Problems Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before March 15, 2010

- 5092: *Proposed by Kenneth Korbin, New York, NY*

  Given equilateral triangle ABC with altitude h and with cevian $\overline{CD}$. A circle with radius x is inscribed in $\triangle ACD$, and a circle with radius y is inscribed in $\triangle BCD$ with $x < y$. Find the length of the cevian $\overline{CD}$ if x, y and h are positive integers with $(x, y, h) = 1$.

- 5093: *Proposed by Worapol Ratanapan (student), Montfort College, Chiang Mai, Thailand*

  $6 = 1 + 2 + 3$ is one way to partition 6, and the product of $1, 2, 3$ is 6. In this case, we call each of $1, 2, 3$ a **part** of 6.

  We denote the maximum of the product of all **parts** of natural number n as $N(n)$.

  As a result, $N(6) = 3 \times 3 = 9$, $N(10) = 2 \times 2 \times 3 \times 3 = 36$, and $N(15) = 3^5 = 243$.

  More generally, $\forall n \in N$, $N(3n) = 3^n$, $N(3n+1) = 4 \times 3^{n-1}$, and $N(3n+2) = 2 \times 3^n$.

  Now let's define $R(r)$ in the same way as $N(n)$, but each **part** of r is positive real. For instance $R(5) = 6.25$ and occurs when we write $5 = 2.5 + 2.5$

  Evaluate the following:

  $$\begin{array}{rl} i) & R(2e) \\ ii) & R(5\pi) \end{array}$$

- 5094: *Proposed by Paolo Perfetti, Mathematics Department, Tor Vergata University, Rome, Italy*

  Let a, b, c be real positive numbers such that $a + b + c + 2 = abc$. Prove that

  $$2(a^2 + b^2 + c^2) + 2(a + b + c) \geq (a + b + c)^2.$$

- 5095: *Proposed by Zdravko F. Starc, Vršac, Serbia*

  Let F_n be the Fibonacci numbers defined by

  $$F_1 = F_2 = 1,\ F_{n+2} = F_{n+1} + F_n\ (n = 1, 2, \cdots).$$

Prove that

$$\sqrt{F_{n-2}F_{n-1}}+1 \le F_n \le \sqrt{(n-2)F_{n-2}F_{n-1}}+1 \ (n=3,4,\cdots).$$

- 5096: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b+\sqrt[4]{ab^3}}+\frac{b}{c+\sqrt[4]{bc^3}}+\frac{c}{a+\sqrt[4]{ca^3}} \ge \frac{3}{2}.$$

- 5097: *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $p \ge 2$ be a natural number. Find the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\lfloor \sqrt[p]{n} \rfloor},$$

where $\lfloor a \rfloor$ denotes the **floor** of a. (Example $\lfloor 2.4 \rfloor = 2$).

Solutions

- 5074: *Proposed by Kenneth Korbin, New York, NY*

Solve in the reals:

$$\sqrt{25+9x+30\sqrt{x}}-\sqrt{16+9x+30\sqrt{x-1}}=\frac{3}{x\sqrt{x}}.$$

Solution by Antonio Ledesma Vila, Requena-Valencia, Spain

Note that the domain of definition is $x \ge 1$, and that the two radicands are perfect squares:

$$\begin{aligned} 25+9x+30\sqrt{x} &= \left(3\sqrt{x}+5\right)^2 \\ 16+9x+30\sqrt{x-1} &= \left(3\sqrt{x-1}+5\right)^2 \end{aligned}$$

So

$$\begin{aligned} \sqrt{25+9x+30\sqrt{x}}-\sqrt{16+9x+30\sqrt{x-1}} &= \frac{3}{x\sqrt{x}} \\ \sqrt{\left(3\sqrt{x}+5\right)^2}-\sqrt{\left(3\sqrt{x-1}+5\right)^2} &= \frac{3}{x\sqrt{x}} \\ \left|3\sqrt{x}+5\right|-\left|3\sqrt{x-1}+5\right| &= \frac{3}{x\sqrt{x}} \\ (3\sqrt{x}+5)-(3\sqrt{x-1}+5) &= \frac{3}{x\sqrt{x}} \end{aligned}$$

$$\begin{aligned}
\sqrt{x}-\sqrt{x-1} &= \frac{1}{x\sqrt{x}}\\
\frac{1}{\sqrt{x}-\sqrt{x-1}} &= x\sqrt{x}\\
\sqrt{x}+\sqrt{x-1} &= x\sqrt{x}\\
\sqrt{x-1} &= (x-1)\sqrt{x}\\
(x-1) &= (x-1)^2x\\
(x-1)\Big(1-(x-1)x\Big) &= 0
\end{aligned}$$

