t\right)\ln t + \frac{t^2}{4} - 4t\right)_0^2 = 6\ln 2 - 2\ln^2 2 - 7$
then $\frac{1}{2}\left(\int\limits_0^1 xu^2(x)\,dx + \int\limits_0^1 xv^2(x)\,dx\right) = \frac{1}{8}\left(8\ln 4 - 12 - 2\left(6\ln 2 - 2\ln^2 2 - 7\right)\right) =$
$\frac{1}{2}\ln 2 + \frac{1}{2}\ln^2 2 + \frac{1}{4}$.
Therefore, $4I = \ln^2 2 - \ln 2 + \frac{1}{2} - \left(\frac{1}{2}\ln 2 + \frac{1}{2}\ln^2 2 + \frac{1}{4}\right) = \frac{1}{2}\ln^2 2 - \frac{3}{2}\ln 2 + \frac{1}{4}$
$I = \frac{1}{8}\ln^2 2 - \frac{3}{8}\ln 2 + \frac{1}{16} \approx -0.137\,37$

Solution B.
Let $u(x) = \ln\left(\sqrt{1+x}+\sqrt{1-x}\right)$, $v(x) = \ln\left(\sqrt{1+x}-\sqrt{1-x}\right)$ and
$I = \int\limits_0^1 xu(x)\,v(x)\,dx.$

Since $u(x)+v(x) = \ln\left(\left(\sqrt{1+x}\right)^2 - \left(\sqrt{1-x}\right)^2\right) = \ln(2x)$ then

$$u(x)\,v(x) = \frac{\ln^2(2x) - u^2(x) - v^2(x)}{2}$$

and, therefore, $2I = \int\limits_0^1 x\ln^2(2x)\,dx - \int\limits_0^1 x\left(u^2(x)+v^2(x)\right)dx.$

Calculation of $\int\limits_0^1 x\left(u^2(x)+v^2(x)\right)dx.$
1. Let $t = \sqrt{1+x}+\sqrt{1-x}$. Then $u^2(x) = \ln^2 t$ and
$t^2 = 2+2\sqrt{1-x^2} \iff \frac{t^2-2}{2} = \sqrt{1-x^2}$
yield $tdt = \frac{-xdx}{\sqrt{1-x^2}} \iff xdx = -\frac{t(t^2-2)}{2}dt.$
Hence, $\int\limits_0^1 xu^2(x)\,dx = -\int\limits_2^{\sqrt{2}} \frac{t(t^2-2)}{2}\ln^2 tdt = \frac{1}{2}\int\limits_{\sqrt{2}}^2 t\left(t^2-2\right)\ln^2 tdt;$
2. Let $t = \sqrt{1+x}-\sqrt{1-x}$. Then $v^2(x) = \ln^2 t$ and
$t^2 = 2-2\sqrt{1-x^2} \iff \frac{2-t^2}{2} = 2\sqrt{1-x^2}$
yield $-tdt = \frac{-x}{\sqrt{1-x^2}}dx \iff xdx = \frac{t(2-t^2)}{2}dt$. Hence,

$$\int\limits_0^1 xu^2(x)\,dx = \int\limits_0^{\sqrt{2}} \frac{t(2-t^2)}{2}\ln^2 tdt = -\frac{1}{2}\int\limits_0^{\sqrt{2}} t\left(t^2-2\right)\ln^2 tdt$$

and we obtain $\int\limits_0^1 x\left(u^2(x)+v^2(x)\right)dx = \frac{1}{2}\int\limits_{\sqrt{2}}^{2} t\left(t^2-2\right)\ln^2 t dt - \frac{1}{2}\int\limits_0^{\sqrt{2}} t\left(t^2-2\right)\ln^2 t dt =$
$\frac{1}{2}\int\limits_0^2 t\left(t^2-2\right)\ln^2 t dt - \int\limits_0^{\sqrt{2}} t\left(t^2-2\right)\ln^2 t dt.$

Using integration by parts twice we obtain we obtain
$\int t\left(t^2-2\right)\ln^2 t dt = \begin{bmatrix} p' = t^3-2t;\ p = \dfrac{t^4}{4}-t^2 \\ q = \ln^2 t;\ q' = \dfrac{2\ln t}{t} \end{bmatrix} =$
$\left(\frac{t^4}{4}-t^2\right)\ln^2 t - \int\left(\frac{t^3}{2}-2t\right)\ln t dt =$
$\left(\frac{t^4}{4}-t^2\right)\ln^2 t - \left(\frac{t^4}{8}-t^2\right)\ln t + \int\left(\frac{t^3}{8}-t\right)dt =$
$\left(\frac{t^4}{4}-t^2\right)\ln^2 t - \left(\frac{t^4}{8}-t^2\right)\ln t + \left(\frac{t^4}{32}-\frac{t^2}{2}\right).$

Hence,
$\int\limits_0^2 t\left(t^2-2\right)\ln^2 t dt = \left(\left(\frac{t^4}{4}-t^2\right)\ln^2 t - \left(\frac{t^4}{8}-t^2\right)\ln t + \left(\frac{t^4}{32}-\frac{t^2}{2}\right)\right)_0^2 = 2\ln 2 - \frac{3}{2},$
$\int\limits_0^{\sqrt{2}} t\left(t^2-2\right)\ln^2 t dt = \left(\frac{\sqrt{2}^4}{4}-\sqrt{2}^2\right)\ln^2\sqrt{2} - \left(\frac{\sqrt{2}^4}{8}-\sqrt{2}^2\right)\ln\sqrt{2} + \left(\frac{\sqrt{2}^4}{32}-\sqrt{2}^2 2\right) =$
$\frac{3}{4}\ln 2 - \frac{1}{4}\ln^2 2 - \frac{7}{8}$ and, therefore,
$\int\limits_0^1 x\left(u^2(x)+v^2(x)\right)dx = \frac{1}{2}\left(2\ln 2-\frac{3}{2}\right) - \left(\frac{3}{4}\ln 2 - \frac{1}{4}\ln^2 2 - \frac{7}{8}\right) =$
$\frac{1}{4}\left(\ln^2 2 + \ln 2 + \frac{1}{2}\right).$

