Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before
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• 5050: Proposed by Kenneth Korbin, New York, NY.
Given \( \triangle ABC \) with integer-length sides, and with \( \angle A = 120^\circ \), and with \((a, b, c) = 1\).
Find the lengths of \( b \) and \( c \) if side \( a = 19 \), and if \( a = 19^2 \), and if \( a = 19^4 \).

• 5051: Proposed by Kenneth Korbin, New York, NY.
Find four pairs of positive integers \((x, y)\) such that \(\frac{(x - y)^2}{x + y} = 8\) with \(x < y\).
Find a formula for obtaining additional pairs of these integers.

• 5052: Proposed by Juan-Bosco Romero Márquez, Valladolid, Spain.
If \(a \geq 0\), evaluate:
\[
\int_0^{+\infty} \arctg \frac{2a(1 + ax)}{x^2(1 + a^2) + 2ax + 1 - a^2} \frac{dx}{1 + x^2}.
\]

• 5053: Proposed by Panagiote Ligouras, Alberobello, Italy.
Let \(a, b\) and \(c\) be the sides, \(r\) the in-radius, and \(R\) the circum-radius of \(\triangle ABC\). Prove or disprove that
\[
\frac{(a + b - c)(b + c - a)(c + a - b)}{a + b + c} \leq 2rR.
\]

• 5054: Proposed by José Luis Díaz-Barrero, Barcelona, Spain.
Let \(x, y, z\) be positive numbers such that \(xyz = 1\). Prove that
\[
\frac{x^3}{x^2 + xy + y^2} + \frac{y^3}{y^2 + yz + z^2} + \frac{z^3}{z^2 + zx + x^2} \geq 1.
\]

• 5055: Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.
Let \(\alpha\) be a positive real number. Find the limit
\[
\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{n + k^\alpha}.
\]

**Solutions**

- 5032: Proposed by Kenneth Korbin, New York, NY.

Given positive acute angles \(A, B, C\) such that

\[
\tan A \cdot \tan B + \tan B \cdot \tan C + \tan C \cdot \tan A = 1.
\]

Find the value of

\[
\frac{\sin A}{\cos B \cdot \cos C} + \frac{\sin B}{\cos A \cdot \cos C} + \frac{\sin C}{\cos A \cdot \cos B}.
\]

**Solution 1 by Brian D. Beasley, Clinton, SC.**

Since \(A, B,\) and \(C\) are positive acute angles with

\[
1 = \frac{\sin A \sin B \cos C + \cos A \sin B \sin C + \sin A \cos B \sin C}{\cos A \cos B \cos C},
\]

we have \(\cos(A + B + C) = 0\) and thus \(A + B + C = 90^\circ\). Then

\[
\frac{\sin A}{\cos B \cos C} + \frac{\sin B}{\cos A \cos C} + \frac{\sin C}{\cos A \cos B} = \frac{\sin A \cos A + \sin B \cos B + \sin C \cos C}{\cos A \cos B \cos C}.
\]

Letting \(N\) be the numerator of this latter fraction, we obtain

\[
N = \sin A \cos A + \sin B \cos B + \cos(A + B) \sin(A + B)
\]

\[
= \sin A \cos A + \sin B \cos B + (\cos A \cos B - \sin A \sin B)(\sin A \cos B + \cos A \sin B)
\]

\[
= \sin A \cos A(1 + \cos^2 B - \sin^2 B) + \sin B \cos B(1 + \cos^2 A - \sin^2 A)
\]

\[
= \sin A \cos A(2 \cos^2 B) + \sin B \cos B(2 \cos^2 A)
\]

\[
= 2 \cos A \cos B(\sin A \cos B + \cos A \sin B)
\]

\[
= 2 \cos A \cos B \sin(A + B)
\]

\[
= 2 \cos A \cos B \cos C.
\]

Hence the desired value is 2.