Therefore, $x=1$ or $x^2-x-1=0 \Longrightarrow x=\dfrac{1\pm\sqrt{5}}{2}$. But $\dfrac{1-\sqrt{5}}{2}$ is an extraneous root. Hence, the only two real solutions are $x=1$ and $x=\dfrac{1+\sqrt{5}}{2}=\phi$, the golden ratio.

Also solved by Daniel Lopez Aguayo, Puebla, Mexico; José Luis Díaz-Barrero, Barcelona, Spain; Brian D. Beasley, Clinton, SC; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Katherine Janell Eyre (student, Angelo State University), San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Michael N. Fried, Kibbutz Revivim, Israel; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA; Ercole Suppa, Teramo, Italy; David C. Wilson, Winston-Salem, NC, and the proposer.

5075: *Proposed by Kenneth Korbin, New York, NY*

An isosceles trapezoid is such that the length of its diagonal is equal to the sum of the lengths of the bases. The length of each side of this trapezoid is of the form $a+b\sqrt{3}$ where a and b are positive integers.

Find the dimensions of this trapezoid if its perimeter is $31+16\sqrt{3}$.

Solution by Michael N. Fried, Kibbutz Revivim, Israel

Let the equal sides be $s=a+b\sqrt{3}$ and the bases be $b_1=p+q\sqrt{3}$ and $b_2=u+v\sqrt{3}$. Since each of its diagonals d is the sum of the bases, we have:

$$d=b_1+b_2=(p+u)+(q+v)\sqrt{3}=y+x\sqrt{3},$$

where a,b,p,q,u,v, and accordingly, y and x are all positive integers.

We begin by making some observations.
I. Since the diagonal $d = b_1 + b_2$, we have $P = 2s + d = 31 + 16\sqrt{3} \quad (1)$

II. From (1), we have,

$$\begin{aligned} s = a + b\sqrt{3} &= \left(\frac{31-y}{2}\right) + \left(\frac{16-x}{2}\right)\sqrt{3} \text{ or} \\ a &= \frac{31-y}{2} \quad (2) \\ b &= \frac{16-x}{2} \quad (3) \end{aligned}$$

And since a and b are positive integers, (2) and (3) imply that *y is odd and x even.*

III. Since any isosceles trapezoid can be inscribed in a circle, we can apply Ptolemy's theorem here to obtain the equation: $d^2 - s^2 = b_1 b_2$ (4). This, together with the fact that $d = b_1 + b_2$, implies that the bases b_1 and b_2 are the solutions of the equation $\mathrm{b}^2 - db + (d^2 - s^2) = 0$. Thus:

$$\begin{aligned} b_1 &= \frac{1}{2}\left(d + \sqrt{4s^2 - 3d^2}\right) \quad (5) \\ b_2 &= \frac{1}{2}\left(d - \sqrt{4s^2 - 3d^2}\right) \quad (6) \end{aligned}$$

IV. Since $b_1 = p + q\sqrt{3}$ and $b_2 = u + v\sqrt{3}$ where $p, q, u,$ and v are integers, it follows from (5) and (6) that

$$4s^2 - 3d^2 = \left(K + L\sqrt{3}\right)^2 = K^2 + 3L^2 + 2KL\sqrt{3} \quad (7)$$

where K and L are integers.

Now, let us find bounds for d and, from those, bounds for y and x. But to start, let us find bounds for $\dfrac{s}{d}$.

