Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before April 15, 2015

- **5337:** *Proposed by Kenneth Korbin, New York, NY*

  Given convex quadrilateral $ABCD$ with sides,

  $$\begin{aligned} \overline{AB} &= 1+3\sqrt{2} \\ \overline{BC} &= 6+4\sqrt{2} \text{ and} \\ \overline{CD} &= -14+12\sqrt{2}. \end{aligned}$$

  Find side $\overline{AD}$ so that the area of the quadrilateral is maximum.

- **5338:** *Proposed by Arkady Alt, San Jose, CA* Determine the maximum value of

  $$F\left(x,y,z\right)=\min\left\{\frac{|y-z|}{|x|},\frac{|z-x|}{|y|},\frac{|x-y|}{|z|}\right\},$$

  where x,y,z are arbitrary nonzero real numbers.

- **5339:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu "George Emil Palade" School, Buzău, Romania*

  Calculate: $\displaystyle\int_0^{\pi/2}\frac{3\sin x+4\cos x}{3\cos x+4\sin x}dx.$

- **5340:** *Proposed by Oleh Faynshteyn, Leipzig, Germany*

  Let a,b and c be the side-lengths, and s the semi-perimeter of a triangle. Show that

  $$\frac{a^2+b^2}{(s-c)^2}+\frac{b^2+c^2}{(s-a)^2}+\frac{c^2+a^2}{(s-b)^2}\geq 24.$$

- **5341:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $z_1,z_2,\cdots,z_n$, and $w_1,w_2,\cdots,w_n$ be sequences of complex numbers. Prove that

  $$Re\left(\sum_{k=1}^{n}z_kw_k\right)\leq\frac{3}{(n+1)(n+2)}\sum_{k=1}^{n}|z_k|^2+\frac{3n^2+6n+1}{20}\sum_{k=1}^{n}|w_k|^2.$$

- **5342:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a, b, c, \alpha > 0$ be real numbers. Study the convergence of the integral

$$I(a,b,c,\alpha) = \int_1^\infty \left(a^{1/x} - \frac{b^{1/x} + c^{1/x}}{2}\right)^\alpha dx.$$

The problem is about studying the conditions which the four parameters, a, b, c, and α, should verify such that the improper integral would converge.

Solutions

- **5319:** *Proposed by Kenneth Korbin, New York, NY*

Let N be an odd integer greater than one. Then there will be a Primitive Pythagorean Triangle with perimeter equal to $\left(N^2+N\right)^2$. For example, if $N=3$, then the perimeter equals$\left(3^2+3\right)^2 = 144$.

Find the sides of the PPT for perimeter $\left(15^2+15\right)^2$ and for perimeter $\left(99^2+99\right)^2$.

Solution by David E. Manes, SUNY College at Oneonta, Oneonta, NY

The Primitive Pythagorean Triangle (a, b, c) with perimeter $(15^2+15)^2$ is $(6975, 24832, 25793)$, and the PPT with perimeter $(99^2+99)^2$ is $(1950399, 48010000, 48049601)$. One may easily verify that these triangles satisfy the conditions of the problem.

If $m > n$ are relatively prime positive integers of opposite parity, then they generate a PPT

$$(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2),$$

with perimeter $P = 2m(m+n)$. If P is a square, then $m = 2q^2$ and $m + n = p^2$ for some positive integers p and q. Therefore,

$$(m, n) = (2q^2, p^2 - 2q^2)$$

and

$$\begin{aligned} a &= m^2 - n^2 = p^2(4q^2 - p^2), \\ b &= 2mn \qquad = 4q^2(p^2 - 2q^2), \\ c &= m^2 + n^2 = p^4 - 4p^2q^2 + 8q^4. \end{aligned}$$

Note that p is odd, $\sqrt{2}q < p < 2q$ since $4q^2 - p^2 > 0$ and $p^2 - 2q^2 > 0$, and $\gcd(p, q) = 1$. Furthermore, the perimeter P is $4p^2q^2 = (2pq)^2$.

