Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before
March 15, 2011

- **5140:** *Proposed by Kenneth Korbin, New York, NY*

  Given equilateral triangle ABC with an interior point P such that

$$\begin{aligned} \overline{AP} &= 22+16\sqrt{2} \\ \overline{BP} &= 13+9\sqrt{2} \\ \overline{CP} &= 23+16\sqrt{2}. \end{aligned}$$

  Find $\overline{AB}$.

- **5141:** *Proposed by Kenneth Korbin, New York, NY*

  A quadrilateral with sides $259, 765, 285, 925$ is constructed so that its area is maximum. Find the size of the angles formed by the intersection of the diagonals.

- **5142:** *Proposed by Michael Brozinsky, Central Islip, NY*

  Let CD be an arbitrary diameter of a circle with center O. Show that for each point A distinct from O, C, and D on the line containing CD, there is a point B such that the line from D to any point P on the circle distinct from C and D bisects angle APB.

- **5143:** *Proposed by Valmir Krasniqi (student), Republic of Kosova*

  Show that

$$\sum_{n=1}^{\infty} Cos^{-1}\, \frac{1+\sqrt{n^2+2n}\cdot\sqrt{n^2-1}}{n\,(n+1)} \quad = \quad \frac{\pi}{2}.$$

$$(Cos^{-1} = Arccos)$$

- **5144:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Compute

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left[1+\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right)^{1/n} \right].$$

- **5145:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $k \geq 1$ be a natural number. Find the sum of

$$\sum_{n=1}^{\infty} \left(\frac{1}{1-x} - 1 - x - x^2 - \cdots - x^n \right)^k, \quad \text{for } |x| < 1.$$

Solutions

- **5122:** *Proposed by Kenneth Korbin, New York, NY*

Partition the first 32 non-negative integers from 0 to 31 into two sets A and B so that the sum of any two distinct integers from set A is equal to the sum of two distinct integers from set B and vice versa.

Solution 1 by Michael N. Fried, Kibbutz Revivim, Israel

Suppose A contains 0. This means that any other number in A must be the sum of two numbers in B. The next number in A, therefore, must be at least 3 since 3 is the smallest number that is the sum of two positive integers. On the other hand, the next number in A cannot be greater than 3, for 1 and 2 must still be in B. This group of four numbers forms a kind of unit, which we can represent graphically as follows:

$$\begin{matrix} 0 & & 3 \\ & \sqcup & \\ 1 & & 2 \end{matrix} \qquad \text{or} \qquad \begin{matrix} 1 & & 2 \\ & \sqcap & \\ 0 & & 3 \end{matrix}$$

The symmetry of the unit reflects the fact that $a + b = c + d$ if and only if $b - d = a - c$, that is if and only if there is some number k such that $c = a + k$ and $d = b - k$. Thus any four consecutive integers forming such a figure will have the property that the sum of the top pair of numbers equals the sum of the bottom pair.

(This makes the problem almost a geometrical one, for arranging the numbers in set A and B in parallel lines as in the figure above, the condition of the problem becomes that every pair of numbers in the first line corresponds to a pair of numbers in the second line.)

So our strategy for the problem will be to assemble units such as those above to produce larger units satisfying in each case the condition of the problem.

Let us then start with two. The first, as before is:

$$\begin{matrix} 0 & & 3 \\ & \sqcup & \\ 1 & & 2 \end{matrix}$$

And as we have already argued, the first two numbers of A and B *must* be arranged in this way. The second unit, then, will be either

$$\begin{matrix} 4 & & 7 \\ & \sqcup & \\ 5 & & 6 \end{matrix} \qquad \text{or} \qquad \begin{matrix} 5 & & 6 \\ & \sqcap & \\ 4 & & 7 \end{matrix}$$

The symmetrical combination,

$$\begin{matrix} 0 & & 3 \\ & \sqcup & \\ 1 & & 2 \end{matrix} \qquad \begin{matrix} 4 & & 7 \\ & \sqcup & \\ 5 & & 6 \end{matrix}$$

fails, because the pair $(0, 4)$ in the upper row has no matching pair in the second row.

However, the non-symmetrical combination works:

$$\begin{array}{ccc} 0 & & 3 \\ & \sqcup & \\ 1 & & 2 \end{array} \qquad \begin{array}{ccc} 5 & & 6 \\ & \sqcap & \\ 4 & & 7 \end{array}$$

Again, these two form a new kind of unit, and, as before, any eight consecutive integers forming a unit such as the above, will have the property that any pair of numbers in the top row will have the same sum as some pair in the bottom row.

So, let us try and fit together two units of this type, and let us call them R and S. As before, there are two possibilities, one symmetric and one anti-symmetric.
Since the anti-symmetric option worked before, let us try it again and call the top row **A** and the bottom row **B**.

$$\overbrace{\begin{array}{ccc} 0 & & 3 \\ & \sqcup & \\ 1 & & 2 \end{array} \quad \begin{array}{ccc} 5 & & 6 \\ & \sqcap & \\ 4 & & 7 \end{array}}^{R} \qquad\qquad \overbrace{\begin{array}{ccc} 8 & & 11 \\ & \sqcap & \\ 9 & & 10 \end{array} \quad \begin{array}{ccc} 13 & & 14 \\ & \sqcup & \\ 12 & & 15 \end{array}}^{S}$$

$$\begin{array}{rcl} \mathbf{A} & = & \{0,3,5,6,8,11,13,14\} \\ \mathbf{B} & = & \{1,2,4,7,9,10,12,15\} \end{array}$$

Now, to check whether this combination works we do not have to check $\binom{8}{2} = 28$ pairs of numbers.

