Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
May 1, 2007

- 4954: *Proposed by Kenneth Korbin, New York, NY.*

  Find four pairs of positive integers (a, b) that satisfy

  $$\frac{a+i}{a-i} \cdot \frac{b+i}{b-i} = \frac{111+i}{111-i}$$

  with $a < b$.

- 4955: *Proposed by Kenneth Korbin, New York, NY.*

  Between 100 and 200 pairs of red sox are mixed together with between 100 and 200 pairs of blue sox. If three sox are selected at random, then the probability that all three are the same color is 0.25. How many pairs of sox were there altogether?

- 4956: *Proposed by Kenneth Korbin, New York, NY.*

  A circle with radius $3\sqrt{2}$ is inscribed in a trapezoid having legs with lengths of 10 and 11. Find the lengths of the bases.

- 4957: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $\{a_n\}_{n\geq 0}$ be the sequence defined by $a_0 = 1, a_1 = 2, a_2 = 1$ and for all $n \geq 3$, $a_n^3 = a_{n-1}a_{n-2}a_{n-3}$. Find $\lim_{n\to\infty} a_n$.

- 4958: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $f : [a, b] \to R$ ($0 < a < b$) be a continuous function on $[a, b]$ and derivable in (a, b).

Prove that there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{1}{c\sqrt{ab}} \cdot \frac{\ln(ab/c^2)}{\ln(c/a) \cdot \ln(c/b)}.$$

- 4959: *Proposed by Juan-Bosco Romero Márquez, Valladolid, Spain.*

  Find all numbers $N = ab$, were $a, b = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, such that

$$[S(N)]^2 = S(N^2),$$

where S(N)=a+b is the sum of the digits. For example:

$$\begin{array}{rcll} N & = & 12 & N^2 = 144 \\ S(N) & = & 3 & S(N^2) = 9 \quad \text{and } [\text{S(N)}]^2 = \text{S(N}^2). \end{array}$$

Solutions

- 4918: *Proposed by Kenneth Korbin, New York, NY.*

  Find the dimensions of an isosceles triangle that has integer length inradius and sides and which can be inscribed in a circle with diameter 50.

Solution by Paul M. Harms, North Newton, KS.

Put the circle on a coordinate system with center at $(0, 0)$ and the vertex associated with the two equal sides at $(0, 25)$. Also make the side opposite the $(0, 25)$ vertex parallel to the x-axis. Using (x, y) as the vertex on the right side of the circle, we have $x^2 + y^2 = 25^2 = 625$. Let d be the length of the equal sides. Using the right triangle with vertices at $(0, 25), (0, y)$, and (x, y) we have $(25 - y)^2 + x^2 = d^2$.

Then $d^2 = (25 - y)^2 + (25^2 - y^2) = 1250 - 50y$; the semi-perimeter $s = x + d$ and the inradius $r = \sqrt{\dfrac{x^2(d - x)}{d + x}}$. Using $x^2 + y^2 = 25^2$, we will check to see if $x = 24$ and $y = 7$ satisfies the problem. The number $d^2 = 900$, so $d = 30$. The inradius $r = \sqrt{\dfrac{24^2(6)}{54}} = 8$. Thus the isosceles triangle with side lengths $30, 30, 48$ and $r = 8$ satisfies the problem. If $x = 24$ and $y = -7$, then $d = 40$ and $r = 12$. The isosceles triangle with side lengths $40, 40, 48$ and $r = 12$ also satisfies the problem.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; David Stone and John Hawkins, Statesboro, GA; David C. Wilson, Winston-Salem, NC, and the proposer.

- 4919: *Proposed by Kenneth Korbin, New York, NY.*

  Let x be any even positive integer. Find the value of

$$\sum_{k=0}^{x/2} \binom{x-k}{k} 2^k.$$

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX.