From equation (4), we have:

$$\begin{aligned} \frac{s^2}{d^2} = 1 - \frac{b_1 b_2}{d^2} &= 1 - \frac{b_1 b_2}{(b_1 + b_2)^2} \\ &= 1 - \frac{1}{4}\left(\frac{(b_1+b_2)^2 - (b_1 - b_2)^2}{(b_1+b_2)^2}\right) \\ &= \frac{3}{4} + \frac{1}{4}\left(\frac{b_1 - b_2}{b_1 + b_2}\right)^2 \end{aligned}$$

Thus,

$$\frac{3}{4} < \frac{s^2}{d^2} < 1$$

or

$$\frac{\sqrt{3}}{2} < \frac{s}{d} < 1.$$

From this, we can write,

$$1 + \sqrt{3} < \frac{2s + d}{d} < 3.$$

By (1), we can substitue $31 + 16\sqrt{3}$ for $2s + d$, thus eliminating s. With that, we obtain:

$$\frac{31 + 16\sqrt{3}}{3} < d < \frac{31 + 16\sqrt{3}}{1 + \sqrt{3}} \qquad (8)$$

Replacing d by $y + x\sqrt{3}$, we can rewrite (8) as bounds for y in terms of x:

$$\frac{31 + (16 - 3x)\sqrt{3}}{3} < y < \frac{(31 - 3x) + (16 - x)\sqrt{3}}{1 + \sqrt{3}} \qquad (9)$$

Since y must be a positive integer, x cannot exceed 11, otherwise y will be either negative or less than 1. Also, recalling observation II, x must be even and y must be odd. Replacing x successively by $2, 4, 6, 8$, and 10, then, we find by (9) that the corresponding values of y will be $17, 13, 11, 7$, and 3. From these values, in turn, we can then find a and b by equations (2) and (3). The five possibilities we have to check are summarized in the following table.

$$\left\{\begin{array}{cc} d = y + x\sqrt{3} & s = a + b\sqrt{3} \\ x = 2 \quad y = 17 & a = 7 \quad b = 7 \\ x = 4 \quad y = 13 & a = 9 \quad b = 6 \\ x = 6 \quad y = 11 & a = 10 \quad b = 5 \\ x = 8 \quad y = 7 & a = 12 \quad b = 4 \\ x = 10 \quad y = 3 & a = 14 \quad b = 3 \end{array}\right\}$$

Now, in observation IV, we found $4s^2 - 3d^2 = \left(K + L\sqrt{3}\right)^2 = K^2 + 3L^2 + 2KL\sqrt{3}$ which of course must be a positive number. This immediately eliminates the first and last possibilities, $d = 17 + 2\sqrt{3}, s = 7 + 7\sqrt{2}$, and $d = 3 + 10\sqrt{3}, s = 14 + 3\sqrt{2}$ since the rational part of $4s^2 - 3d^2$ (that is, the part not multiplying $\sqrt{3}$) is negative for these pairs.

This leaves only the second, third, and fourth possibilities. The rational parts of $4s^2 - 3d^2$ for these are, respectively, $105, 13$, and 45. It is then easy to check that only $13 = 1^2 + 3 \times 2^2$ corresponding to $d = 11 + 6\sqrt{3}, s = 10 + 5\sqrt{3}$ can be written in the form $K^2 + 3L^2$, and the irrational part is also $4 = 2KL$.

Hence, these together with equations (5) and (6), give us our solution:

$$\begin{array}{rcl} s & = & 10 + 5\sqrt{3} \\ b_1 & = & 6 + 4\sqrt{3} \\ b_2 & = & 5 + 2\sqrt{3} \end{array}$$

Also solved by Mayer Goldberg, Beer-Sheva, Israel; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

