# Problems

Ted Eisenberg, Section Editor

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*This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:*

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

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*Solutions to the problems stated in this issue should be posted before*
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- 4960: *Proposed by Kenneth Korbin, New York, NY.*

  Equilateral triangle $ABC$ has an interior point $P$ such that

  $$\overline{AP}=\sqrt{5},\ \overline{BP}=\sqrt{12},\ \text{and}\ \overline{\text{CP}}=\sqrt{17}.$$

  Find the area of $\triangle APB$.

- 4961: *Proposed by Kenneth Korbin, New York, NY.*

  A convex hexagon is inscribed in a circle with diameter $d$. Find the area of the hexagon if its sides are $3, 3, 3, 4, 4$ and 4.

- 4962: *Proposed by Kenneth Korbin, New York, NY.*

  Find the area of quadrilateral $ABCD$ if the midpoints of the sides are the vertices of a square and if $AB=\sqrt{29}$ and $CD=\sqrt{65}$.

- 4963: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Calculate

  $$\lim_{n\to\infty}\sum_{1\le i<j\le n}\frac{1}{3^{i+j}}.$$

- 4964: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $x, y$ be real numbers and we define the law of composition

  $$x\perp y = x\sqrt{1+y^2}+y\sqrt{1+x^2}.$$

Prove that $(R, +)$ and $(R, \perp)$ are isomorphic and solve the equation $x \perp a = b$.

- 4965: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $h_a, h_b, h_c$ be the heights of triangle $ABC$. Let $P$ be any point inside $\triangle ABC$. Prove that

$$(a)\quad \frac{h_a}{d_a} + \frac{h_b}{d_b} + \frac{h_c}{d_c} \geq 9, \quad (b)\quad \frac{d_a^2}{h_a^2} + \frac{d_b^2}{h_b^2} + \frac{d_c^2}{h_c^2} \geq \frac{1}{3},$$

  where $d_a, d_b, d_c$ are the distances from $P$ to the sides $BC, CA$ and $AB$ respectively.

*Solutions*

- 4930: *Proposed by Kenneth Korbin, New York, NY.*

  Find an acute angle $y$ such that $\cos(y) + \cos(3y) - \cos(5y) = \dfrac{\sqrt{7}}{2}$.

  **Solution by Brian D. Beasley, Clinton, SC.**

  Given an acute angle $y$, let $c = \cos(y) > 0$. We use $\cos(3y) = 4c^3 - 3c$ and $\cos(5y) = 16c^5 - 20c^3 + 5c$ to transform the given equation into

$$-16c^5 + 24c^3 - 7c = \frac{\sqrt{7}}{2}.$$

  Since this equation in turn is equivalent to

$$32c^5 - 48c^3 + 14c + \sqrt{7} = (8c^3 - 4\sqrt{7}c^2 + \sqrt{7})(4c^2 + 2\sqrt{7}c + 1) = 0,$$

  we need only determine the positive zeros of $f(x) = 8x^3 - 4\sqrt{7}x^2 + \sqrt{7}$. Applying $\cos(7y) = 64c^7 - 112c^5 + 56c^3 - 7c$, we note that the six zeros of

$$64x^6 - 112x^4 + 56x^2 - 7 = f(x)(8x^3 + 4\sqrt{7}x^2 - \sqrt{7})$$

  are $\cos(k\pi/14)$ for $k \in \{1, 3, 5, 9, 11, 13\}$. We let $g(x) = 8x^3 + 4\sqrt{7}x^2 - \sqrt{7}$ and use $g'(x) = 24x^2 + 8\sqrt{7}x$ to conclude that $g$ is increasing on $(0, \infty)$, and hence has at most one positive zero. But $g(1/2) > 0$, $\cos(\pi/14) > 1/2$, and $\cos(3\pi/14) > 1/2$, so $\cos(\pi/14)$ and $\cos(3\pi/14)$ must be zeros of $f(x)$ instead. Thus we may take $y = \pi/14$ or $y = 3\pi/14$ in the original equation.

  **Also solved by: Dionne Bailey, Elsie Campbell, and Charles Dimminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayete, IN; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Satesboro, GA, and the proposer.**

- 4931: *Proposed by Kenneth Korbin, New York, NY.*

  A Pythagorean triangle and an isosceles triangle with integer length sides both have the same length perimeter $P = 864$. Find the dimensions of these triangles if they both have the same area too.

  **Solution by David Stone and John Hawkins (jointly), Statesboro, GA.**

  Surprisingly, there exists only one such pair of triangles: the (primitive) Pythagorean tiangle $(135, 352, 377)$ and the isosceles triangle $(366, 366, 132)$. Each has a perimeter 864 and area $23,760$.