To simplify matters, let $x = 2n$ and

$$S(n) = \sum_{k=0}^{n} \binom{2n-k}{k} 2^k.$$

Since

$$\binom{m}{i} = \binom{m-1}{i-1} + \binom{m-1}{i}$$

for $m \geq 2$ and $1 \leq i \leq m-1$, we have

$$\begin{aligned}
\binom{2n+4-k}{k} &= \binom{2n+3-k}{k-1} + \binom{2n+3-k}{k} \\
&= \binom{2n+3-k}{k-1} + \binom{2n+2-k}{k-1} + \binom{2n+2-k}{k} \\
&= \binom{2n+3-k}{k-1} + \binom{2n+3-k}{k-1} - \binom{2n+2-k}{k-2} + \binom{2n+2-k}{k} \\
&= \binom{2n+2-k}{k} + 2\binom{2n+3-k}{k-1} - \binom{2n+2-k}{k-2}
\end{aligned}$$

for $n \geq 1$ and $2 \leq k \leq n+1$.

Therefore, for $n \geq 1$,

$$\begin{aligned}
S(n+2) &= \sum_{k=0}^{n+2} \binom{2n+4-k}{k} 2^k \\
&= 1 + (2n+3)\cdot 2 + \sum_{k=2}^{n+1} \binom{2n+4-k}{k} 2^k + 2^{n+2} \\
&= 1 + (2n+3)\cdot 2 + \sum_{k=2}^{n+1} \binom{2n+2-k}{k} 2^k + 2\sum_{k=2}^{n+1} \binom{2n+3-k}{k-1} 2^k \\
&\quad - \sum_{k=2}^{n+1} \binom{2n+2-k}{k-2} 2^k + 2^{n+2} \\
&= 4 + \sum_{k=0}^{n+1} \binom{2n+2-k}{k} 2^k + 2\sum_{k=1}^{n} \binom{2n+2-k}{k} 2^{k+1} - \sum_{k=0}^{n-1} \binom{2n-k}{k} 2^{k+2} + 2^{n+2} \\
&= S(n+1) + 4\sum_{k=0}^{n+1} \binom{2n+2-k}{k} 2^k - \sum_{k=0}^{n-1} \binom{2n-k}{k} 2^{k+2} - 2^{n+2} \\
&= 5S(n+1) - 4\sum_{k=0}^{n} \binom{2n-k}{k} 2^k \\
&= 5S(n+1) - 4S(n).
\end{aligned}$$

To solve for $S(n)$, we use the usual techniques for solving homogeneous linear difference equations with constant coefficients. If we look for a solution of the form $S(n) = t^n$, with $t \neq 0$, then

$$S(n+2) = 5S(n+1) - 4S(n)$$

becomes

$$t^2 = 5t - 4,$$

whose solutions are $t = 1, 4$. This implies that the general solution for $S(n)$ is

$$S(n) = A \cdot 4^n + B \cdot 1^n = A \cdot 4^n + B,$$

for some constants A and B. The initial conditions $S(1) = 3$ and $S(2) = 11$ yield $A = \frac{2}{3}$ and $B = \frac{1}{3}$. Hence,

$$S(n) = \frac{2}{3} \cdot 4^n + \frac{1}{3} = \frac{2^{2n+1} + 1}{3}$$

for all $n \geq 1$. The final solution is

$$\sum_{k=0}^{x/2} \binom{x-k}{k} 2^k = \frac{2^{x+1} + 1}{3}$$

for all even positive integers x.

Also solved by David E. Manes, Oneonta, NY, David Stone, John Hawkins, and Scott Kersey (jointly), Statesboro, GA, and the proposer.

- 4920: *Proposed by Stanley Rabinowitz, Chelmsford, MA.*

Find positive integers a, b, and c (each less than 12) such that

$$\sin \frac{a\pi}{24} + \sin \frac{b\pi}{24} = \sin \frac{c\pi}{24}.$$

Solution by John Boncek, Montgomery, AL.