**Solution 2 by Kee-Wai Lau, Hong Kong, China.**

The condition \(\tan A \tan B + \tan B \tan C + \tan C \tan A = 1\) is equivalent to \(\cot A + \cot B + \cot C = \cot A \cot B \cot C\). Since it is well known that

\[
\cos(A + B + C) = -\sin A \sin B \sin C \left(\cot A + \cot B + \cot C - \cot A \cot B \cot C\right),
\]

so \(\cos(A + B + C) = 0\) and \(A + B + C = \frac{\pi}{2}\). Hence,

\[
\sin 2A + \sin 2B + \sin 2C = 2 \sin(A + B) \cos(A - B) + 2 \sin C \cos C
\]
\[
2 \cos C \left( \cos(A - B) + \cos(A + B) \right) = 4 \cos A \cos B \cos C.
\]

If follows that
\[
\frac{\sin A}{\cos B \cos C} + \frac{\sin B}{\cos A \cos C} + \frac{\sin C}{\cos A \cos B} = \frac{\sin 2A + \sin 2B + \sin 2C}{2 \cos A \cos B \cos C} = 2.
\]

Solution 3 by Boris Rays, Chesapeake, VA.

\[
tan A \tan B + tan B \tan C + tan C \tan A = 1 \text{ implies,}
\]
\[
tan B(tan A + tan C) = 1 - tan A tan C
\]
\[
tan A + tan C
\]
\[
1 - tan A tan C
\]
\[
tan(A + C) = \cot B = \tan(90^\circ - B).
\]

Similarly, we obtain:
\[
tan(B + C) = \frac{1}{\tan A} = \cot A = \tan(90^\circ - A)
\]
\[
tan(A + B) = \frac{1}{\tan C} = \cot C = \tan(90^\circ - C), \text{ which implies}
\]
\[
A = 90^\circ - (B + C)
\]
\[
B = 90^\circ - (A + C)
\]
\[
C = 90^\circ - (A + B).
\]

Therefore,
\[
\frac{\sin A}{\cos B \cos C} + \frac{\sin B}{\cos A \cos C} + \frac{\sin C}{\cos A \cos B} = \frac{\sin(90^\circ - (B + C))}{\cos B \cos C} + \frac{\sin(90^\circ - (A + C))}{\cos A \cos C} + \frac{\sin(90^\circ - (A + B))}{\cos A \cos B}
\]
\[
= \frac{\cos(B + C)}{\cos B \cos C} + \frac{\cos(A + C)}{\cos A \cos C} + \frac{\cos(A + B)}{\cos A \cos B}
\]
\[
= \frac{\cos B \cos C - \sin B \sin C}{\cos B \cos C} + \frac{\cos A \cos C - \sin A \sin C}{\cos A \cos C} + \frac{\cos A \cos B - \sin A \sin B}{\cos A \cos B}
\]
\[
= \left(1 - \tan B \tan C\right) + \left(1 - \tan A \tan C\right) + \left(1 - \tan A \tan B\right)
\]
\[
= 1 + 1 + 1 - (\tan A \tan B + \tan B \tan C + \tan A \tan C)
\]
\[
= 3 - 1 = 2.
\]

Also solved by Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie (jointly), San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M.
Harms, North Newton, KS; John Hawkins, and David Stone (jointly), Statesboro, GA; Valmir Krasniqi, Prishtin, Kosovo; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; David C. Wilson, Winston-Salem, NC, and the proposer.

- 5033: Proposed by Kenneth Korbin, New York, NY.

Given quadrilateral $ABCD$ with coordinates $A(-3,0), B(12,0), C(4,15),$ and $D(0,4)$. Point $P$ is on side $AB$ and point $Q$ is on side $CD$. Find the coordinates of $P$ and $Q$ if area $\triangle PCD = \triangle QAB = \frac{1}{2}$ area quadrilateral $ABCD$. (1)

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

$P$ is on side $AB$: $y = 0 \Rightarrow P(p, 0)$.

$Q$ is on side $CD$: $y = \frac{11}{4}x + 4 \Rightarrow Q(4q, 11q + 4)$.

Area quadrilateral $ABCD = \triangle ABD + \triangle BCD$, so

$\begin{align*}
\frac{1}{2} \begin{vmatrix}
p & 0 & 1 \\
4 & 15 & 1 \\
0 & 4 & 1
\end{vmatrix} &= \frac{1}{2} \begin{vmatrix}
4q & 11q + 4 & 1 \\
-3 & 0 & 1 \\
12 & 0 & 1
\end{vmatrix} \\
&= \frac{1}{2} \begin{vmatrix}
-3 & 0 & 1 \\
12 & 0 & 1 \\
0 & 4 & 1
\end{vmatrix} + \frac{1}{2} \begin{vmatrix}
12 & 0 & 1 \\
4 & 15 & 1 \\
0 & 4 & 1
\end{vmatrix} \\
&\leftrightarrow 11p + 16 = 30 + 74 = 15(11q + 4) \Rightarrow 11p + 16 = \pm 104 = 15(11q + 4)
\end{align*}$

$\leftrightarrow P_1(8,0)$ or $P_2(-120/11,0)$ and $Q_1(16/15,104/15)$ or $Q_2(-656/165,-104/15)$.