  By Heron's Formula (or geometry), an isosceles triangle with given perimeter $P$ and sides

$(a, a, b)$ has area

$$A = \frac{b}{4}\sqrt{4a^2 - b^2} = \frac{P-2a}{4}\sqrt{P(4a-P)}, \text{ where } \frac{P}{4} \leq a \leq \frac{P}{2}.$$

In our problem, $P = 864$. We can analyze possibilities to reduce the number of cases to check or we can use a calculator or computer to check all possibilities. In any case, there are only a few such triangles with integer length sides:

$$\left\{\begin{matrix} a & b & A \\ 222 & 420 & 15,120 \\ 240 & 384 & 27,648 \\ 270 & 324 & 34,992 \\ 312 & 240 & 34,560 \\ 366 & 132 & 23,760 \end{matrix}\right\}$$

Now, if $(a, b, c)$ is a Pythaorean triangle with given perimeter $P$ and given area $A$, we can solve the equations

$$\begin{aligned} P &= a + b + c \\ c^2 &= a^2 + b^2 \\ A &= \frac{ab}{2} \end{aligned}$$

to obtain $a = \dfrac{(P^2 + 4A) \pm \sqrt{P^4 - 24P^2A + 16A^2}}{4P}, \quad b = \dfrac{2A}{a}, \ c = P - a - \dfrac{2A}{a}.$

We substitute $P = 864$ and the values for $A$ from the above table. Only with $A = 23,760$ do we find a solutions $(135, 352, 377)$. (Note that the two large values of $A$ each produce a negative under the radical because those values of $A$ are too large to be hemmed up by a perimeter of 864, while the first two values of $A$ produce right triangles with non-integer sides.)

**Also solved by Brain D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; Amihai Menuhin, Beer-Sheva, Israel, Harry Sedinger, St. Bonaventure, NY, and the proposer.**

- 4932: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $ABC$ be a triangle with semi-perimeter $s$, in-radius $r$ and circum-radius $R$. Prove that

$$\sqrt[3]{r^2} + \sqrt[3]{s^2} \leq 2\sqrt[3]{2R^2}$$

and determine when equality holds.

**Solution by the proposer.**

From Euler's inequality for the triangle $2r \leq R$, we have $r/R \leq 1/2$ and

$$\left(\frac{r}{R}\right)^{2/3} \leq \left(\frac{1}{2}\right)^{2/3} \qquad (1)$$

Next, we will see that

$$\frac{s}{R} \leq \frac{3\sqrt{3}}{2} \qquad (2)$$

In fact, from Sine's Law

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

we have

$$\frac{a+b+c}{\sin A + \sin B + \sin C} = 2R$$

or

$$\frac{s}{R} = \frac{a+b+c}{2R} = \sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$$

as claimed. Notice that the last inequality is an immediate consequence of Jensen's inequality applied to the function $f(x) = \sin x$ that is concave in $[0, \pi]$.

Finally, from (1) and (2), we have

$$\left(\frac{r}{R}\right)^{2/3} + \left(\frac{s}{R}\right)^{2/3} \le \left(\frac{1}{2}\right)^{2/3} + \left(\frac{3\sqrt{3}}{2}\right)^{2/3} = 2\sqrt[3]{2}$$

from which the statement immediately follows as desired. Note that equality holds when $\triangle ABC$ is equilateral, as immediately follows from (1) and (2).

- 4933: *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Barcelona, Spain.*

Let $n$ be a positive integer. Prove that

$$\frac{1}{n}\sum_{k=1}^{n} k\binom{n}{k}^{1/2} \le \frac{1}{2}\sqrt{(n+1)2^n}.$$

**Solution by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX .**

By the Binomial Theorem,

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}x^k &= (1+x)^n \\
\frac{d}{dx}\sum_{k=0}^{n}\binom{n}{k}x^k &= \frac{d}{dx}(1+x)^n \\
\sum_{k=1}^{n}k\binom{n}{k}x^{k-1} &= n(1+x)^{n-1} \\
\sum_{k=1}^{n}k\binom{n}{k}x^{k} &= nx(1+x)^{n-1} \\
\frac{d}{dx}\sum_{k=1}^{n}k\binom{n}{k}x^{k} &= \frac{d}{dx}\left[nx(1+x)^{n-1}\right] \\
\sum_{k=1}^{n}k^2\binom{n}{k}x^{k-1} &= n(1+x)^{n-2}(nx+1) \qquad (1).
\end{aligned}$$