Recall the standard trigonometric identity:

$$\sin(x+y) + \sin(x-y) = 2 \sin x \cos y.$$

Let $x + y = \frac{a\pi}{24}$ and $x - y = \frac{b\pi}{24}$. Then

$$\sin \frac{a\pi}{24} + \sin \frac{b\pi}{24} = 2 \sin \frac{(a+b)\pi}{48} \cos \frac{(a-b)\pi}{48}.$$

We can make the right hand side of this equation equal to $\sin \frac{c\pi}{24}$ if we let $a - b = 16$ and $a + b = 2c$, or in other words, by choosing a value for c and then taking $a = 8 + c$ and $b = c - 8$.

Since we want positive solutions, we start by taking $c = 9$. This gives us $a = 17$ and $b = 1$. Since $\sin \frac{17\pi}{24} = \sin \frac{7\pi}{24}$, replace $a = 17$ by $a = 7$ and we have a solution $a = 7, b = 1$ and $c = 9$.

By taking $c = 10$ and $c = 11$ and using the same analysis, we obtain two additional triples that solve the problem, namely: $a = 6, b = 2, c = 10$ and $a = 5, b = 3, c = 11$.

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Kenneth Korbin, NY, NY; Peter, E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4921: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Evaluate $\displaystyle\int_0^{\pi/2} \frac{\cos^{2006} x + 2006\sin^2 x}{2006 + \sin^{2006} x + \cos^{2006} x} dx.$

Solution by Michael C. Faleski, Midland, MI.

Call this integral I. Now, substitute $\sin^2 x = 1 - \cos^2 x$ and add to the numerator $\sin^{2006} x - \sin^{2006} x$ to give

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{2006 + \sin^{2006} x + \cos^{2006} x - (2006\cos^2 x + \sin^{2006} x)}{2006 + \sin^{2006} x + \cos^{2006} x} dx \\ &= \int_0^{\pi/2} dx - \int_0^{\pi/2} \frac{2006\cos^2 x + \sin^{2006} x}{2006 + \sin^{2006} x + \cos^{2006} x} dx. \end{aligned}$$

The second integral can be transformed with $u = \pi/2 - x$ to give

$$\int_0^{\pi/2} \frac{2006\cos^2 x + \sin^{2006} x}{2006 + \sin^{2006} x + \cos^{2006} x} dx = -\int_{\pi/2}^0 \frac{\cos^{2006} u + 2006\sin^2 u}{2006 + \sin^{2006} u + \cos^{2006} u} du = I.$$

Hence, $I = \int_0^{\pi/2} dx - I \Longrightarrow 2I = \dfrac{\pi}{2} \Longrightarrow I = \dfrac{\pi}{4}.$

$$\int_0^{\pi/2} \frac{\cos^{2006} x + 2006\sin^2 x}{2006 + \sin^{2006} x + \cos^{2006} x} dx = \frac{\pi}{4}.$$

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Ovidiu Furdui, Kalamazoo, MI; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4922: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let a, b be real numbers such that $0 < a < b$ and let $f : [a, b] \to R$ be a continuous function in $[a, b]$ and derivable in (a, b). Prove that there exists $c \in (a, b)$ such that

$$cf(c) = \frac{1}{\ln b - \ln a} \int_a^b f(t)\, dt.$$

Solution by David E. Manes, Oneonta, NY.

For each $x \in [a, b]$, define the function $F(x)$ so that $F(x) = \int_a^x f(t)dt$. Then $F(b) = \int_a^b f(t)dt, F(a) = 0$ and, by the Fundamental Theorem of Calculus, $F'(x) = f(x)$ for each $x \in (a, b)$.

Let $g(x) = \ln(x)$ be defined on $[a, b]$. Then both functions F and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) and $g'(x) = \dfrac{1}{x} \neq 0$

for each $x \in (a,b)$. By the Extended Mean-Value Theorem, there is at least one number $c \in (a,b)$ such that

$$\frac{F'(c)}{g'(c)} = \frac{F(b)-F(a)}{g(b)-g(a)} = \frac{\int_a^b f(t)dt}{\ln b - \ln a}.$$

Since $\dfrac{F'(c)}{g'(c)} = cf(c)$, the result follows.

