Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
May 15, 2008

- 5008: *Proposed by Kenneth Korbin, New York, NY.*

  Given isosceles trapezoid $ABCD$ with $\angle ABD = 60^o$, and with legs $\overline{BC} = \overline{AD} = 31$.

  Find the perimeter of the trapezoid if each of the bases has positive integer length with $\overline{AB} > \overline{CD}$.

- 5009: *Proposed by Kenneth Korbin, New York, NY.*

  Given equilateral triangle ABC with a cevian $\overline{CD}$ such that $\overline{AD}$ and $\overline{BD}$ have integer lengths. Find the side of the triangle $\overline{AB}$ if $\overline{CD} = 1729$ and if $(\overline{AB}, 1729) = 1$.

- 5010: *Proposed by José Gibergans-Báguena and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let α, β, and γ be real numbers such that $0 < \alpha \leq \beta \leq \gamma < \pi/2$. Prove that

  $$\frac{\sin 2\alpha + \sin 2\beta + \sin 2\gamma}{(\sin\alpha + \sin\beta + \sin\gamma)(\cos\alpha + \cos\beta + \cos\gamma)} \leq \frac{2}{3}.$$

- 5011: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $\{a_n\}_{n\geq 0}$ be the sequence defined by $a_0 = a_1 = 2$ and for $n \geq 2,\ a_n = 2a_{n-1} - \dfrac{1}{2}a_{n-2}$. Prove that

  $$2^p a_{p+q} + a_{q-p} = 2^p a_p a_q$$

  where $p \leq q$ are nonnegative integers.

- 5012 *Richard L. Francis, Cape Girardeau, MO.*

  Is the incenter of a triangle the same as the incenter of its Morley triangle?

- 5013: *Proposed by Ovidiu Furdui, Toledo, OH.*

  Let $k \geq 2$ be a natural number. Find the sum

$$\sum_{n_1,n_2,\cdots,n_k\geq 1} \frac{(-1)^{n_1+n_2+\cdots+n_k}}{n_1+n_2+\cdots+n_k}.$$

Solutions

- 4990: *Proposed by Kenneth Korbin, New York, NY.*

  Solve

$$40x+42\sqrt{1-x^2}=29\sqrt{1+x}+29\sqrt{1-x}$$

  with $0 < x < 1$.

  Solution by Boris Rays, Chesapeake, VA.

  Let $x=\cos\alpha$, where $\alpha \in (0,\pi/2)$. Then

$$\begin{aligned}
40\cos\alpha+42\sqrt{1-\cos^2\alpha} &= 29\sqrt{1+\cos}+29\sqrt{1-\cos\alpha}\\
&= 29\sqrt{2}\left(\sqrt{\frac{1+\cos\alpha}{2}}+\sqrt{\frac{1-\cos\alpha}{2}}\right)\\
&= 29\cdot\frac{2}{\sqrt{2}}\left(\sqrt{\frac{1+\cos\alpha}{2}}+\sqrt{\frac{1-\cos\alpha}{2}}\right)\\
&= 29\cdot 2\left(\frac{1}{\sqrt{2}}\cos\frac{\alpha}{2}+\frac{1}{\sqrt{2}}\sin\frac{\alpha}{2}\right)\\
&= 58\left(\cos\frac{\pi}{4}\cos\frac{\alpha}{2}+\sin\frac{\pi}{4}\sin\frac{\alpha}{2}\right)=58\cos(\frac{\pi}{4}-\frac{\alpha}{2}). \quad \text{Therefore,}
\end{aligned}$$

$$\begin{aligned}
40\cos\alpha+42\sin\alpha &= 58\cos(\frac{\pi}{4}-\frac{\alpha}{2}).\\
\frac{40}{58}\cos\alpha+\frac{42}{58}\sin\alpha &= \cos(\frac{\pi}{4}-\frac{\alpha}{2})\\
\frac{20}{29}\cos\alpha+\frac{21}{29}\sin\alpha &= \cos(\frac{\pi}{4}-\frac{\alpha}{2}).
\end{aligned}$$

  Let $\cos\alpha_0=\dfrac{20}{29}$. Then $\sin\alpha_0=\sqrt{1-\left(\dfrac{20}{29}\right)^2}=\dfrac{21}{29}$.

$$\cos\alpha_0\cos\alpha+\sin\alpha_0\sin\alpha \quad = \quad \cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$$

$$\cos(\alpha_0 - \alpha) \quad = \quad \cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right).$$