If $P = (15^2 + 15)^2$, then $2pq = 240$. Therefore $pq = 120$ and the only factors of 120 that statisfy p as being odd and $\sqrt{2}q < p < 2q$ are $p = 15$ and $q = 8$. For these values of p and q,

$$\begin{aligned} a &= 15^2\left(4 \cdot 8^2 - 15^2\right) = 6975, \\ b &= 4 \cdot 8^2\left(15^2 - 2 \cdot 8^2\right) = 24832, \\ c &= 15^4 - 4 \cdot 15^2 \cdot 8^2 + 8 \cdot 8^4 = 25793. \end{aligned}$$

If $P = (99^2 + 99)^2$, then $2pq = 99^2 + 99 = 9900$. Therefore $pq = 4950$ and the only factors of 4950 that satisfy p as being odd and $\sqrt{2}q < p < 2q$ are $p = 99$ and $q = 50$. Then

$$\begin{aligned} a &= 99^2\left(4 \cdot 50^2 - 99^2\right) = 1950399, \\ b &= 4 \cdot 50^2\left(99^2 - 2 \cdot 50^2\right) = 48010000, \\ c &= 99^4 - 4 \cdot 99^2 \cdot 50^2 + 8 \cdot 50^4 = 480449601. \end{aligned}$$

Also solved by Ashland University Undergraduate Problem Solving Group, Ashland, OH; Brian D. Beasley, Presbyterian College, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kee-Wai Lau, Hong Kong, China; Corneliu Mănescu-Avram, Transportation High School Ploiesti, Romania; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănesti, Romania (jointly with) Neculai Stanciu, "Geroge Emil Palade School," Buzău, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5320:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

It is fairly well known that if (a, b, c) is a Primitive Pythagorean Triple (PPT), then the product abc is divisible by 60. Find infinitely many PPT's (a, b, c) such that the sum $(a + b + c)$ is also divisible by 60.

Solution 1 by Bruno Salgueiro Fanego,Viveiro Spain

It is know that a, b and c are the respective legs and hypothenuse of a PPT if and only if $a = m^2 - n^2,\ b = 2mn$, and $c = m^2 + n^2$ for some positive integers m and n such that $m > n$ and $\gcd(m, n) = 1$ and $m - n$ is odd.

Hence, the perimeter, $a + b + c = 2m(m + n)$, will be divisible by 60 if, for example, m is divisible by 30 because in that case, $2m$ and hence $2m(m + n)$ would each be divisible by 60.

Thus, we can find infinitely many PPT's $(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2)$ such that the sum $a + b + c$ is also divisible by 60, if we take $m = 30k$, with k being a positive integer and, for example, $n = 1$ because in that case, $m = 30k > 1$, $\gcd(m, 1) = 1$, and $m - n = 30k - 1$ is odd. A possible infinite set of PPT's is given by

$$(a, b, c) = (900k^2 - 1, 60k, 900k^2 + 1), \text{ where } k \text{ is a positive integer.}$$

Solution 2 by Paul M. Harms, North Newton, KS

Consider the Pythagorean Triangle $\{n^2 + 1, n^2 - 1, 2n\}$ where n is a positive even integer. Then the odd integers $(n^2 + 1)$ and $(n^2 - 1)$, do not have 2 as a factor. Since their difference is 2 units, these two integers have no common prime factor greater than one. Thus the triple $(n^2 + 1, n^2 - 1, 2n)$ represents the sides of a PPT when n is a positive even integer. The sum of the three side is $2n^2 + 2n = 2n(n + 1)$. Let $n = 30K$ where K is a positive integer. Then n is a positive even integer and the sum of the three sides is divisible by 60. Using different $K's$ we see that there are infinitely many PPT's satisfying the problem whose sides have the form $(n^2 + 1, n^2 - 1, 2n)$ and $n = 30K$. In these cases the sum of the three sides is $2n(n + 1) = 60K(30K + 1)$.