Also solved by Michael Brozinsky, Central Islip, NY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4923: *Proposed by Michael Brozinsky, Central Islip, NY.*

  Show that if $n \geq 6$ and is composite, then n divides $(n-2)!$.

  Solution by Brian D. Beasley, Clinton, SC.

  Let n be a composite integer with $n \geq 6$. We consider two cases:

  (i) Assume n is not the square of a prime. Then we may write $n = ab$ for integers a and b with $1 < a < b < n-1$. Thus a and b are distinct and are in $\{2,3,\ldots,n-2\}$, so $n = ab$ divides $(n-2)!$.

  (ii) Assume $n = p^2$ for some odd prime p. Then $n - 2 = p^2 - 2 \geq 2p$, since $p > 2$. Hence both p and $2p$ are in $\{3,4,\ldots,n-2\}$, so $n = p^2$ divides $(n-2)!$.

  Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Luke Drylie (student, Old Dominion U.), Chesapeake, VA; Kenneth Korbin, NY, NY; Paul M. Harms, North Newton, KS; Jahangeer Kholdi, Portsmouth, VA; N. J. Kuenzi, Oshkosh, WI; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4924: *Proposed by Kenneth Korbin, New York, NY.*

  Given $\displaystyle\sum_{N=1}^{\infty} \frac{F_N}{K^N} = 3$ where F_N is the N^{th} Fibonacci number. Find the value of the positive number K.

  Solution by R. P. Sealy, Sackville, New Brunswick, Canada.

  The ratio test along with the fact that $\displaystyle\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$ implies $\displaystyle\sum_{n=1}^{\infty} \frac{F_n}{K^n}$ converges for $K > \dfrac{1+\sqrt{5}}{2}$. Then

$$\begin{aligned}
3 = \sum_{n=1}^{\infty} \frac{F_n}{K^n} &= \frac{1}{K} + \frac{1}{K^2} + \sum_{n=3}^{\infty} \frac{F_n}{K^n} \\
&= \frac{1}{K} + \frac{1}{K^2} + \sum_{n=3}^{\infty} \frac{F_{n-1}+F_{n-2}}{K^n} \\
&= \frac{1}{K} + \frac{1}{K^2} + \frac{1}{K}\sum_{n=3}^{\infty} \frac{F_{n-1}}{K^{n-1}} + \frac{1}{K^2}\sum_{n=3}^{\infty} \frac{F_{n-2}}{K^{n-2}}
\end{aligned}$$

$$= \frac{1}{K} + \frac{1}{K^2} + \frac{1}{K}\left(3 - \frac{1}{K}\right) + \frac{3}{K^2}$$
$$= \frac{4}{K} + \frac{3}{K^2} \Rightarrow K = \frac{2+\sqrt{13}}{3}.$$

Also solved by Brian D. Beasley, Clinton, SC; Sam Brotherton (student, Rockdale Magnet School For Science and Technology), Conyers, GA; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; José Luis Díaz-Barrero, Barcelona, Spain; Luke Drylie (student, Old Dominion U.), Chesapeake, VA; Paul M. Harms, North Newton, KS; Jahangeer Kholdi and Boris Rays (jointly), Portsmouth, VA & Chesapeake,VA (respectively); N. J. Kuenzi, Oshkosh, WI; Tom Leong, Scotrun, PA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4925: *Proposed by Kenneth Korbin, New York, NY.*

In the expansion of

$$\frac{x^4}{(1-x)^3(1-x^2)} = x^4 + 3x^5 + 7x^6 + 13x^7 + \cdots$$

find the coefficient of the term with x^{20} and with x^{21}.

Solution 1 by Brian D. Beasley, Clinton, SC.