Therefore we obtain from the above,

$$\begin{aligned}
1) \quad \alpha_0 - \alpha_1 &= \frac{\pi}{4} - \frac{\alpha_1}{2} \\
\alpha_1 &= 2\alpha_0 - \frac{\pi}{2}, \text{ where } \alpha_0 = \arccos\frac{20}{29}. \\
2) \quad \alpha_0 - \alpha_2 &= -\left(\frac{\pi}{4} - \frac{\alpha_2}{2}\right) = \frac{\alpha_2}{2} - \frac{\pi}{4} \\
\frac{3}{2}\alpha_2 &= \alpha_0 + \frac{\pi}{4} \\
\alpha_2 &= \frac{2}{3}\alpha_0 + \frac{\pi}{6}, \text{ where } \alpha_0 = \arccos\frac{20}{29}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
1) \quad x_1 &= \cos\left(2\alpha_0 - \frac{\pi}{2}\right) = \cos(2\alpha_0)\cos\frac{\pi}{2} + \sin(2\alpha_0)\sin\frac{\pi}{2} \\
&= 2\sin\alpha_0\cos\alpha_0 \cdot 1 = 2 \cdot \frac{21}{29} \cdot \frac{20}{29} = \frac{840}{841}. \\
2) \quad x_2 &= \cos\left(\frac{2}{3}\alpha_0 + \frac{\pi}{6}\right) = \cos\left(\frac{2}{3}\arccos\left(\frac{20}{29}\right) + \frac{\pi}{6}\right).
\end{aligned}$$

The solution is:

$$x_1 = \frac{840}{841} \qquad x_2 = \cos\left(\frac{2}{3}\arccos\left(\frac{20}{29}\right) + \frac{\pi}{6}\right).$$

Remark: This solution is an adaptation of the solution to SSM problem 4966, which is an adaptation of the solution on pages 13-14 of *Mathematical Miniatures* by Savchev and Andreescu.

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; José Hernández Santiago (student at UTM), Oaxaca, México; Kee-Wai Lau, Hong Kong, China; Peter E. Liley, Lafayette, IN; John Nord, Spokane, WA; Paolo Perfetti, Math Dept., U. of Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4991: *Proposed by Kenneth Korbin, New York, NY.*

Find six triples of positive integers (a, b, c) such that

$$\frac{9}{a} + \frac{a}{b} + \frac{b}{9} = c.$$

Solution by David Stone and John Hawkins, Statesboro, GA,(with comments by editor).

David Stone and John Hawkins submitted a six page densely packed analysis of the problem, but it is too long to include here. Listed below is their solution and the gist of their analysis as to how they solved it. (Interested readers may obtain their full analysis by writing to David at <dstone@georgiasouthern.edu> or to me at <eisenbt@013.net>. Others who solved the problem programmed a computer.

David and John began by listing what they believe to be all ten solutions to the problem.

$$\begin{pmatrix} a & b & c \\ 2 & 12 & 6 \\ 9 & 9 & 3 \\ 14 & 588 & 66 \\ 18 & 36 & 5 \\ 54 & 12 & 6 \\ 162 & 4 & 41 \\ 378 & 588 & 66 \\ 405 & 25 & 19 \\ 11826 & 21316 & 2369 \\ 29565 & 133225 & 14803 \end{pmatrix}$$

The analysis in their words:
Rewriting the equation, we seek positive integer solutions to

$$(1) \qquad 81b + 9a^2 + ab^2 = 9abc.$$

<u>Theorem</u>. A solution must have the form $a = 3^i A, b = 3^j A^2$, where $(A, 3) = 1,\ i, j \geq 0$. At least one of i, j must be ≥ 1.

<u>Proof</u>. From equation (1), we see that 9 divides all terms but ab^2, so 9 divides ab^2, so 3 divides a or b so at least one of i, j must be ≥ 1.

Also from equation (1), it is clear that if p is a prime different from 3, then p divides a if and only if p divides b.

Suppose p is such a prime and $a = 3^i p^m C, b = 3^j p^n D$, where $m, n \geq 1$, and C and D are not divisible by 3 or p. Then equation (1) becomes

$$81\left(3^j p^n D\right) + 9\left(3^i p^m C\right)^2 + \left(3^i p^m C\right)\left(3^j p^n D\right)^2 = 9\left(3^i p^m C\right)\left(3^j p^n D\right)c,$$

or

$$(\#) \ \ 3^{j+4} p^n D + 3^{2i+2} p^{2m} C^2 + 3^{i+2j} p^{m+2n} C D^2 = 3^{i+j+2} p^{m+n} C D c.$$

If $n < 2m$, we can divide equation (#) by p^n to obtain

$$3^{j+4} D + 3^{2i+2} p^{2m-n} C^2 + 3^{i+2j} p^{m+n} C D^2 = 3^{i+j+2} p^m C D c.$$

But then p divides each term after the first, so p divides $3^{j+4}D$, which is impossible.