All of the subunits will satisfy the condition of the problem. Indeed, we do not have to check pairs contained in the first and second, second and third and third and fourth terms, because they represent eight consecutive integers as discussed above. And we do not have to check pairs from the first and fourth terms because these also behave like a single unit R (where for example the pair (0,13) corresponds to (1,12) just as (0,5) corresponded to (1,4). So we only have to check pairs of numbers coming from the first and third elements and the second and fourth. But here we find a problem, for (2,10) in **B** cannot have a corresponding pair in **A.**

Let us then check the symmetrical arrangement:

$$\overbrace{\begin{array}{ccc} 0 & & 3 \\ & \sqcup & \\ 1 & & 2 \end{array} \quad \begin{array}{ccc} 5 & & 6 \\ & \sqcap & \\ 4 & & 7 \end{array}}^{R} \qquad\qquad \overbrace{\begin{array}{ccc} 9 & & 10 \\ & \sqcup & \\ 8 & & 11 \end{array} \quad \begin{array}{ccc} 12 & & 15 \\ & \sqcap & \\ 13 & & 14 \end{array}}^{S}$$

$$\begin{array}{rcl} \mathbf{A} & = & \{0,3,5,6,9,10,12,15\} \\ \mathbf{B} & = & \{1,2,4,7,8,11,13,14\} \end{array}$$

As in the anti-symmetrical arrangement, we need not check pairs of numbers in R or S, or, in this case, pairs if the first and third elements or second and fourth, which behave exactly as R and S individually. We need only check non-symmetrical pairs in the first and fourth elements and in the second and third. For the former this means (3,15) and (0,12) in **A** and (1,13) and (2,14) in **B.** For these we have corresponding pairs (3,15) to (7,8), (0,12) to (4,8), (1,13) to (5,9) and (2,14) to (6,10). Similarly, corresponding pairs exist for each non-smmetric pair in **A** and **B** in the second and third elements.

The above arrangement is then a new unit of 16 consecutive numbers satisfying the condition that every pair in the upper row **A,** has a correspnding pair of numbers in the second row **B,** with the same sum.

Finally, then, we want to join together two units, each of 16 consecutive integers as above, to partition the set of 32 consecutive integers $\{0, 1, 2, \ldots, 31\}$.
Reasoning as above, and checking only the critical elements in the unit for corresponding sums, we see that the symmetric case works.

$$\text{The symmetric case :}$$
$$\overbrace{\begin{matrix} 0 & & 3 & 5 & & 6 & 9 & & 10 & 12 & & 15 \\ & \sqcup & & & \sqcap & & & \sqcap & & & \sqcup & \\ 1 & & 2 & 4 & & 7 & 8 & & 11 & 13 & & 14 \end{matrix}} \quad \text{and} \quad \overbrace{\begin{matrix} 16 & & 19 & 21 & & 22 & 25 & & 26 & 28 & & 31 \\ & \sqcup & & & \sqcap & & & \sqcap & & & \sqcup & \\ 17 & & 18 & 20 & & 23 & 24 & & 27 & 29 & & 30 \end{matrix}}$$

Thus,

$$\begin{aligned} \mathbf{A} &= \{0, 3, 5, 6, 9, 10, 12, 15, 16, 19, 21, 22, 25, 26, 28, 31\} \\ \mathbf{B} &= \{1, 2, 4, 7, 8, 11, 13, 14, 17, 18, 20, 23, 24, 27, 29, 30\} \end{aligned}$$

Editor's comment: In Michael's solution each element in the set of four consecutive integers was written as being the vetex of an isosceles trapezoid. (The trapezoids were oriented with the bases being parallel to the top and bottom edges of page; Michael then manitpulated the trapezoids by flipping their bases.)

Adoración Martínez Ruiz of the Mathematics Club of the Institute of Secondary Education (No. 1) in Requena-Valencia, Spain also approached the problem geometrically in an almost identical manner as Michael. I adopted Adoración Martínez' notation of "cups" $\sqcup$ and "caps" $\sqcap$ instead of Michael's isosceles trapezoids in writing-up Michael's solution. (If the shorter base of the trapezoid was closer to the bottom edge of the page than the longer base, then that trapezoid became a cup, $\sqcup$; whereas if the shorter base of the trapezoid was closer to the top edge of the page than the longer base, then that trapezoid became a cap, $\sqcap$.