We have

$$\begin{aligned} \frac{1}{(1-x)^3(1-x^2)} &= \frac{1}{(1-x)^4(1+x)} \\ &= (1-x+x^2-x^3+\cdots)(1+x+x^2+x^3+\cdots)^4 \\ &= (1-x+x^2-x^3+\cdots)(1+2x+3x^2+4x^3+\cdots)^2 \\ &= (1-x+x^2-x^3+\cdots)(1+4x+10x^2+20x^3+\cdots), \end{aligned}$$

where the coefficients of the second factor in the last line are the binomial coefficients $C(k,3)$ for $k = 3,4,5,\ldots$. Hence, allowing for the x^4 in the original numerator, the desired coefficient of x^{20} is

$$\sum_{k=3}^{19} C(k,3)(-1)^{19-k} = 525.$$

Similarly, the desired coefficient of x^{21} is

$$\sum_{k=3}^{20} C(k,3)(-1)^{20-k} = 615.$$

Solution 2 by Tom Leong, Scotrun, PA.

Equivalently, we find the coefficients of x^{16} and x^{17} in

$$\frac{1}{(1-x)^3(1-x^2)}. \qquad (1)$$

We use the following well-known generating functions:

$$\begin{aligned}\frac{1}{1-x^2} &= 1+x^2+x^4+x^6+\cdots\\ \frac{1}{(1-x)^{m+1}} &= \binom{m}{m}+\binom{m+1}{m}x+\binom{m+2}{m}x^2+\binom{m+3}{m}x^3+\cdots.\end{aligned}$$

A decomposition of (1) is

$$\frac{1}{(1-x)^3(1-x^2)}=\frac{1}{2}\frac{1}{(1-x)^4}+\frac{1}{4}\frac{1}{(1-x)^3}+\frac{1}{8}\frac{1}{(1-x)^2}+\frac{1}{8}\frac{1}{(1-x)}.$$

Thus the coefficient of x^n is

$$\frac{1}{2}\binom{n+3}{3}+\frac{1}{4}\binom{n+2}{2}+\frac{1}{8}\binom{n+1}{1}+\frac{1}{8}=\frac{(n+2)(n+4)(2n+3)}{24}\quad \text{if } n \text{ is even}$$

or

$$\frac{1}{2}\binom{n+3}{3}+\frac{1}{4}\binom{n+2}{2}+\frac{1}{8}\binom{n+1}{1}=\frac{(n+1)(n+3)(2n+7)}{24}\quad \text{if } n \text{ is odd.}$$

So the coefficient of x^{16} is $\dfrac{18\cdot 20\cdots 35}{24}=525$ and the coefficient of x^{17} is $\dfrac{18\cdot 20\cdots 41}{24}=$ 615.

Solution 3 by Paul M. Harms, North Newton, KS.

When

$$-1<x<1,\quad \frac{1}{1-x}=1+x+x^2+\cdots.$$

Taking two derivatives, we obtain for

$$-1<x<1,\quad \frac{2}{(1-x)^3}=2+3(2)x+4(3)x^2+\cdots.$$

When

$$-1<x<1,\quad \frac{x^4}{1-x^2}=x^4+x^6+x^8+\cdots.$$

The series for $\dfrac{x^4}{(1-x)^3(1-x^2)}$ can be found by multiplying

$$\frac{1}{2}\cdot\frac{2}{(1-x)^3}\cdot\frac{x^4}{(1-x^2)}=\frac{1}{2}\Big[2+3(2)x+4(3)x^2+\cdots+18(17)x^{16}+19(18)x^{17}+\cdots\Big]\Big[x^4+x^6+x^8+\cdots\Big].$$

The coefficient of x^{20} is

$$\frac{1}{2}\Big[18(17)+16(15)+14(13)+\cdots 4(3)+2\Big]=525.$$

The coefficient of x^{21} is

$$\frac{1}{2}\Big[19(18)+17(16)+15(14)+\cdots 5(4)+3(2)\Big]=615.$$

Comment: Jahangeer Kholdi and Boris Rays noticed that the coefficients in $x^4+3x^5+7x^6+13x^7+22x^8+34x^9+50x^{10}+\cdots$, are the partial sums of the alternate triangular numbers. I.e., $1, 3, 1+6, 3+10, 1+6+15, 3+10+21, \cdots$, which leads to the coefficients of x^{20} and x^{21} being 525 and 615 respectively.