If $n > 2m$, we can divide through equation (#) by p^{2m} to obtain

$$\begin{aligned} 3^{j+4} p^{n-2m} D + 3^{2i+2} C^2 + 3^{i+2j} p^{2n-m} C D^2 &= 3^{i+j+2} p^{n-m} C D c \\ 81 p^{m-2n} D + 9C^2 + p^m C D^2 &= 9 p^{m-n} C D c. \end{aligned}$$

Noting that $2n > 4m > m$ and $n > 2m > m$, we see that p divides each term except $3^{2i+2}C^2$, so p divides $3^{2i+2}C^2$, which is impossible.

Therefore $n = 2m$.

That is, a and b have the same prime divisors, and in b, the power on each such prime is twice the corresponding power in a; therefore, in b, the product of all divisors other than 3 is the square of the analogous product in a. So the proof is complete.

They then substituted this result into equation (1) obtaining

$$81\left(3^j A^2\right) + 9\left(3^i A^2\right) + \left(3^i A\right)\left(3^j A^2\right)^2 = 9\left(3^i A\right)\left(3^j A^2\right)c,$$

or

$$(2) \qquad \left(2^{j+4} + 3^{2i+2}\right) + 3^{i+2j}A^3 = 3^{i+j+2}Ac$$

and started looking for values of i, j, A and c satisfying this equation.

Analyzing the cases **(1)** where 3 divides b but not a; **(2)** where 3 divides a but not b; and **(3)** where 3 divides a and b led to the solutions listed above.
They ended their submission with comments about the patterns they observed in solving analogous equations of the form $\dfrac{N}{a} + b + \dfrac{c}{N} = c$ for various integral values of N.

Also solved by Charles Ashbacher, Marion, IA; Britton Stamper (student at Saint George's School), Spokane, WA, and the proposer.

- 4992: *Proposed by Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie, San Angelo, TX.*

A closed circular cone has integral values for its height and base radius. Find all possible values for its dimensions if its volume V and its total area (including its circular base) A satisfy $V = 2A$.

Solution by R. P. Sealy, Sackville, New Brunswick, Canada.

$$\begin{aligned} \frac{1}{3}\pi r^2 h &= 2(\pi r^2 + \pi r\sqrt{r^2+h^2}) \text{ or} \\ rh &= 6r + 6\sqrt{r^2+h^2}. \end{aligned}$$

Squaring and simplifying gives $r^2 = 36\dfrac{h}{h-12}$. Therefore, $\dfrac{h}{h-12}$ is a square, and $\dfrac{h}{h-12} \in \{1, 4, 9, 16, \ldots\}$. Note that $f(h) = \dfrac{h}{h-12}$ is a decreasing function of h for $h > 12$ and that $h(16) = 4$. Note also that $f(13)$, $f(14)$ and $f(15)$ are not squares of integers. Therefore $(h, r) = (16, 24)$ is the only solution.

Also solved by Paul M.Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA; Britton Stamper (student at Saint George's School), Spokane, WA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4993: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Find all real solutions of the equation

$$126x^7 - 127x^6 + 1 = 0.$$

Solution by N. J. Kuenzi, Oshkosh, WI.

Both 1 and 1/2 are easily seen to be positive rational roots of the given equation. So $(x-1)$ and $(2x-1)$ are both factors of the polynomial $126x^7-127x^6+1$. Factoring yields

$$126x^2-127x^6+1=(x-1)(2x-1)(63x^5+31x^4+15c^3+7x^2+3x+1).$$

The equation $(63x^5+31x^4+15c^3+7x^2+3x+1)$ does not have any rational roots (Rational Roots Theorem) nor any positive real roots (Descartes' Rule of Signs).
Using numerical techniques one can find that -0.420834167 is the approximate value of a real root.
The four other roots are complex with approximate values:

$$0.1956354060+0.4093830251i \qquad 0.1956354060-0.4093830251i$$

$$-0.2312499936+0.3601917120i \quad -0.2312499936-0.3601917120i$$

So the real solutions of the equation $126x^7-127x^6+1=0$ are 1, 1/2 and -0.420834167.

Also solved by Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro GA, and the proposer.