5076: *Proposed by M.N. Deshpande, Nagpur, India*

Let a, b, and m be positive integers and let F_n satisfy the recursive relationship

$$F_{n+2} = mF_{n+1} + F_n, \text{ with } F_0 = a, F_1 = b, n \geq 0.$$

Furthermore, let $a_n = F_n^2 + F_{n+1}^2, n \geq 0$. Show that for every a, b, m, and n,

$$a_{n+2} = (m^2+2)a_{n+1} - a_n.$$

Solution 1 by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX

From the given,

$$\begin{aligned}
a_{n+2} &= F_{n+2}^2 + F_{n+3}^2 \\
&= F_{n+2}^2 + (mF_{n+2} + F_{n+1})^2 \\
&= F_{n+2}^2 + m^2F_{n+2}^2 + mF_{n+1}F_{n+2} + mF_{n+1}F_{n+2} + F_{n+1}^2 \\
&= F_{n+2}^2 + m^2F_{n+2}^2 + mF_{n+1}F_{n+2} + mF_{n+1}(F_n + mF_{n+1}) + F_{n+1}^2 \\
&= F_{n+2}^2 + m^2F_{n+2}^2 + mF_{n+1}(F_{n+2} + F_n) + m^2F_{n+1}^2 + F_{n+1}^2 \\
&= F_{n+2}^2 + m^2F_{n+2}^2 + (F_{n+2} - F_n)(F_{n+2} + F_n) + m^2F_{n+1}^2 + F_{n+1}^2 \\
&= F_{n+2}^2(m^2+2) + F_{n+1}^2(m^2+1) - F_n^2 \\
&= (F_{n+2}^2 + F_{n+1}^2)(m^2+2) - (F_n^2 + F_{n+1}^2) \\
&= (m^2+2)a_{n+1} - a_n.
\end{aligned}$$

Solution 2 by G. C. Greubel, Newport News, VA

Changing the terms slightly we shall use the more familiar Fibonacci polynomial terminology. The fibonacci polynomials are given by

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x).$$

The Binet form of the Fibonacci polynomials is given by

$$F_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where

$$\begin{aligned}
\alpha &= \alpha(x) = \frac{1}{2}\left(x + \sqrt{x^2+4}\right) \\
\beta &= \beta(x) = \frac{1}{2}\left(x - \sqrt{x^2+4}\right).
\end{aligned}$$

Also, the Lucas polynomials are given by

$$L_n(x) = \alpha^n + \beta^n$$

and satisfies the recurrence relation

$$L_{n+2}(x) = xL_{n+1}(x) + L_n(x).$$

The term to be considered is

$$a_n = F_{n+1}^2(x) + F_n^2(x).$$

It can be seen that

$$F_n^2(x) = \frac{1}{x^2+4}\left(L_{2n}(x) - 2(-1)^n\right).$$

This leads to the relation

$$a_n = \frac{1}{x^2+4}\left(L_{2n+1}(x) + L_{2n}(x)\right).$$

The relation being asked to show is given by

$$a_{n+2} = \left(x^2+2\right)a_{n+1} - a_n.$$

Let $\phi_n = (x^2+2)\,a_{n+1} - a_n$ for the purpose of demonstration. With the use of the above equations we can see the following:

$$\begin{aligned}
\left(x^2+4\right)\phi_n &= \left(x^2+4\right)\left[\left(x^2+2\right)a_{n+1} - a_n\right] \\
&= \left(x^2+2\right)\left(L_{2n+3} + L_{2n+2}\right) - \left(L_{2n+1} + L_{2n}\right) \\
&= \left(x^2+2\right)\left(\left(x^2+x+1\right)L_{2n+1} + (x+1)L_{2n}\right) - \left(L_{2n+1} + L_{2n}\right) \\
&= \left(x^4+x^3+3x^2+2x+1\right)L_{2n+1} + \left(x^3+x^2+2x+1\right)L_{2n} \\
&= \left(x^3+x^2+2x+1\right)L_{2n+2} + \left(x^2+x+1\right)L_{2n+1} \\
&= \left(x^2+x+1\right)L_{2n+3} + (x+1)L_{2n+2} \\
&= xL_{2n+4} + L_{2n+4} + L_{2n+3} \\
&= L_{2n+5} + L_{2n+4}. \quad \mathbf{(1)}
\end{aligned}$$

From the equation $a_n = \frac{1}{x^2+4}\left(L_{2n+1}(x) + L_{2n}(x)\right)$ we have

$$\left(x^2+4\right)a_{n+2} = L_{2n+5} + L_{2n+4}. \quad \mathbf{(2)}$$

Comparing the result of (1) to that of (2) leads to $\phi_n = a_{n+2}$. Thus we have the relation

$$a_{n+2} = \left(x^2+2\right)a_{n+1} - a_n$$

and this provides the relation being sought.