Also solved by Michael Brozinsky, Central Islip, NY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; José Luis Díaz-Barrero, Barcelona, Spain; Jahangeer Kholdi and Boris Rays (jointly), Portsmouth, VA & Chesapeake,VA (respectively); Peter E. Liley, Lafayette, IN; John Nord, Spokane, WA; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4926: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Calculate
$$\sum_{n=1}^{\infty} \frac{nF_n^2}{3^n}$$
where F_n is the n^{th} Fibonacci number defined by $F_1 = F_2 = 1$ and for $n \geq 3, F_n = F_{n-1} + F_{n-2}$.

Solution by David Stone and John Hawkins, Statesboro, GA.

By Binet's Formula, $F_n = \dfrac{\alpha^n - \beta^n}{\sqrt{5}}$, where α and β are the solutions of the quadratic equation $x^2 - x - 1 = 0$; $\alpha = \dfrac{1+\sqrt{5}}{2}$, $\beta = \dfrac{1-\sqrt{5}}{2}$.

Note that $a - b = \sqrt{5}$, $\alpha \cdot \beta = -1$, $\alpha^2 + \beta^2 = 3$, and $\alpha^6 + \beta^6 = 18$. Also recall from calculus that $\displaystyle\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $|x| < 1$. Thus we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{nF_n^2}{3^n} &= \sum_{n=1}^{\infty} \frac{n}{3^n}\frac{\alpha^{2n} - 2\alpha^n\beta^n + \beta^{2n}}{5} \\
&= \sum_{n=1}^{\infty} \frac{n}{3^n}\frac{\alpha^{2n} - 2(-1)^n + \beta^{2n}}{5} \\
&= \frac{1}{5}\left\{ \sum_{n=1}^{\infty} n\left(\frac{\alpha^2}{3}\right)^n - 2\sum_{n=1}^{\infty} n\left(\frac{-1}{3}\right)^n + \sum_{n=1}^{\infty} n\left(\frac{\beta^2}{3}\right)^n \right\} \\
&= \frac{1}{5}\left\{ \frac{\frac{\alpha^2}{3}}{\left[1-\frac{\alpha^2}{3}\right]^2} - 2\frac{\frac{-1}{3}}{\left[1+\frac{1}{3}\right]^2} + \frac{\frac{\beta^2}{3}}{\left[1-\frac{\beta^2}{3}\right]^2} \right\}, \quad \text{valid because } \frac{\beta^2}{3} < \frac{\alpha^2}{3} < 1; \\
&= \frac{1}{5}\left\{ \frac{3\alpha^2}{[3-\alpha^2]^2} + \frac{3}{8} + \frac{3\beta^2}{[3-\beta^2]^2} \right\} \\
&= \frac{3}{5}\left\{ \frac{\alpha^2}{[\beta^2]^2} + \frac{1}{8} + \frac{\beta^2}{[\alpha^2]^2} \right\} \text{ because } \alpha^2 + \beta^2 = 3, \\
&= \frac{3}{5}\left\{ \frac{1}{8} + \frac{\alpha^6 + \beta^6}{\alpha^4\beta^4} \right\} \text{ by algebra,}
\end{aligned}$$

$$= \frac{3}{5}\left\{\frac{1}{8}+\frac{18}{1}\right\} = \frac{87}{8}.$$

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Tom Leong, Scotrun, PA, and the proposer.