Michael's solution and Adoración Martínez' solution were identical to one another up until the last step. At that point Michael took the symmetric extension in moving from the first 16 non-negative integers to the first 32 non-negative integers, whereas Adoración Martínez took the anti-symmetric extension, and surprisingly (at least to me), each solution worked.

$$\text{Adoración Martínez}' \text{ anti} - \text{symmetric case :}$$
$$\overbrace{\begin{matrix} 0 & & 3 & 5 & & 6 & 9 & & 10 & 12 & & 15 \\ & \sqcup & & & \sqcap & & & \sqcap & & & \sqcup & \\ 1 & & 2 & 4 & & 7 & 8 & & 11 & 13 & & 14 \end{matrix}} \quad \text{and} \quad \overbrace{\begin{matrix} 17 & & 18 & 20 & & 23 & 24 & & 27 & 29 & & 30 \\ & \sqcap & & & \sqcup & & & \sqcup & & & \sqcap & \\ 16 & & 19 & 21 & & 22 & 25 & & 26 & 28 & & 31 \end{matrix}}$$

$$\begin{aligned} \mathbf{A} &= \{0, 3, 5, 6, 9, 10, 12, 15, 17, 18, 20, 23, 24, 27, 29, 30\} \\ \mathbf{B} &= \{1, 2, 4, 7, 8, 11, 13, 14, 16, 19, 21, 22, 25, 26, 28, 31\} \end{aligned}$$

So now we have two solutions to the problem, each motivated by geometry, and it was assumed (at least by me) that their were no other solutions. Michael challenged **Mayer Goldberg,** a colleague in CS here at BGU, to find other solutions, and he did; many of them! Following is his approach.

Solution 2 by Mayer Goldberg, Beer-Sheva, Israel

Notation: For any set S of integers, the set $aS + b$ is the set $\{ak + b : k \in S\}$.

Construction: We start with the set $A_0 = \{0, 4\}, b_0 = \{1, 2\}$. We define A_n, B_n inductively as follows:

$$A_{n+1} = (2A_n + 1) \cup (2B_n)$$

$$B_{n+1} = (2A_n) \cup (2B_n + 1)$$

Claim: The sets A_n, B_n partition the set $\{0, \ldots, 2^{n+2}\}$ according to the requirements of the problem.

Proof: By Induction. The sets $A_0 B_0$ satisfy the requirement trivially, since they each contain one pair, and by inspection, we see that the sums are the same. Assume that A_n, B_n satisfy the requirement. Pick $x_1, x_2 \in A_{n+1}$.

- **Case I:** $x_1 = 2x_3 + 1, x_2 = 2x_4 + 1$, for $x_3, x_4 \in A_n$. Then by the induction hypothesis (IH), there exists $y_3, y_4 \in B_n$, such that $x_3 + x_4 = y_3 + y_4$. Consequently,

$$x_1 + x_2 = 2(x_3 + x_4) + 2 = 2(y_3 + y_4) + 2 = (2y_3 + 1) + (2y_4 + 1).$$

So let $y_1 = 2y_3 + 1, y_2 = 2y_4 + 1 \in B_{n+1}$.

- **Cases II & III:** $x_1 = 2x_3 + 1, x_2 = 2y_4$, for $x_3 \in A_n, y_4 \in B_n$.

$$x_1 + x_2 = 2(x_3 + 1) + 2y_4 = 2x_3 + (2y_4 + 1).$$

So let $y_1 = 2x_3, y_2 = 2y_4 + 1 \in B_{n+1}$.

- **Case IV:** $x_1 = 2y_3, x_2 = 2y_4$, for $y_3, y_4 \in B_n$.Then by the IH, there exists $x_3, x_4 \in A_n$, suc that $y_3 + y_4 = x_3 + x_4$. Consequently,

$$x_1 + x_2 = 2y_3 + 2y_4 = 2(y_3 + y_4) = 2(x_3 + x_4) = 2x_3 + 2x_4.$$

So let $y_1 = 2x_3, y_2 = 2y_3 \in B_{n+1}$

Editor: This leads to potentially thousands of such pairs of sets that satisfy the criteria of the problem. Mayer listed about one hundred such examples, a few of which are reproduced below:

$$\begin{aligned} A &= \{0, 3, 5, 6, 9, 10, 13, 15, 17, 18, 19, 23, 24, 27, 29, 30\}, \\ B &= \{1, 2, 4, 7, 8, 11, 12, 14, 16, 20, 21, 22, 25, 26, 28, 31\} \end{aligned}$$

$$\begin{aligned} A &= \{0, 3, 5, 6, 9, 10, 11, 15, 17, 18, 19, 23, 24, 27, 29, 30\}, \\ B &= \{1, 2, 4, 7, 8, 12, 13, 14, 16, 20, 21, 22, 25, 26, 28, 31\} \end{aligned}$$

$$\begin{aligned} A &= \{0, 3, 5, 6, 9, 10, 14, 15, 17, 18, 19, 23, 24, 27, 29, 30\}, \\ B &= \{1, 2, 4, 7, 8, 11, 12, 13, 16, 20, 21, 22, 25, 26, 28, 31\} \end{aligned}$$