Brian D. Beasley, Clinton, SC; Valmir Bucaj (student, Texas Lutheran University) Seguin, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA; David C. Wilson, Winston-Salem, NC, and the proposer.

5077: *Proposed by Isabel Iriberri Díaz and José Luis Díaz-Barrero, Barcelona, Spain*

Find all triplets (x, y, z) of real numbers such that

$$\left.\begin{array}{l} xy(x+y-z)=3, \\ yz(y+z-x)=1, \\ zx(z+x-y)=1. \end{array}\right\}$$

Solution by Ercole Suppa, Teramo, Italy

From the second and third equation it follows that

$$yz(y+z) = zx(z+x) \qquad \Longleftrightarrow \qquad (x-y)(x+y+z) = 0.$$

If $x+y+z=0$ the first two equations yield $-2xyz = 3$ and $-xyz = 1$ which is impossible.

If $x = y$ then the system can rewritten as

$$\begin{array}{rcl} x^2(2x-z) & = & 3 \\ z^2y & = & 1 \\ z^2x & = & 1 \end{array}$$

Thus $x = \dfrac{1}{z^2}$ and

$$\begin{array}{rcl} \dfrac{1}{z^4}\left(\dfrac{2}{z^2} - z\right) & = & 3 \\ 3z^6 + z^3 - 2 & = & 0 \\ (3z^3-2)(z^3+1) & = & 0 \end{array}$$

The equation $(3z^3-2)(z^3+1) = 0$ factors into

$$\left(3^{1/3}z - 2^{1/3}\right)\left(3^{2/3}z^2 + (3^{1/3}\cdot 2^{1/3})z + 2^{2/3}\right)(z+1)(z^2-z+1) = 0.$$

Setting each factor equal to zero we see that only the first and third factors give real roots for the unknown z. So, the real roots are $z = \sqrt[3]{\dfrac{2}{3}}$ and $z = -1$. And since $x = y = \dfrac{1}{z^2}$ we see that $(1, 1, -1)$ and $\left(\sqrt[3]{\dfrac{9}{4}}, \sqrt[3]{\dfrac{9}{4}}, \sqrt[3]{\dfrac{2}{3}}\right)$ are the only real triplets (x, y, z) that satisfy the given system.

Also solved by Daniel Lopez Aguayo, Puebla, Mexico; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; M. N. Deshpande, Nagpur, India; Bruno Salgueiro Fanego, Viveiro, Spain; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY;

Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA; Antonio Ledesma Vila, Requena-Valencia, Spain, and the proposers.

5078: *Proposed by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy*

Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{a}{\sqrt{b}(b+c)}+\frac{b}{\sqrt{c}(a+c)}+\frac{c}{\sqrt{a}(a+b)}\geq \frac{3}{2}\frac{1}{\sqrt{ab+ac+cb}}.$$

Solution by Kee-Wai Lau, Hong Kong, China

For $x>0$, let $f(x)$ be the convex function x^{-1} so that we have

$$\begin{aligned} & \frac{a}{\sqrt{b}(b+c)}+\frac{b}{\sqrt{c}(a+c)}+\frac{c}{\sqrt{a}(a+b)} \\ = \; & af\Big(\sqrt{b}(b+c)\Big)+bf\Big(\sqrt{c}(a+c)\Big)+cf\Big(\sqrt{a}(a+b)\Big) \\ \geq \; & f\Big(a\sqrt{b}(b+c)+b\sqrt{c}(a+c)+c\sqrt{a}(a+b)\Big) \\ = \; & \frac{1}{a\sqrt{b}(b+c)+b\sqrt{c}(a+c)+c\sqrt{a}(a+b)}. \qquad (1) \end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & a\sqrt{b}(b+c)+b\sqrt{c}(a+c)+c\sqrt{a}(a+b) \\ = \; & \Big(\sqrt{ab(b+c)}\Big)\Big(\sqrt{a(b+c)}\Big)+\Big(\sqrt{bc(a+c)}\Big)\Big(\sqrt{b(a+c)}\Big)+\Big(\sqrt{ca(a+b)}\Big)\Big(\sqrt{c(a+b)}\Big) \\ \leq \; & \Big(\sqrt{ab(b+c)+bc(a+c)+ca(a+b)}\Big)\Big(\sqrt{a(b+c)+b(a+c)+c(a+b)}\Big) \\ = \; & \Big(\sqrt{ab^2+bc^2+ca^2+3abc}\Big)\Big(\sqrt{2(ab+bc+ca)}\Big). \qquad (2) \end{aligned}$$