Solution 3, a generalization by Brian D. Beasley, Presbyterian College, Clinton, SC

We may generalize the given problem as follows: Given any positive integer m, find infinitely many PPT's (a, b, c) such that the sum $(a + b + c)$ is divisible by m.
Fix any positive integer m. If m is even, then for each positive integer k, we let $s = mk$ and $t = 1$ to produce the PPT

$$(a, b, c) = (m^2k^2 - 1, 2mk, m^2k^2 + 1),$$

for which $a + b + c = 2mk(mk + 1)$. If m is odd, then for each positive integer k, we let $s = 2mk$ and $t = 1$ to produce the PPT

$$(a, b, c) = (4m^2k^2 - 1, 4mk, 4m^2k^2 + 1),$$

for which $a + b + c = 4mk(2mk + 1)$.

Also solved by Adnan Ali (Student in A.E.C.S-4), Mumbai, India; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX; Ed Gray, Highland Beach, FL; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Moti Levy, Rehovot, Israel; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Corneliu Mănescu-Avram, Transportation High School Ploiesti, Romania; Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Titu Zvonaru, Comănesti, Romania (jointly with) Neculai Stanciu, "Geroge Emil Palade School," Buzău, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5321:** *Proposed by Lawrence Lesser, University of Texas at El Paso, TX*

On pop quizzes during the fall semester, Al gets 1 out of 3 questions correct, while Bob gets 3 of 8 correct. During the spring semester, Al gets 3/5 questions correct, while Bob gets 2/3 correct. So Bob did better each semester $(3/8 > 1/3$ and $2/3 > 3/5)$ but worse for the overall academic year $(5/11 < 4/8)$. The total number of questions involved in the above example was $3+8+5+3=19$, and the author conjectures (in his chapter in the 2001 Yearbook of the National Council of Teachers of Mathematics) that this is the smallest dataset with nonzero numerators in which this reversal (Simpson's Paradox) happens. If we allow zeros, the smallest dataset is conjectured to be $9:\ 0/1 < 1/4$ and $2/3 < 1/1$, but $2/4 > 2/5$.

Prove these conjectures or find counterexamples.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

I wrote a small computer program that did an exhaustive search. It turned out that the first conjecture is wrong. The smallest value in the first case is 13 and not 19, and these are the solutions:

$$1/1 > 6/7, \quad 1/2 > 1/3, \quad 2/3 < 7/10$$

$$1/1 > 4/5, \quad 1/3 > 1/4, \quad 2/4 < 5/9$$

$$1/1 > 6/7, \quad 2/3 > 1/2, \quad 3/4 < 7/9$$

$$1/1 > 4/5, \quad 2/4 > 1/3, \quad 3/5 < 5/8$$

$$1/1 > 3/4, \quad 2/5 > 1/3, \quad 3/6 < 4/7$$

$$1/1 > 4/5, \quad 3/5 > 1/2, \quad 4/6 < 5/7$$

$$1/2 > 1/3, \quad 1/1 > 6/7, \quad 2/3 < 7/10$$

$$2/3 > 1/2, \quad 1/1 > 6/7, \quad 3/4 < 7/9$$

$$1/3 > 1/4, \quad 1/1 > 4/5, \quad 2/4 < 5/9$$

$$2/4 > 1/3, \quad 1/1 > 4/5, \quad 3/5 < 5/8$$

$$3/5 > 1/2, \quad 1/1 > 4/5, \quad 4/6 < 5/7$$

$$2/5 > 1/3, \quad 1/1 > 3/4, \quad 3/6 < 4/7$$

The smallest value in the second case is indeed 9 and these are the solutions:

$$1/1 > 3/4, \quad 1/3 > 0/1, \quad 2/4 < 3/5$$

$$1/1 > 2/3, \quad 1/4 > 0/1, \quad 2/5 < 2/4$$

$$1/3 > 0/1, \quad 1/1 > 3/4, \quad 2/4 < 3/5$$

$$1/4 > 0/1, \quad 1/1 > 2/3, \quad 2/5 < 2/4$$

Solution 2 by David E. Manes, SUNY College at Oneonta, Oneonta, NY

The conjecture is false for nonzero numerators since $4/5 < 1/1$ and $1/4 < 1/3$ but $5/9 > 2/4$, and the data set is $13 < 19$.