- 4994: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Let a,b,c be three nonzero complex numbers lying on the circle $C=\{z\in\mathbf{C}:|\mathrm{z}|=\mathrm{r}\}$. Prove that the roots of the equation $az^2+bz+c=0$ lie in the ring shaped region $D=\left\{z\in\mathbf{C}:\dfrac{1-\sqrt{5}}{2}\le|\mathrm{z}|\le\dfrac{1+\sqrt{5}}{2}\right\}$.

Solution by Kee-Wai Lau, Hong Kong, China.

By rewriting the equation as $az^2=-bz-c$, we obtain

$$\begin{aligned}|a||z|^2 &= |az^2|=|bz+c|\le|b||z|+|c| \text{ or } |z|^2-|z|-1\le 0\\ &\text{or}\quad \left(|z|+\frac{\sqrt{5}-1}{2}\right)\left(|z|-\frac{\sqrt{5}+1}{2}\right)\le 0 \text{ so that } |z|\le\frac{1+\sqrt{5}}{2}.\end{aligned}$$

By rewriting the equation as $c=-az^2-bz$, we obtain

$$\begin{aligned}|c| &= |-az^2-bz|\le|a||z|^2+|b||z| \text{ or } |z|^2+|z|-1\ge 0\\ &\text{or}\quad \left(|z|+\frac{\sqrt{5}+1}{2}\right)\left(|z|-\frac{\sqrt{5}-1}{2}\right)\ge 0 \text{ so that } |z|\ge\frac{\sqrt{5}-1}{2}.\end{aligned}$$

This finishes the solution.

Also solved by Michael Brozinsky, Central Islip, NY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Russell Euler and Jawad Sadek (jointly), Maryville, MO; Boris Rays, Chesapeake, VA; José Hernández Santiago (student at UTM) Oaxaca, México; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- **4995:** *Proposed by K. S. Bhanu and M. N. Deshpande, Nagpur, India.*

Let A be a triangular array $a_{i,j}$ where $i = 1, 2, \cdots,$ and $j = 0, 1, 2, \cdots, i$. Let

$$a_{1,0} = 1,\ a_{1,1} = 2,\ \text{and } a_{i,0} = T(i+1) - 2 \text{ for } i = 2, 3, 4, \cdots,$$

where $T(i+1) = (i+1)(i+2)/2$, the usual triangular numbers. Furthermore, let $a_{i,j+1} - a_{i,j} = j + 1$ for all j. Thus, the array will look like this:

$$\begin{array}{cccccc}
 & & 1 & 2 & & \\
 & & 4 & 5 & 7 & \\
 & 8 & 9 & 11 & 14 & \\
13 & 14 & 16 & 19 & 23 & \\
19 & 20 & 22 & 25 & 29 & 34
\end{array}$$

Show that for every pair (i, j), $4a_{i,j} + 9$ is the sum of two perfect squares.

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX.

If we allow $T(0) = 0$, then for $i \geq 1$ and $j = 0, 1, \ldots, i$, it's clear from the definition of $a_{i,j}$ that

$$\begin{aligned}
a_{i,j} &= a_{i,0} + T(j) \\
&= T(i+1) - 2 + T(j) \\
&= \frac{i^2 + 3i - 2 + j^2 + j}{2}.
\end{aligned}$$

Therefore, for every pair (i, j),

$$\begin{aligned}
4a_{i,j} + 9 &= 2\left(i^2 + 3i - 2 + j^2 + j\right) + 9 \\
&= 2\left(i^2 + 3i + j^2 + j\right) + 5 \\
&= (i + j + 2)^2 + (i - j + 1)^2.
\end{aligned}$$

Solution 2 by Carl Libis, Kingston, RI.

For every pair $(i, j), 4a(i, j) + 9 = (i - j + 1)^2 + (i + j + 2)^2$ since

$$\begin{aligned}
4a(i,j) + 9 &= 4\left[a(i,0) + \frac{j(j+1)}{2}\right] + 9 = 4\left[\frac{(i+1)(i+2)}{2} - 2 + \frac{j(j+1)}{2}\right] + 9 \\
&= 2(i+1)(i+2) - 8 + 2j(j+1) + 9 \\
&= 2i^2 + 6i + 4 + 2j^2 + 2j + 1 \\
&= (i - j + 1)^2 + (i + j + 2)^2.
\end{aligned}$$

Also solved by Paul M. Harms, North Newton, KS; N. J. Kuenzi, Oshkosh, WI; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro GA; José Hernándz Santiago (student at UTM), Oaxaca, México, and the proposers.