By (1) and (2), it suffices for us to show that $ab^2+bc^2+ca^2+3abc \leq \frac{2}{9}$. In fact,

$$ab^2+bc^2+ca^2+3abc$$

$$
\begin{aligned}
&= \left(a+b+c-\frac{2}{3}\right)(ab+bc+ca)+\frac{a+b+c}{9}-b\left(a-\frac{1}{3}\right)^2-c\left(b-\frac{1}{3}\right)^2-a\left(c-\frac{1}{3}\right)^2\\
&\le \frac{ab+bc+ca}{3}+\frac{1}{9}\\
&= \frac{(a+b+c)^2}{9}-\frac{(a-b)^2+(b-c)^2+(c-a)^2}{18}+\frac{1}{9}\\
&\le \frac{2}{9}.
\end{aligned}
$$

This completes the solution.

Also solved by Boris Rays, Brooklyn, NY, and the proposer.

5079: *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $x \in (0,1)$ be a real number. Study the convergence of the series

$$\sum_{n=1}^{\infty} x^{\sin\frac{1}{1}+\sin\frac{1}{2}+\cdots+\sin\frac{1}{n}}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

For positive integers n and $x \in (0,1)$, let $a_n = a_n(x) = x^{\sin\frac{1}{1}+\sin\frac{1}{2}+\cdots+\sin\frac{1}{n}}$.

Since $\sin\dfrac{1}{n+1} = \dfrac{1}{n}+O\left(\dfrac{1}{n^2}\right)$ as n tends to infinity, so

$$
\begin{aligned}
\left|\frac{a_n}{a_{n+1}}\right| &= \exp\left(\left(\sin\frac{1}{n+1}\right)\left(\ln\frac{1}{x}\right)\right)\\
&= 1+\left(\sin\frac{1}{n+1}\right)\left(\ln\frac{1}{x}\right)+\sum_{m=2}^{\infty}\frac{\left(\left(\sin\frac{1}{n+1}\right)\left(\ln\frac{1}{x}\right)\right)^m}{m!}\\
&= 1+\frac{1}{n}\ln\left(\frac{1}{x}\right)+O\left(\frac{1}{n^2}\right),
\end{aligned}
$$

where the constant implied by the last O depends at most on x. Hence, by Gauss' test, the series of the problem is convergent if $0 < x < \dfrac{1}{e}$ and is divergent if $\dfrac{1}{e} \le x < 1$.

Solution 2 by David Stone and John Hawkins (jointly), Statesboro, GA

Our answer: we have convergence if $0 < x < \dfrac{1}{e}$ and divergence if $\dfrac{1}{e} \leq x < 1$.

We start by looking at the sum $\displaystyle\sum_{i=1}^{n} \sin\frac{1}{k}$. Each term of the sum, $\sin\dfrac{1}{k}$, can be expanded in an alternating series $\sin\dfrac{1}{k} = \dfrac{1}{k} - \dfrac{1}{3!}\left(\dfrac{1}{k}\right)^3 + \cdots$. The error from terminating the series after the first term does not exceed the second term. Thus we have

$$\begin{aligned}
\left|\sin\frac{1}{k} - \frac{1}{k}\right| &< \frac{1}{3!}\left(\frac{1}{k}\right)^3, \text{ so}\\
-\frac{1}{6k^3} &< \sin\frac{1}{k} - \frac{1}{k} < \frac{1}{6k^3}\\
\frac{1}{k} - \frac{1}{6k^3} &< \sin\frac{1}{k} < \frac{1}{k} + \frac{1}{6k^3}. \text{ Therefore,}\\
\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k^3} &< \sum_{k=1}^{n}\sin\frac{1}{k} < \sum_{k=1}^{n}\frac{1}{k} + \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k^3}.
\end{aligned}$$

The series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^3}$ is known to be convergent, say to L, which is grater than any of its partial sums.