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$$\begin{aligned} A &= \{0, 3, 5, 6, 9, 10, 14, 15, 17, 18, 22, 23, 24, 27, 29, 30\}, \\ B &= \{1, 2, 4, 7, 8, 11, 12, 13, 16, 19, 20, 21, 25, 26, 28, 31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,9,10,15,16,17,18,19,23,24,27,29,30\}, \\ B &= \{1,2,4,7,8,11,12,13,14,20,21,22,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,9,10,15,16,17,18,22,23,24,27,29,30\}, \\ B &= \{1,2,4,7,8,11,12,13,14,19,20,21,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,8,9,13,15,17,18,20,23,24,27,29,30\}, \\ B &= \{1,2,4,7,10,11,12,14,16,19,21,22,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,8,9,11,15,17,18,20,23,24,27,29,30\}, \\ B &= \{1,2,4,7,10,12,13,14,16,19,21,22,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,8,9,15,16,17,18,20,23,24,27,29,30\}, \\ B &= \{1,2,4,7,10,11,12,13,14,19,21,22,25,26,28,31\} \end{aligned}$$

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$$\begin{aligned} A &= \{0,3,5,6,9,10,14,15,17,20,22,23,24,27,29,30\}, \\ B &= \ 1,2,4,7,8,11,12,13,16,18,19,21,25,26,28,31\} \end{aligned}$$

Editor (again): **Edwin Gray of Highland Beach, FL** working together with **John Kiltinen of Marquette, MI** claimed and proved by induction the following more general theorem:

Let $S = \{0,1,2,3,\ldots,2^n-1\},\ n > 1$. Then there is a partition of S, say A, B such that

1) $A \cup B = S,\ A \cap B = \emptyset,$ and

2) For all $x, y \in A$, there is an $r, s \in B,$ such that $x + y = r + s,$ and vice versa.

That is, the sum of any two elements in B has two elements in A equal to their sum.

David Stone and John Hawkins both of Statesboro, GA also claimed and proved a more general statement: They showed that: for $n \geq 2$, the set $S_n = \{0,1,2,\ldots,2^n-1\}$ consists of the non-negaitve integers which can be written with n or fewer binary digits. E.g.,

$$S_2 = \{0,1,2,3\} = \{00,01,10,11\} \text{ and}$$

$$S_3 = \{0,1,2,3,4,5,6,7\} = \{000,001,010,011,100,101,110,111\}$$

Their proof consisted of partitioning S_n into two subsets: E_n: those elements of S_n whose binary representation uses an even number of ones, and O_n: those numbers in S_n whose binary reprsentation uses an odd number of ones. Hence, for any $x \neq y$ in E_n, $x+y$ can be written as $x+y=w+z$ for some $w \neq z$ in O_n, and vice versa. This lead them to Adoración Martínez' solution, and they speculated on its uniqueness.

All of this seemed to be getting out-of-hand for me; at first I thought the solution is unique; then I thought that there are only two solutions, and then I thought that there are many solutions to the problem. **Shai Covo's** solutio/Users/admin/Desktop/SSM/For Jan 11/For Jan 11; Jerry.texn however, shows that the answer can be unique if one uses a notion of *sum multiplicity.*

Solution 3 by Shai Covo, Kiryat-Ono, Israel

We give two solutions, the first simple and original, the second sophisticated and more interesting, thanks to the Online Encyclopedia of Integer sequences(OEIS).

Assuming that $0 \in A$, one checks that we must have either

$$\begin{array}{ccc} \{0,3,5,6\} \cup \{25,26,28,31\} \subset A & \text{and} & \{1,2,4,7\} \cup \{24,27,29,30\} \subset B \\ & \text{or} & \\ \{0,3,5,6\} \cup \{24,27,29,30\} \subset A & \text{and} & \{1,2,4,7\} \cup \{25,26,28,31\} \subset B. \end{array}$$

In view of the first possibility, it is natural to examine the following sets:

$$A = \{0,3,5,6,9,10,13,14,17,18,21,22,25,26,28,31\}$$

$$B = \{1,2,4,7,8,11,12,15,16,19,20,23,24,27,29,30\}.$$

To see why this is natural, connect the numbers with arrows, in increasing order, starting with a vertical arrow pointing down to 1. Now, define

$$C = \{a_1 + a_2 \mid a_1, a_2 \in A, a_1 \neq a_2\} \subset \{3,4,5,\ldots,59\} \text{ and}$$

$$D = \{b_1 + b_2 \mid b_1, b_2 \in B, b_1 \neq b_2\} \subset \{3,4,5,\ldots,59\}.$$

We want to show that $C = D$, or equivalently, for every $x \in \{3,4,5,\ldots,59\}$ either $x \in C \cap D$ or $x \notin C \cup D$. Checking each x value, we find that

$$C \cap D = \{3,4,5,\ldots,59\} \setminus \{4,7,55,58\} \text{ and } \{4,7,55,58\} \cap (\mathrm{C} \cup \mathrm{D}) = \emptyset.$$

Thus, $C = D$, and so the problem is solved with A and B as above.

We now turn to the second solution. OEIS sequences $A001969$ (numbers with an even number of 1's in their binary expansion) and $A000069$ (numbers with an odd number of 1's in their binary expansion) "give the unique solution to the problem of splitting the nonnegative integers into two classes in such a way that sums of pairs of distinct elements from either class occur *with the same multiplicities.* [Lambek and Moser]." We have verified (by computer) that, in the case at hand, the sets

$$A = \{A001969(n) : A001969(n) \leq 30\}$$

$$
\begin{aligned}
&= \{0,3,5,6,9,10,12,15,17,18,20,23,24,27,29,30\} \text{ and}\\
B &= \{A000069(n): \ A000069(n) \le 31\}\\
&= \{1,2,4,7,8,11,13,14,16,19,21,22,25,26,28,31\}
\end{aligned}
$$

split the first 32 nonnegative integers from 0 to 31 in the manner stated for splitting the nonnegative integers. (The number 32 plays an important role here.) However, this is not the case for the sets A and B from the previous solution (consider, for example,12=3+9 versus 12=1+11, 12=4+8; there are seven more such examples.)