Observations by Ken Korbin. The following four points are on a straight line:

midpoint of $AC$, midpoint of $BD$, $P_1$, and $Q_1$. Moreover, the midpoint of $P_1P_2$ is the midpoint of $Q_1Q_2$ = the intersection point of lines $AB$ and $CD$.

Also solved by Brian D. Beasley, Clinton, SC; Michael N. Fried, Kibbutz Revivim, Israel; John Hawkins and David Stone (jointly), Statesboro, GA; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; David C. Wilson, Winston-Salem, NC, and the proposer.

- 5034: Proposed by Roger Izard, Dallas, TX.

In rectangle $MDCB$, $MB \perp MD$. $F$ is the midpoint of $BC$, and points $N$, $E$ and $G$ lie on line segments $DC, DM$ and $MB$ respectively, such that $NC = GB$. Let the area of quadrilateral $MGFC$ be $A_1$ and let the area of quadrilateral $MGFE$ be $A_2$. Determine the area of quadrilateral $EDNF$ in terms of $A_1$ and $A_2$.

Solution by Paul M. Harms, North Newton, KS.

Put the rectangle $MDCB$ on a coordinate system. Assume all nonzero coordinates are positive with coordinates
The coordinates satisfy $e < c$ and $g < b$. The area $A_1$ of the quadrilateral $MGFC = \text{area of } \triangle MGF + \text{area of } \triangle MFC$. Then

$$A_1 = \frac{1}{2}g(c/2) + \frac{1}{2}(c/2)(b + g).$$

The area $A_2$ of the quadrilateral $MGFE = \text{area of } \triangle MGF + \text{area of } \triangle MEF$. Then

$$A_2 = \frac{1}{2}g(c/2) + \frac{1}{2}eb.$$

The area of the quadrilateral $EDNF = \text{area of } \triangle EFD + \text{area of } \triangle FDN$. The area of the quadrilateral $EDNF$ is then

$$= \frac{1}{2}(c - e)b + \frac{1}{2}g(c/2)$$

$$= 2\left(\frac{1}{2}\right)(c/2)b - \frac{1}{2}eb + \frac{1}{2}g(c/2)$$

$$= 2A_1 - A_2.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; John Hawkins and David Stone (jointly), Statesboro, GA; Kenneth Korbin, New York, NY; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA, and the proposer.

• 5035: Proposed by José Luis Díaz-Barrero, Barcelona, Spain.

Let $a, b, c$ be positive numbers. Prove that

$$(a^ab^c, c^a, a^c, b^a, c^b)^2(a^{-b+c} + b^{-c+a} + c^{-a+b})^3 \geq 27.$$

Solution 1 by David E. Manes, Oneonta, NY.

Note that the inequality is equivalent to

$$\frac{3}{a^{b+c} + b^{c+a} + c^{a+b}} \leq \frac{3}{\sqrt[3]{a^{2a}b^{2b}c^{2c}}}.$$

Since the problem is symmetrical in the variables $a, b, c$, we can assume $a \geq b \geq c$. Therefore, $\ln a \geq \ln b \geq \ln c$. By the Rearrangement Inequality

$$a \ln a + b \ln b + c \ln c \geq b \ln a + c \ln b + a \ln c$$

and

$$a \ln a + b \ln b + c \ln c \geq c \ln a + a \ln b + b \ln c.$$

Adding the two inequalities yields

$$2a \ln a + 2b \ln b + 2c \ln c \geq (b + c) \ln a + (c + a) \ln b + (a + b) \ln c.$$
Therefore,

\[
\ln \left( a^{2a} b^{2b} c^{2c} \right) \geq \ln \left( a^{b+c} b^{c+a} c^{a+b} \right) \text{ or } \\
\sqrt[3]{a^{2a} b^{2b} c^{2c}} \geq \sqrt[3]{a^{b+c} b^{c+a} c^{a+b}}.
\]

By the Harmonic-Geometric Mean Inequality

\[
\frac{3}{a^{\frac{1}{3a}} + b^{\frac{1}{3b}} + c^{\frac{1}{3c}}} \leq \sqrt[3]{a^{b+c} b^{c+a} c^{a+b}} \leq \sqrt[3]{a^{2a} b^{2b} c^{2c}}.
\]