Since (using integration by parts again)
$\int\limits_0^1 x\ln^2(2x)dx = \frac{1}{4}\int\limits_0^1 2x\ln^2(2x)\cdot 2dx = \frac{1}{4}\int\limits_0^2 t\ln^2 t dt = \frac{1}{4}\left(\frac{t^2}{2}\left(\ln^2 t - \ln t + \frac{1}{2}\right)\right)_0^2 =$
$\frac{1}{2}\left(\ln^2 2 - \ln 2 + \frac{1}{2}\right)$ then $I = \frac{1}{2}\left(\frac{1}{2}\left(\ln^2 2 - \ln 2 + \frac{1}{2}\right) - \frac{1}{4}\left(\ln^2 2 + \ln 2 + \frac{1}{2}\right)\right) =$
$\frac{1}{8}\left(\ln^2 2 - 3\ln 2 + \frac{1}{2}\right) \approx -0.137\,37.$

Solution 3 by Kee-Wai Lau, Hong Kong, China

Denote the integral of the problem by I. We show that

$$I = \frac{2\ln^2 2 - 6\ln 2 + 1}{16}. \qquad (1)$$

Let $I_1 = \int_0^1 x\ln^2(2x)dx, \quad I_2 = \int_0^1 x\ln^2\left(\sqrt{1+x}-\sqrt{1-x}\right)dx$ and

$I_3 = \int_0^1 x \ln^2\left(\sqrt{1+x} - \sqrt{1-x}\right) dx$. Using the identity $ab = \dfrac{(a+b)^2 - a^2 - b^2}{2}$ with $a = \ln\left(\sqrt{1+x} - \sqrt{1-x}\right)$ and $b = \ln\left(\sqrt{1+x} + \sqrt{1-x}\right)$, we see that

$$I = \frac{1}{2}\left(I_1 - I_2 - I_3\right). \qquad (2)$$

To evaluate I_1, I_2, and I_3, we need the known result, readily proved by differentiation, that for nonnegative integer n,

$$\int x^n \ln^2 x dx = x^{n+1}\left(\frac{\ln^2 x}{n+1} - \frac{2\ln x}{(n+1)^2} + \frac{2}{(n+1)^3}\right) + \text{constant} \qquad (3)$$

Since $I_1 = \frac{1}{4}\int_0^2 x \ln^2 x dx$, so by (3) we have

$$I_1 = \frac{2\ln^2 - 2\ln 2 + 1}{4}. \qquad (4)$$

Since $(\sqrt{1+x} - \sqrt{1-x})^2 = 2(1 - \sqrt{1-x^2})$, so

$$I_2 = \frac{1}{4}\int_0^1 x \ln^2\left(2\left(1 - \sqrt{1-x^2}\right)\right) = \frac{1}{8}\int_0^1 \ln^2\left(2\left(1 - \sqrt{1-x}\right)\right) dx.$$

By the substitution $y = 2(1 - \sqrt{1-x})$, so that $x = y - \frac{y^2}{4}$, we obtain $I_2 = \frac{1}{16}\int_0^2 (2-y)\ln^2 y dy$. By (3) we have

$$I_2 = \frac{2\ln^2 2 - 6\ln 2 + 7}{16}. \qquad (5)$$

By using the identity $\left(\sqrt{1+x} + \sqrt{1-x}\right)^2 = 2\left(1 + \sqrt{1-x^2}\right)$, we obtain

$$I_3 = \frac{1}{4}\int_0^1 x \ln^2\left(2\left(1 + \sqrt{1-x^2}\right)\right) dx = \frac{1}{8}\int_0^1 x \ln^2\left(2\left(1 + \sqrt{1-x}\right)\right) dx.$$

By the substitution $y = 2\left(1 + \sqrt{1-x}\right)$, so that $x = y - \frac{y^2}{4}$, we obtain $I_3 = \frac{1}{16}\int_2^4 (y-2)\ln^2 y \, dy$. By (3),we have

$$I_3 = \frac{2\ln^2 2 + 10\ln 2 - 5}{16}. \qquad (6)$$

Now by $(2), (4), (5)$ and (6), we obtain (1) and this completes the solution.

Editor's comment: **Ed Gray of Highland Beach, FL** transformed the given integral into

$$\frac{1}{4}\int_2^{\sqrt{2}} \left(2y - y^3\right) \ln y \left(\ln(2-y) + \ln(2+y)\right) dy$$

and then he converted the various ln functions into series expansions to obtain a polynomial in y. This gave the approximate value of the integral as listed above.

Also solved (in closed form) by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy, and the proposer.

Comment by the proposer, Ovidiu Furdui: It is worth mentioning this logarithmic integral is missing from the book by Gradshteyn and Ryzhik, *Tables of Integrals, Series and Products,* Sixth Edition, Academic Press, 2000.

Late Solutions

Late solutions to 5271 and to 5273 were received by **Paul M. Harms of North Newton, KS** and from **David E. Manes, SUNY College at Oneonta, NY.** Their solutions were mailed on time but they got caught up in the Christmas rush mail, and arrived on my desk after the solutions to these problems had been published.