Editor (still again): I did not understand the notion about sums having the *same multiplicity,* but this is the key for having *a unique solution* to the problem, as it states in the OEIS. So I asked Shai to elaborate on this notion. Here is what he wrote:

The point is that "given the unique solution to the problem of splitting the nonnegative integers..." refers to the infinite set $\{0,1,2,3,...\}$ and not the finite set $\{0,1,2,...,31\}$. I should have stressed this point in my solution. As far as I can recall, I considered doing so, but decided not to, based on the following: "... the manner stated for splitting the nonnegative integers" only refers to "splitting the nonnegative integers into two classes in such a way that sums of pairs of distinct elements from either class occur with the same multiplicities," and not to "give the unique solution to the problem of splitting the nonnegative integers...".

In explaining the notion of itself, Shai wrote:

Consider Michael Fried's sets:

$$
\begin{aligned}
A &= \{0,3,5,6,9,10,12,15,16,19,21,22,25,26,28,31\}\\
B &= \{1,2,4,7,8,11,13,14,17,18,20,23,24,27,29,30\}.
\end{aligned}
$$

For set A, the number 16 can be decomposed as 0+16 and 6+10; hence the multiplicity is 2. For set B, on the other hand, 16 can only be decomposed as 2+14 (8+8 does not count, since we consider distinct elements only); hence the multiplicity is 1.

Also solved by Brian D. Beasley, Clinton, SC; Edwin Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; John Kiltinen, Marquette, MI; Charles McCracken, Dayton, OH; Adoración Martínez Ruiz, Requena-Valencia, Spain; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5123:** *Proposed by Kenneth Korbin, New York, NY*

Given isosceles triangle ABC with $\overline{AB} = \overline{BC} = 2011$ and with cevian $\overline{BD}$. Each of the line segments $\overline{AD}$, $\overline{BD}$, and $\overline{CD}$ have positive integer length with $\overline{AD} < \overline{CD}$.

Find the lengths of those three segments when the area of the triangle is minimum.

Solution by Shai Covo, Kiryat-Ono, Israel

We begin by observing that $\overline{AC} \in \{3, 4, \ldots, 4021\}$. This follows from $\overline{AC} < \overline{AB} + \overline{BC} = 4022$ and the assumption that $\overline{AC} = \overline{AD} + \overline{CD}$ is the sum of the distinct positive integers. The area S of triangle ABC can be expressed in terms of $\overline{AC}$ as

$$S = S\left(\overline{AC}\right) = \frac{\overline{AC}}{2}\sqrt{2011^2 - \left(\frac{\overline{AC}}{2}\right)^2}.$$

Define $f(x) = x^2(2011^2 - x^2), x \in [0, 2011]$. Then $S\left(\overline{AC}\right) = \sqrt{f\left(\overline{AC}/2\right)}$. It is readily verified that the function f (and hence $\sqrt{f}$) is unimodal with mode $m = 2011/\sqrt{2}$; that is, it is increasing for $x \le m$ and decreasing for $x \ge m$. If thus follows from $f(4021/2) < f(127/2)$ that $S(4021) < S(k)$ for any integer $127 \le k \le 4020$. Next by the law of cosines, we find that

$$\overline{BD}^2 = 2011^2 + \overline{AD}^2 - 2 \cdot 2011 \cdot \overline{AD} \cdot \frac{\overline{AC}/2}{2011}.$$

Hence,

$$\overline{AD}^2 - \overline{AC} \cdot \overline{AD} + \left(2011^2 - \overline{BD}^2\right) = 0.$$

The roots of this quadratic equation are given by the standard formula as

$$\overline{AD}_{1,2} = \frac{\overline{AC} \pm \sqrt{\overline{AC}^2 - 4\left(2011^2 - \overline{BD}^2\right)}}{2}.$$

However, we are given that $\overline{AD} < \overline{CD}$; hence $\overline{AD} = \overline{AD_2}$ and $\overline{CD} = \overline{AD_1}$, and we must have $\overline{AC}^2 > 4\left(2011^2 - \overline{BD}^2\right)$. Since, obviously, $\overline{BD} \le 2010$, we must have $\overline{AC^2} > 4\left(2011^2 - 2010^2\right) = 4 \cdot 4021$; hence, $127 \le \overline{AC} \le 4021$.

Thus, under the condition that S is minimum, we wish to find an integer value of $\overline{BD}(\le 2010)$ that makes $\overline{AD_{1,2}}$ (that is, $\overline{CD}$ and $\overline{AD}$) distinct integers when $\overline{AC}$ is set to 4021.