If zero numerator are allowed, then we will show that the smallest data set is indeed mine, that is if $\dfrac{a_2}{a_1} < \dfrac{A_2}{A_1}$ and $\dfrac{b_2}{b_1} < \dfrac{B_2}{B_1}$, then $\dfrac{a_2+b_2}{a_1+b_1} > \dfrac{A_2+B_2}{A_1+B_1}$ is impossible if $A_1+B_1+a_1+b_1 \le 8$ and $a_2 = 0$. To do so , we will maximize $\dfrac{a_2+b_2}{a_1+b_1}$ while minimizing $\dfrac{A_2+B_2}{A_1+B_1}$ Then $a_1 = 1$ and the maximum value of A_1 is 4.

If $A_1 = 4$, then maximizing $\dfrac{a_2+b_2}{a_1+b_1}$ and minimizing $\dfrac{A_2+B_2}{A_1+B_1}$ yields the following

$$\left.\begin{matrix} 0/1 < 1/4 \\ 1/2 < 1/1 \end{matrix}\right\} \Longrightarrow 1/3 < 2/5.$$

Note that for other values of A_2, the fraction $\dfrac{A_2+B_2}{A_1+B_1} > \dfrac{2}{5}$ while $\dfrac{a_2+b_2}{a_1+b_1} = \dfrac{1}{3}$.

If $A_1 = 3$, then $b_1 + B_1 \le 4$ implies b_1 is 2 or 3. If $b_1 = 2$, then maximizing $\dfrac{a_2+b_2}{a_1+b_1}$, one obtains

$$\left.\begin{matrix} 0/1 < 1/3 \\ 1/2 < 1/1 \end{matrix}\right\} \Longrightarrow 1/3 < 2/4.$$

If $b_1 = 3$, then maximizing $\dfrac{a_2+b_2}{a+1+b+1}$ yields

$$\left.\begin{matrix} 0/1 < 1/3 \\ 2/3 < 1/1 \end{matrix}\right\} \Longrightarrow 2/4 = 2/4.$$

If $A_1 = 2$, then $A_2 = 1$ and $b_1 + B_1 \le 5$ implies b_1 is 2,3, or 4. If $b_1 = 2$, then $b_2 = 1$ and minimizing B_2/B_1 so that $b_2/b_1 < B_2/B_1$ implies $B_1 = 3$ and $B_2 = 2$. Thus,

$$\left.\begin{matrix} 0/1 < 1/2 \\ 1/2 < 2/3 \end{matrix}\right\} \Longrightarrow 1/3 < 3/5.$$

If $b_1 = 3$ then $b_2 = 2$ and minimizing B_2/B_1 implies $B_1 = 1 = B_2$. Therefore,

$$\left.\begin{matrix} 0/1 < 1/2 \\ 2/3 < 1/1 \end{matrix}\right\} \Longrightarrow 2/4 < 2/3.$$

If $b_1 = 4$ then $b_2 = 3$ and $B_1 = B_2 = 1$. Therefore,

$$\left.\begin{matrix} 0/1 < 1/2 \\ 3/4 < 1/1 \end{matrix}\right\} \Longrightarrow 3/5 < 2/3.$$