**Solution 2** by Paolo Perfetti, Mathematics Department, University “Tor Vergata,” Rome, Italy.

Taking the logarithm we obtain,

\[
2 \sum_{\text{cyc}} \ln a + 3 \ln \left( \sum_{\text{cyc}} a^{-(b+c)} \right) \geq 3 \ln 3.
\]

The concavity of the logarithm yields,

\[
2 \sum_{\text{cyc}} \ln a + 3 \left( \ln 3 - \sum_{\text{cyc}} (b + c) \ln a \right) \geq 3 \ln 3.
\]

Defining \( s = a + b + c \) gives,

\[
\sum_{\text{cyc}} (3a - s) \ln a \geq 0.
\]

Since the second derivative of the function \( f(x) = (3x - s) \ln x \) is positive for any \( x \) and \( s \), \( f''(x) = 3/x + s/x^2 \) it follows that,

\[
\sum_{\text{cyc}} (3a - s) \ln a \geq \sum_{\text{cyc}} (3a - s) \ln a \bigg|_{a=s/3} = 0.
\]

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Boris Rays, Chesapeake, VA, and the proposer.

• 5036: Proposed by José Luis Díaz-Barrero, Barcelona, Spain.

Find all triples \((x, y, z)\) of nonnegative numbers such that

\[
\begin{cases} 
 x^2 + y^2 + z^2 = 1 \\
 3^x + 3^y + 3^z = 5
\end{cases}
\]

**Solution 1** by John Hawkins and David Stone, Statesboro, GA.

We are looking for all first octant points of intersection of the unit sphere with the surface \( 3^x + 3^y + 3^z = 5 \). Clearly, the intercept points \((1, 0, 0), (0, 1, 0)\) and \((0, 0, 1)\) are solutions. We claim there no other solutions.
Consider the traces of our two surfaces in the $xy$-plane: the unit circle and the curve give by $3^x + 3^y = 4$. Our only concern is in the first quadrant, where we have a unit quarter circle and the curve $y = \frac{\ln(4 - 3^x)}{\ln 3}$. The two curves meet on the coordinate axes; otherwise graphing software shows that the logarithmic curve lies inside the quarter circle.

By the symmetry of the variables, we have the same behavior when we look at the traces in the $xz$- and $yz$-planes. That is, at our boundaries of concern, the exponential surface starts inside the sphere. By implicit differentiation of $3^x + 3^y + 3^z = 5$, we have

$$\frac{\partial z}{\partial x} = -\frac{3x}{3^z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{3y}{3^z},$$

which are both negative for nonnegative $x, y$ and $z$. Therefore, the exponential surface descends from a trace inside the sphere to a trace which lies within the sphere. So the two surfaces have no points of intersection within the interior of the first octant.

**Solution 2 by Ovidiu Furdui, Campia Turzii, Cluj, Romania.**

Such triples are $(x, y, z) = (1, 0, 0), (0, 1, 0), (0, 0, 1)$. We note that the first equation implies that $x, y, z \in [0, 1]$. On the other hand, using Bernoulli’s inequality we obtain that

$$\begin{align*}
3^x &= (1 + 2)^x \leq 1 + 2x \\
3^y &= (1 + 2)^y \leq 1 + 2y \\
3^z &= (1 + 2)^z \leq 1 + 2,
\end{align*}$$

and hence, $5 = 3^x + 3^y + 3^z \leq 3 + 2(x + y + z)$. It follows that $1 \leq x + y + z$. This implies that $x^2 + y^2 + z^2 \leq x + y + z$, and hence, $x(1 - x) + y(1 - y) + z(1 - z) \leq 0$.

Since the left hand side of the preceding inequality is nonnegative we obtain that $x(1 - x) = y(1 - y) = z(1 - z) = 0$ from which it follows that $x, y, z$ are either 0 or 1. This combined with the first equation of the system shows that exactly one of $x, y,$ and $z$ is 1 and the other two are 0, and the problem is solved.

**Solution 3 by the proposer.**

By inspection we see that $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ are solutions of the given system. We claim that they are the only solutions of the system. In fact, for all $t \in [0, 1]$ the function $f(t) = 3^t$ is greater than or equal to the function $g(t) = 2t^2 + 1$, as can be easily proven, for instance, by drawing their graphs when $0 \leq t \leq 1$.