- 4927: *Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Barcelona, Spain.*

Let k be a positive integer and let

$$A = \sum_{n=0}^{\infty} \frac{(-1)^n}{2k(2n+1)} \qquad \text{and} \qquad B = \sum_{n=0}^{\infty}(-1)^n \left\{\sum_{m=0}^{2k} \frac{(-1)^m}{(4k+2)n+2m+1}\right\}.$$

Prove that $\dfrac{B}{A}$ is an even integer for all $k \geq 1$.

Solution by Tom Leong, Scotrun, PA.

Note that inside the curly braces in the expression for B, the terms of the (alternating) sum are the reciprocals of the consecutive odd numbers from $(4k+2)n+1$ to $(4k+2)n+(4k+1)$. As $n = 0, 1, 2, \ldots$, the reciprocal of every positive odd number appears exactly once in this sum (disregarding its sign). Thus

$$B = \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{2k} \frac{(-1)^{m+n}}{(4k+2)n+2m+1}\right\} = \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1}$$

from which we find $\dfrac{B}{A} = 2k$. (In fact, it is well-known that $B = \pi/4$.)

Comment by Editor: This problem was incorrectly stated when it was initially posted in the May, 06 issue of SSM. The authors reformulated it, and the correct statement of the problem and its solution are listed above. The corrected version was also solved by **Paul M. Harms of North Newton, KS.**

- 4928: *Proposed by Yair Mulian, Beer-Sheva, Israel.*

Prove that for all natural numbers n

$$\int_0^1 \frac{2x^{2n+1}}{x^2-1}dx = \int_0^1 \frac{x^n}{x-1} + \frac{1}{x+1}dx.$$

Comment by Editor: The integrals in their present form do not exist, and I did not see this when I accepted this problem for publication. Some of the readers rewrote the problem in what they described as "its more common form;" i.e., to show that $\displaystyle\int_0^1 \frac{2x^{2n+1}}{x^2-1} - \left(\frac{x^n}{x-1} + \frac{1}{x+1}\right)dx = 0$. But I believe that one cannot legitimately recast the problem in this manner, because the $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$ if, and only if, $f(x)$ and $g(x)$ is each integrable over these limits. So as I see it, the problem as it was originally stated is not solvable. Mea culpa, once again.

- 4929: *Proposed by Michael Brozinsky, Central Islip, NY.*

An archaeological expedition uncovered 85 houses. The floor of each of these houses was a rectangular area covered by mn tiles where $m \leq n$. Each tile was a 1 unit by 1 unit square. The tiles in each house were all white, except for a (non-empty) square configuration of blue tiles. Among the 85 houses, all possible square configurations of blue tiles appeared once and only once. Find all possible values of m and n.

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX.

Assume that each configuration of blue tiles is a $k \times k$ square. Since $m \leq n$ and each such configuration was non-empty, it follows that $k = 1, 2, \ldots, m$. For each value of k, there are $(m-k+1)(n-k+1)$ possible locations for the $k \times k$ configuration of blue tiles. Since each arrangement appeared once and only once among the 85 houses, we have

$$\begin{aligned} 85 &= \sum_{k=1}^{m} (m-k+1)(n-k+1) \\ &= \sum_{k=1}^{m} (m+1)(n+1) - (m+n+2)\sum_{k=1}^{m} k + \sum_{k=1}^{m} k^2 \\ &= m(m+1)(n+1) - (m+n+2)\frac{m(m+1)}{2} + \frac{m(m+1)(2m+1)}{6} \\ &= \frac{m(m+1)}{6}\left[3n - (m-1)\right] \end{aligned}$$

or

$$m(m+1)\left[3n-(m-1)\right] = 510. \qquad (1)$$

This implies that m and $m+1$ must be consecutive factors of 510. By checking all 16 factors of 510, we see that the only possible values of m are $1, 2, 5$. If $m = 2$, (1) does not produce an integral solution for n. If $m = 1$ or 5, equation (1) yields $n = 85$ or 7 (respectively). Therefore, the only solutions are $(m, n) = (1, 85)$ or $(5, 7)$.

Also solved by Tom Leong, Scotrun, PA; Paul M. Harms, North Newton, KS; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.