Evaluating (1) when $x = 1$,

$$\begin{aligned}
\sum_{k=1}^{n}k^2\binom{n}{k} &= n(n+1)2^{n-2} \\
\frac{1}{n}\sum_{k=1}^{n}k^2\binom{n}{k} &= \frac{(n+1)2^n}{4} \qquad (2).
\end{aligned}$$

By the Root Mean Square Inequality and (2),

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} k\binom{n}{k}^{1/2} &\le \sqrt{\frac{\sum_{k=1}^{n} k^2\binom{n}{k}}{n}}\\ &= \sqrt{\frac{(n+1)2^n}{4}}\\ &= \frac{1}{2}\sqrt{(n+1)2^n}.\end{aligned}$$

**Also solved by the proposer.**

- 4934: *Proposed by Michael Brozinsky, Central Islip, NY.*

Mrs. Moriaty had two sets of twins who were always getting lost. She insisted that one set must chose an arbitrary non-horizontal chord of the circle $x^2+y^2=4$ as long as the chord went through $(1,0)$ and they were to remain at the opposite endpoints. The other set of twins was similarly instructed to choose an arbitrary non-vertical chord of the same circle as long as the chord went through $(0,1)$ and they too were to remain at the opposite endpoints. The four kids escaped and went off on a tangent (to the circle, of course). All that is known is that the first set of twins met at some point and the second set met at another point. Mrs. Moriaty did not know where to look for them but Sherlock Holmes deduced that she should confine her search to two lines. What are their equations?

**Solution by R. P. Sealy, Sackville, New Brunswick, Canada**

The equations of the two lines are $x=4$ for the first set of twins and $y=4$ for the second set of twins.

The vertical chord through the point (1,0) meets the circle at points $(1,\sqrt{3})$ and $(1,-\sqrt{3})$. The slopes of the tangent lines are $-\dfrac{1}{\sqrt{3}}$ and $\dfrac{1}{\sqrt{3}}$. So the equations of the tangent lines are

$$y=-\frac{1}{\sqrt{3}}x+\frac{4}{\sqrt{3}} \quad\text{and}\quad y=\frac{1}{\sqrt{3}}x-\frac{4}{\sqrt{3}}.$$

These tangent lines meet at the point (4,0). Otherwise, a non-vertical (and non-horizontal) chord through the point (1,0) intersects the circle at points $(a,b)$ and $(c,d)$, $bd\neq 0$, $b\neq d$. The slopes of the tangent lines are $-\dfrac{a}{b}$ and $-\dfrac{c}{d}$. So the equations of the tangent lines are

$$y=-\frac{a}{b}x+\frac{4}{b} \quad\text{and}\quad y=-\frac{c}{d}x+\frac{4}{d}.$$

The $x$-coordinate of the point of intersection of the tangent lines is $\dfrac{4(d-b)}{ad-bc}$. And since the points $(a,b),(c,d)$ and (1,0) are on the chord, we have

$$\frac{b-0}{a-1}=\frac{d-0}{c-1}$$

or

$$d-b=ad-bc.$$

Therefore, the $x$-coordinate of the point of intersection of the tangent lines is 4.

Similar calculations apply to position of the second set of twins.

**Also solve by Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.**

- 4935: *Proposed by Xuan Liang, Queens, NY and Michael Brozinsky, Central Islip, NY.*

Without using the converse of the Pythagorean Theorem nor the concepts of slope, similar triangles or trigonometry, show that the triangle with vertices $A(-1,0), B(m^2,0)$ and $C(0,m)$ is a right triangle.

**Solution by Harry Sedinger, St. Bonaventure, NY.**

Let $O=(0,0)$. The area of $\triangle ABC$ is

$$\begin{aligned}\frac{1}{2}\Big(|OB|\Big)\Big(|AC|\Big) &= \frac{1}{2}m(m^2+1)=\frac{1}{2}m\sqrt{m^2+1}\sqrt{m^2+1}\\ &= \frac{1}{2}\sqrt{m^4+m^2}\sqrt{m^2+1}=\frac{1}{2}\Big(|BC|\Big)\Big(|AB|\Big).\end{aligned}$$

Thus if $AB$ is considered the base of $\triangle ABC$, its height is $|BC|$, so $AB \perp BC$ and $\triangle ABC$ is a right triangle.

**Also solved by Charles Ashbacher, Cedar Rapis, IA; Brian D. Beasley, Clinton, SC; Grant Evans (student, Saint George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Jahangeer Kholdi, Portsmouth, VA; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA; William Weirich (student Virginia Commonwealth University), Richmond, VA, and the proposers.**

**Editor's comment:** Several readers used the distance formula or the law of cosines, or the dot product of vectors in their solutions; but to the best of my knowledge, these notions are obtained with the use of the Pythagorean Theorem.