Since $x^2 + y^2 + z^2 = 1$, then $x \in [0, 1], y \in [0, 1]$ and $z \in [0, 1]$. Therefore

$$\begin{align*}
3^x &\geq 2x^2 + 1, \\
3^y &\geq 2y^2 + 1, \\
3^z &\geq 2z^2 + 1.
\end{align*}$$

Adding up the preceding expressions yields

$$3^x + 3^y + 3^z \geq 2(x^2 + y^2 + z^2) + 3 \geq 5$$

and we are done.

Also solved by Charles McCracken, Dayton, OH; Paolo Perfetti, Mathematics Department, University “Tor Vergata,” Rome, Italy, and Boris Rays, Chesapeake, VA.
Let $k, p$ be natural numbers. Prove that
\[ 1^k + 3^k + 5^k + \cdots + (2n+1)^k = (1 + 3 + \cdots + (2n+1))^p \]
for all $n \geq 1$ if and only if $k = p = 1$.

**Solution 1 by Carl Libis, Kingston, RI.**

Since \( (1 + 3 + \cdots + (2n+1))^p = \left[ (n+1)^2 \right]^p = (n+1)^{2p} \), it is clear that \( (1 + 3 + \cdots + (2n+1))^p \) is a monic polynomial of degree $2p$.

Let $S_{2n+1}^k = \sum_{i=1}^{2n+1} i^k$. Then
\[
S_{2n+1}^k = \sum_{i=1}^{n+1} (2i-1)^k + \sum_{i=1}^{n} (2i)^k = \sum_{i=0}^{n} (2i+1)^k + 2^n \sum_{i=1}^{n} i^k = \sum_{i=0}^{n} (2i+1)^k + 2^k S_n^k.
\]

Then $S_{2n+1}^k = S_{2n}^k - 2^k S_n^k$. It is well known for sums of powers of integers $S_n^k$, that the leading term of $S_n^k$ is $\frac{n^{k+1}}{k+1}$. Thus the leading term of
\[
1^k + 3^k + 5^k + \cdots + (2n+1)^k
\]
is
\[
\frac{(2n+1)^{k+1}}{k+1} - 2^k \frac{n^{k+1}}{k+1} = \frac{2^{k+1}n^{k+1} - 2^k n^{k+1}}{k+1} = \frac{2^k n^{k+1}}{k+1}.
\]

This is monic if, and only if, $k = 1$. When $k = 1$ we have that
\[
\sum_{i=0}^{n} (2i+1) = S_{2n}^1 - 2S_n^1 = \frac{(2n+1)(2n+2)}{2} - 2 \frac{n(n+1)}{2} = (n+1)^2.
\]

For $k, p$ natural numbers we have that
\[ 1^k + 3^k + 5^k + \cdots + (2n+1)^k = \left( 1 + 3 + \cdots + (2n+1) \right)^p \]
for all $n \geq 1$ if, and only if, $k = p = 1$.

**Solution 2 by Kee-Wai Lau, Hong Kong, China.**

If $k = p = 1$, the equality $1^k + 3^k + 5^k + \cdots + (2n+1)^k = (1 + 3 + \cdots + (2n+1))^p$ is trivial. Now suppose that the equality holds for all $n \geq 1$. By putting $n = 1, 2$, we obtain $1 + 3^k = 4^p$ and $1 + 3^k + 5^k = 9^p$. Hence
\[
3^k = 4^p - 1 \quad \text{and} \quad 5^k = 9^p - 4^p.
\]

Eliminating $k$ from the last two equations, we obtain $9^p = 4^p + (4^p - 1)^{(\ln 5/\ln 3)}$. Hence,
\[
9^p < 2 \left( 4^p (\ln 5/\ln 3) \right)
\]
\[
p \ln 9 < \ln 2 + \frac{p(\ln 4)(\ln 5)}{\ln 3},
\]
and
Thus $p = 1, 2, 3, 4$. But it is easy to check that only the case $p = 1$ and $k = 1$ admits solutions in the natural numbers for the equation $1 + 3^k = 4^p$, and this completes the solution.

**Solution 3 by Paul M. Harms, North Newton, KS.**

Clearly if $k = p = 1$, the equation holds for all appropriate integers $n$. For the *only if* part of the statement consider the contrapositive statement:

If $p \neq 1$ or $k \neq 1$, then for some $n \geq 1$ the equation does not hold.

Consider $n = 1$. Then the equation in the problem is $1^k + 3^k = (1 + 3)^p = 4^p$. If $k = 1$ with $p > 1$, then $4 < 4^p$ so the equation does not hold.