We thus look for $\overline{BD} \in \{1, 2, \ldots, 2010\}$ for which the discriminant $\Delta = 4021^2 - 4(2011^2 - BD^2)$ is a positive perfect square, say $\Delta = j^2$ with $j \in N$ (actually, $j = \overline{CD} - \overline{AD}$). This leads straightforwardly to the following equation:

$$\left(2\overline{BD} + j\right)\left(2\overline{BD} - j\right) = 3 \cdot 7 \cdot 383.$$

Since 3,7, and 383 are primes, we have to consider the following four cases:

- $\left(2\overline{BD} - j\right) = 1$ and $\left(2\overline{BD} + j\right) = 3 \cdot 7 \cdot 383$. This leads to $\overline{BD} = 2011$; however, $\overline{BD}$ must be less than 2011.
- $\left(2\overline{BD} - j\right) = 3$ and $\left(2\overline{BD} + j\right) = 7 \cdot 383$. This leads to $\overline{BD} = 671$ and $j = 1339$, and hence to our first solution:

$$\overline{AD} = 1341,\ \overline{BD} = 671,\ \overline{CD} = 2680.$$

- $\left(2\overline{BD}-j\right)=7$ and $\left(2\overline{BD}+j\right)=3\cdot 383$. This leads to $\overline{BD}=289$ and $j=571$, and hence to our second solution:

$$\overline{AD}=1725,\ \overline{BD}=289,\ \overline{CD}=2296.$$

- $\left(2\overline{BD}-j\right)=3\cdot 7$ and $\left(2\overline{BD}+j\right)=383$. This leads to $\overline{BD}=101$ and $j=181$, and hence to our third solution:

$$\overline{AD}=1920,\ \overline{BD}=101,\ \overline{CD}=2101.$$

Editor: **David Stone and John Hawkins** made two comments in their solution. They started off their solution by letting $r=\overline{AC}$, the length of the triangle's base. By Heron's formula, they obtained the triangle's area: $K=\dfrac{r}{4}\sqrt{4022^2-r^2}$ and then they made the following observations.

- **a)** $\overline{BD}=1$ and $\overline{CD}=2011$ gives us a triangle ABC with area $\left(\dfrac{1}{2}-\dfrac{1}{4\,(2011)^2}\right)\sqrt{4\,(2011^2)-1}\approx 2010.999689$ which is the smallest value that can be obtained **not** requiring $\overline{AD}$ to be an integer.
- **b)** Letting $m=\overline{AD}, n=\overline{CD}, k=\overline{BD}$, (where $1\le m<n$ and $\overline{AC}=m+n\le 4021$), and letting α be the base angle at vertex A (and at C), and dropping an altitude from B to side AC, we obtain a right triangle and see that

$$\cos\alpha=\frac{\overline{AC}/2}{2011}=\frac{m+n}{2\cdot 2011}.$$

Using the Law of Cosines in triangle BDC, we have

$$k^2=n^2+2011^2-2\cdot 2011\cdot n\cos\alpha=2011^2+n^2-n(m+n),$$

so we have a condition which the integers m,n and k must satisfy

$$k^2=2011^2-mn \qquad\qquad (\mathbf{1})$$

There are many triangles satisfying condition (**1**), some with interesting characteristics. There are no permissible triangles with base 4020, five with base 4019 and six with base 4018. All have larger areas than the champions listed above.

The altitude of each triangle in our winners group is 44.8 so the "shape ratio", altitude/base, is very small: 0.011. A wide flat triangle indeed!

One triangle with base 187 has a relatively small area: 187,825.16. This is as close as we can come to a tall, skinny triangle with small area. Its altitude/base ratio is 10.7.

In general, the largest isosceles triangle is an isosceles right triangle. With side lengths 2011, this would require a hypotenuse (our base) of $2011\sqrt{2}\approx 2843.98$. There are no permissible triangles with $r=2844$. Letting $r=2843$, we find the two largest permissible triangles:

$$m = 291,\ n=2552, \text{cevian}=1817 \text{ and area } 2,022,060.02$$

$$m = 883,\ n=1960, \text{cevian}=1521 \text{ and area } 2,022,060.02$$

The triangle with $m = 3, n = 2680$ (hence base =2683) has a large area: 2,009,788,52. The cevian has length 2009; it is very close to the side AB.

The triangle with $m = 1524, n = 1560$ and cevian=1291, comes closer than any other we found to having the cevian bisect the base. Its area is 1,990,528.49

Also solved by Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5124:** *Proposed by Michael Brozinsky, Central Islip, NY*

If $n > 2$ show that $\sum_{i=1}^{n} \sin^2\left(\frac{2\pi i}{n}\right) = \frac{n}{2}$.

Solution 1 by Piriyathumwong P. (student, Patumwan Demonstration School), Bangkok, Thailand

Since $\cos 2\theta = 1 - 2\sin^2\theta$, we have

$$\begin{aligned}\sum_{i=1}^{n} \sin^2\left(\frac{2\pi i}{n}\right) &= \frac{1}{2}\sum_{i=1}^{n}\left(1 - \cos\left(\frac{4\pi i}{n}\right)\right)\\ &= \frac{n}{2} - \frac{1}{2}\sum_{i=1}^{n}\cos\left(\frac{4\pi i}{n}\right)\end{aligned}$$

We now have to show that $S = \sum_{i=1}^{n}\cos\left(\frac{4\pi i}{n}\right) = 0$.