Moreover, by looking at the graph of $y = 1/x$ we see that

$$\begin{aligned}
\frac{1}{k} &< \int_{k-1}^{k}\frac{1}{u}du = \ln k - \ln(k-1), \text{ and}\\
\frac{1}{k} &> \int_{k}^{k+1}\frac{1}{u}du = \ln(k+1) - \ln(k).
\end{aligned}$$

Using these for our bound on the partial sum of $\sin\dfrac{1}{k}$, we obtain

$$\sum_{k=1}^{n}\Big(\ln(k+1) - \ln k\Big) - \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k^3} < \sum_{k=1}^{n}\frac{1}{k} - \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k^3} < \sum_{k=1}^{n}\sin\frac{1}{k}, \text{ so}$$

$$\ln(n+1) - \frac{1}{6}L < \sum_{k=1}^{n}\frac{1}{k} - \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k^3} < \sum_{k=1}^{n}\sin\frac{1}{k}.$$

On the other hand,

$$\sum_{k=1}^{n}\sin\frac{1}{k} < \sum_{k=1}^{n}\frac{1}{k} + \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k^3} < 1 + \ln n + \frac{1}{6}L.$$

Thus we have bounds on the sine sum:

$$\ln(n+1) - \frac{1}{6}L < \sum_{i=1}^{n}\sin\frac{1}{k} < 1 + \ln n + \frac{1}{6}L.$$

We use this to investigate the convergence so the series $\sum_{n=1}^{\infty} x^{\sin\frac{1}{1}+\sin\frac{1}{2}+\cdots+\sin\frac{1}{n}}$.

Since $0 < x < 1$, we know that x^u is a decreasing function of u. Thus

$$x^{-\frac{1}{6}L+\ln(n+1)} > x^{\sum_{k=1}^{n}\sin\frac{1}{k}} > x^{\frac{1}{6}L+\ln n}$$

and we have

$$x^{\frac{1}{6}L+1}\sum_{n=1}^{t} x^{\ln n} < \sum_{n=1}^{t} x^{\sum_{k=1}^{n}\sin\frac{1}{k}} < x^{-\frac{1}{6}L}\sum_{n=1}^{t} x^{\ln(n+1)}.$$

Noting that

$$x^{\ln n} = e^{\ln(x^{\ln n})} = e^{(\ln n)(\ln x)} = e^{\ln n^{\ln x}} = n^{\ln x}$$

we can rewrite the outside sums to obtain

$$x^{\frac{1}{6}L+1}\sum_{n=1}^{t} n^{\ln x} < \sum_{n=1}^{t} x^{\sum_{k=1}^{n}\sin\frac{1}{k}} < x^{-\frac{1}{6}L}\sum_{n=1}^{t} (n+1)^{\ln x}.$$

It is well known that the series $\sum_{n=1}^{\infty} n^{\alpha}$ diverges if $\alpha \geq -1$. Hence, if $\ln x \geq -1$, the series $\sum_{n=1}^{\infty} x^{\sum_{k=1}^{n}\sin\frac{1}{k}}$ dominates the divergent series $\sum_{n=1}^{\infty} x^{\ln x}$ and thus diverges. That is, we have divergence if $1 > x \geq \frac{1}{e}$.

Likewise, it is well known that $\sum_{n=1}^{\infty}(n+1)^{\alpha}$ converges if $\alpha < -1$. So if $\ln x < -1$, the series $\sum_{n=1}^{\infty} x^{\sum_{k=1}^{n}\sin\frac{1}{k}}$ is dominated by the convergent series $\sum_{n=1}^{\infty}(n+1)^{\ln x}$ and thus converges.

That is, we have convergence if $0 < x < \frac{1}{e}$.

Also solved by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, and the proposer.