If $k > 1$ with $p = 1$, then $1^k + 3^k > 4$ so the equation does not hold.

Now consider both $p > 1$ and $k > 1$ using the equation in the form

$$3^k = 4^p - 1^k = (2^p - 1)(2^p + 1).$$

If $p > 1$, then $2^p - 1 > 1$ and $2^p + 1 > 1$. Also, the expressions $2^p - 1$ and $2^p + 1$ are 2 units apart so that if 3 is a factor of one of these expressions then 3 is not a factor of the other expression. Since both expressions are greater than one, if 3 is a factor of one of the expressions, then the other expression has a prime number other than 3 as a factor. Thus $(2^p - 1)(2^p + 1)$ has a prime number other than 3 as a factor and cannot be equal to $3^k$, a product of just the prime number 3. Thus the equation does not hold when both $p > 1$ and $k > 1$.

**Solution 4 by John Hawkins and David Stone, Statesboro, GA.**

Denote $1^k + 3^k + 5^k + \cdots + (2n + 1)^k = (1 + 3 + \cdots + (2n + 1))^p$ by (#). The condition requesting *all* $n \geq 1$ is overkill. Actually, we can prove the following are equivalent:

(a) condition (#) holds for all $n \geq 1$,

(b) condition (#) holds for all $n = 1$,

(c) $k = p = 1$.

Clearly, (a)$\Rightarrow$ (b).

Also (c)$\Rightarrow$ (a), for if $k = p = 1$, then (#) becomes the identity

$$1 + 3 + 5 + \cdots + (2n + 1) = (1 + 3 + \cdots + (2n + 1)).$$

Finally, we prove that (b)$\Rightarrow$ (c). Assuming the truth of (#) for $n = 1$ tells us that $3^k = 4^p - 1$.

If $k = 1$, we immediately conclude that $p = 1$ and we are finished.

Arguing by contradiction, suppose $k \geq 2$, so $3^k$ is actually a multiple of 9. Thus $4^p \equiv 1$ (mod 9). Now consider the powers of 4 modulo 9:

$$4^0 \equiv 1$$

$$4^1 \equiv 4$$
\[ 4^2 \equiv 7 \pmod{9} \]
\[ 4^3 \equiv 1 \pmod{9} \]

That is, 4 has order 3(mod 9), so \(4^p \equiv 1 \pmod{9}\) if and only if \(p\) is a multiple of 3. Based upon some numerical testing, we consider \(4^p\) modulo 7: \(4^p = 4^3t \equiv 1^t \equiv 1 \pmod{7}\). That is, 7 divides \(4^p - 1\), so \(4^p - 1\) cannot be a power of 3. We have reached a contradiction.

**Solution 5 by the proposer.**

One implication is easy to prove. To prove the other implication we note that

\[
1 + 3 + \cdots + (2n + 1) = \sum_{k=1}^{n+1}(2k-1) = 2 \sum_{k=1}^{n+1}k - (n + 1) = (n+1)(n+2)-(n+1) = (n+1)^2.
\]

It follows that

\[
1^k + 3^k + 5^k + \cdots + (2n + 1)^k = (n + 1)^{2p}.
\]

We multiply the preceding relation by \(2/(2n + 1)^{k+1}\) and we get that

\[
\frac{2}{2n+1} \left( \left( \frac{1}{2n+1} \right)^k + \left( \frac{3}{2n+1} \right)^k + \cdots + \left( \frac{2n + 1}{2n+1} \right)^k \right) = 2 \frac{(n + 1)^{2p}}{(2n + 1)^{k+1}}. \tag{1}
\]

Letting \(n \to \infty\) in (1) we get that

\[
\int_0^1 x^k dx = \frac{1}{k+1} = \lim_{n \to \infty} 2 \frac{(n + 1)^{2p}}{(2n + 1)^{k+1}}.
\]

It follows that \(2p = k + 1\) and that \(\frac{1}{k+1} = \frac{1}{2^p}\). However, the equation \(k + 1 = 2^k\) has a unique positive solution namely \(k = 1\). This can be proved by applying Bernoulli’s inequality as follows

\[ 2^k = (1 + 1)^k \geq 1 + k \cdot 1 = k + 1, \]

with equality if and only if \(k = 1\). Thus, \(k = p = 1\) and the problem is solved.

**Also solved by Boris Rays, Chesapeake, VA.**

**Late Solutions**

Late solutions were received from David C. Wilson of Winston-Salem, NC to problems 5026, 5027, and 5028.