Multiplying both sides of S by $2\sin\left(\frac{2\pi}{n}\right)$, gives

$$\begin{aligned}2\sin\left(\frac{2\pi}{n}\right)\cdot S &= 2\sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{4\pi}{n}\right) + 2\sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{8\pi}{n}\right) + \ldots + 2\sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{4n\pi}{n}\right)\\ &= \left(\sin\left(\frac{6\pi}{n}\right) - \sin\left(\frac{2\pi}{n}\right)\right) + \left(\sin\left(\frac{10\pi}{n}\right) - \sin\left(\frac{6\pi}{n}\right)\right) + \ldots\\ &\quad + \left(\sin\left(\frac{(4n+2)\pi}{n}\right) - \sin\left(\frac{(4n-2)\pi}{n}\right)\right)\\ &= \sin\left(\frac{(4n+2)\pi}{n}\right) - \sin\left(\frac{2\pi}{n}\right)\\ &= 0\end{aligned}$$

Hence, $S = 0$, and we are done.

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

To avoid confusion with the complex number $i = \sqrt{-1}$, we will consider

$$\sum_{k=1}^{n} \sin^2\left(\frac{2\pi k}{n}\right).$$

If $R = e^{(4\pi i/n)}$, with $n > 2$, then $R \neq 1$ and $R^n = e^{4\pi i} = 1$. Then, using the formula for a geometric sum, we get

$$\sum_{k=1}^{n} R^k = R\frac{R^n - 1}{R - 1} = 0,$$

and hence,

$$\sum_{k=1}^{n} \cos\left(\frac{4\pi k}{n}\right) = \sum_{k=1}^{n} Re\left(R^k\right) = Re\left(\sum_{k=1}^{n} R^k\right) = 0.$$

Therefore, by the half-angle formula,

$$\sum_{k=1}^{n} \sin^2\left(\frac{2\pi k}{n}\right) = \frac{1}{2}\sum_{k=1}^{n}\left[1 - \cos\left(\frac{4\pi k}{n}\right)\right] = \frac{n}{2}.$$

Also solved by Daniel Lopez Aguayo (student, Institute of Mathematics, UNAM), Morelia, Mexico; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Shai Covo, Kiryat-Ono, Israel; Bruno Salgueiro Fanego, Viveiro, Spain; Michael N. Fried, Kibbutz Revivim, Israel; G.C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Pedro H. O. Pantoja, Natal-RN, Brazil; Paolo Perfetti, Department of Mathematics, University of Rome, Italy; Boris Rays, Brooklyn, NY; Raúl A. Simón, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5125:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{ab}{2(c+a)+5b} + \frac{bc}{2(a+b)+5c} + \frac{ca}{2(b+c)+5a} < \frac{11}{32}.$$

Solution by Kee-Wai Lau, Hong Kong, China

We prove the sharp inequality

$$\frac{ab}{2(c+a)+5b} + \frac{bc}{2(a+b)+5c} + \frac{ca}{2(b+c)+5a} < \frac{1}{3}. \qquad (1)$$

Let $x = \dfrac{a}{a+b+c}$, $y = \dfrac{b}{a+b+c}$, $z = \dfrac{c}{a+b+c}$ so that (1) can be written as

$$(a+b+c)\left(\frac{xy}{3y+2} + \frac{yz}{3z+2} + \frac{zx}{3x+2}\right) \leq \frac{1}{3}. \qquad (2)$$

Since

$$a+b+c = \sqrt{3\left(a^2+b^2+c^2\right) - (a-b)^2 - (b-c)^2 - (c-a)^2} \leq \sqrt{3\left(a^2+b^2+c^2\right)} = 3$$

so to prove (2), we need only prove that

$$\frac{xy}{3y+2}+\frac{yz}{3z+2}+\frac{zx}{3x+2}\le\frac{1}{9}. \qquad (3)$$

whenever x, y, z are positive and $x+y+z=1$. It is easy to check that (3) is equivalent to

$$\frac{x}{3y+2}+\frac{y}{3z+2}+\frac{z}{3x+2}\ge\frac{1}{3}. \qquad (4)$$

By the convexity of the function $\frac{1}{t}$, for $t>0$ and Jensen's inequality, we have

$$\frac{x}{3y+2}+\frac{y}{3z+2}+\frac{z}{3x+2}\ge\frac{1}{x(3y+2)+y(3z+2)+z(3x+2)}=\frac{1}{3\,(xy+yz+zx)+2}.$$

Now

$$xy+yz+zx=\frac{2\,(x+y+z)^2-(x-y)^2-(y-z)^2-(z-x)^2}{6}\le\frac{1}{3}$$

and so (4) holds. This proves (1) and equality holds when $a=b=c=1$.

Also solved by Shai Covo, Kiryat-Ono, Israel; Paolo Perfetti, Department of Mathematics, University of Rome, Italy, and the proposer.

- **5126:** *Proposed by Pantelimon George Popescu, Bucharest, Romania and José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c, d be positive real numbers and $f:[a,b]\to[c,d]$ be a function such that $|f(x)-f(y)|\ge|g(x)-g(y)|$, for all $x, y\in[a,b]$, where $g:R\to R$ is a given injective function, with $g(a),\ g(b)\in\{c,d\}$.

Prove

(**i**) $f(a)=c$ and $f(b)=d,$ or $f(a)=d$ and $f(b)=c.$

(**ii**) If $f(a)=g(a)$ and $f(b)=g(b),$ then $f(x)=g(x)$ for $a\le x\le b.$

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

To avoid trivial situations, we will assume that $a<b$. Then, since $g\,(x)$ is injective and $g\,(a)\,,g\,(b)\in\{c,d\}$, it follows that $c<d$ also.

First of all, the fact that $f\,(x)\in[c,d]$ for all $x\in[a,b]$ implies that

$$|f\,(x)-f\,(y)|\le d-c$$

for all $x, y\in[a,b]$.

(i) In particular, since $g\,(a)\,,g\,(b)\in\{c,d\}$, we have

$$d-c\ge|f\,(a)-f\,(b)|\ge|g\,(a)-g\,(b)|=d-c.$$

Hence, $|f\,(a)-f\,(b)|=d-c$ with $c\le f\,(a)\,,f\,(b)\le d$, and we get $f\,(a)=c$ and $f\,(b)=d$, or $f\,(a)=d$ and $f\,(b)=c$.

(ii) Suppose $f(a) = g(a) = c$ and $f(b) = g(b) = d$. The proof in the other case is similar. Then, since $c \leq f(x) \leq d$ for all $x \in [a, b]$, we obtain

$$\begin{aligned} d - c &= (d - f(x)) + (f(x) - c) \\ &= |d - f(x)| + |f(x) - c| \\ &= |f(b) - f(x)| + |f(x) - f(a)| \\ &\geq |g(b) - g(x)| + |g(x) - g(a)| \\ &= |d - g(x)| + |g(x) - c| \\ &\geq |d - c| \\ &= d - c. \end{aligned}$$

Thus, for all $x \in [a, b]$,

$$|d - f(x)| = |d - g(x)| \text{ and } |f(x) - c| = |g(x) - c|.$$

If there is an $x_0 \in [a, b]$ such that $f(x_0) \neq g(x_0)$, then

$$d - f(x_0) = g(x_0) - d \text{ and } f(x_0) - c = c - g(x_0)$$

and hence,

$$2d = f(x_0) + g(x_0) = 2c.$$

This is impossible since $c \neq d$. Therefore, $f(x) = g(x)$ for all $x \in [a, b]$.

Remark. The condition that $a, b, c, d > 0$ seems unnecessary for the solution of this problem.

Editor: **Shai Covo** suggested that the problem can be made more interesting by adding a third condition. Namely:

iii) If $f(a) \neq g(a)$ (or equivalently, $f(b) \neq g(b)$), then $f(x) + g(x) = c + d$ for all $x \in [a, b]$ and, hence, $f(x) - f(y) = g(y) - g(x)$ for all $x, y \in [a, b]$.

Also solved by Shai Covo, Kiryat-Ono, Israel; Paolo Perfetti, Department of Mathematics, University of Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- **5127:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $n \geq 1$ be an integer and let $T_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!}$, denote the $(2n-1)$th Taylor polynomial of the sine function at 0. Calculate

$$\int_0^\infty \frac{T_n(x) - \sin x}{x^{2n+1}} dx.$$

Solution by Paolo Perfetti, Department of Mathematics, University of Rome, Italy

Answer: $\dfrac{\pi(-1)^{n-1}}{2(2n)!}$

Proof: Integrating by parts:

$$\begin{aligned}
\int_0^\infty \frac{T_n(x) - \sin x}{x^{2n+1}} dx &= -\frac{1}{2n} \int_0^\infty (T_n(x) - \sin x)(x^{-2n})' dx \\
&= \frac{T_n(x) - \sin x}{-2nx^{2n}} \Big|_0^\infty + \frac{1}{2n} \int_0^\infty \frac{T_n'(x) - \cos x}{x^{2n}} dx \\
&= \frac{1}{2n} \int_0^\infty \frac{T_n'(x) - \cos x}{x^{2n}} dx
\end{aligned}$$

using $T_n(x) - \sin x = -\sum_{k=n+1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!}$ in the last equality.

After writing $T_n'(x) - \cos x = -\sum_{k=n+1}^{\infty} (-1)^{k-1} \frac{x^{2k-2}}{(2k-2)!}$, we do the second step.

$$\begin{aligned}
\int_0^\infty \frac{T_n'(x) - \cos x}{(2n)x^{2n}} dx &= \frac{-1}{2n(2n-1)} \int_0^\infty (T_n'(x) - \cos x)(x^{-2n+1})' dx \\
&= \frac{T_n'(x) - \cos x}{-2n(2n-1)x^{2n-1}} \Big|_0^\infty + \frac{1}{2n(2n-1)} \int_0^\infty \frac{T_n''(x) + \sin x}{x^{2n-1}} dx \\
&= \frac{1}{2n(2n-1)} \int_0^\infty \frac{T_n''(x) + \sin x}{x^{2n-1}} dx.
\end{aligned}$$

After $2n$ steps we obtain

$$\frac{(-1)^{n-1}}{(2n)!} \int_0^\infty \frac{\sin}{x} dx = \frac{\pi(-1)^{n-1}}{2(2n)!}$$

Also solved by Shai Covo, Kiryat-Ono, Israel; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.