If $A_1 = 1$ then $A_2 = 1$ and $b_1 + B_1 \le 6$. Therefore $b_1$is 2,3, or 4. If $b_1 = 2$, then $b_2 = 1$ and minimizing B_2/B_1 implies $B_1 = 3$ and $B_2 = 2$. Therefore,

$$\left.\begin{matrix} 0/1 < 1/1 \\ 1/2 < 2/3 \end{matrix}\right\} \Longrightarrow 1/3 < 3/4.$$

If $b_1 = 3$, then $b_2 = 2$ and $b_2/b_1 < B_2/B_1$ implies $B_1 = B_2 = 1$ since $B_1 \le 3$. Thus,

$$\left.\begin{array}{l} 0/1 < 1/1 \\ 2/3 < 1/1 \end{array}\right\} \Longrightarrow 2/4 < 1/1.$$

Note if $b_1 = 3$ and $b_2 = 1$, then minimizing B_2/B_1 implies $B_1 = 2$ and $B_2 = 1$. Therefore,

$$\left.\begin{array}{l} 0/1 < 1/1 \\ 1/3 < 1/2 \end{array}\right\} \Longrightarrow 1/4 < 2/3.$$

If $b_1 = 4$, then the only case when $\dfrac{A_2 + B_2}{A_1 + B_1} \neq 1$ is when $b_2 =$. Then $B_1 = 2$ and $B_2 = 1$. Then

$$\left.\begin{array}{l} 0/1 < 1/1 \\ 1/4 < 1/2 \end{array}\right\} \Longrightarrow 1/5 < 2/3.$$

Hence, if zero numerators are allowed, then the smallest dataset in which Simpson's Paradox can happen is nine.

Comments by the **Michael N Fried of Kibbutz Revivim, Israel** and by **Lawrence Lesser,** the proposer.

Michael: The inequalities need not be strict, we have for example Bob 1/1, 2/10 and Al 1/2, 1/5. So Bob does better OR AS WELL as Al, while the total for Bob is 3/11, is worse than the total for Al, 2/7. Under this assumption, the total number of questions is 1+10+2+5=18<19.

Michael went on to say that these numbers can be represented as slopes of lines, i.e., the slopes of the lines from (0,0) to (1,1) and (10,2) are great than those from (0,0) to (2,1) and (5,1), while the slope of the line given by the vector sum of (1,1) and (10,2) is less than that given by the vector sum of (2,1) and (5,1).

Lawrence: By allowing equality we could actually get it all the way down to 9 (e.g, Bob 1/1, 2/4; Al 1/2, 1/2) but almost every formulation of the problem that I have seen maintains strict inequality.

Slopes of lines is one of many representations of problem that I complied in my chapter in the 2001 NCTM yearbook, <http://www.statlit.org/pdf/2001LesserNCTM. pdf>

- **5322:** *Proposed by D.M. Bătinetu-Girugiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu "G.E. Palade", School, Buzău, Romania*

If $\displaystyle\lim_{n\to\infty}\left(-\frac{3}{2}\sqrt[3]{n^2} + \sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}\right) = a > 0$, then compute $\displaystyle\lim_{n\to\infty}\left(\frac{-\frac{3}{2}\sqrt[3]{n^2} + \sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}}{a}\right)^{\sqrt[3]{n}}$.

Solution 1 by Kee-Wai Lau, Hong Kong, China

Let $[x]$ be the greatest integer not exceeding x. It is easy to prove by induction that for positive integers n,

$$\sum_{k=1}^{n} k^{-1/3} - \frac{3}{2}n^{2/3} = b + \frac{1}{2}n^{-1/3} + \frac{1}{3}\int_{n}^{\infty}\left(t - [t] - \frac{1}{2}\right)t^{-4/3}dt \qquad (1)$$

where $b = -\left(\frac{1}{2} + \frac{1}{3}\int_1^\infty (t-[t])\, t^{-4/3} dt\right)$. The constant b is finite since

$\left|\int_1^\infty (t-[t])\, t^{-4/3} dt\right| \leq \int_1^\infty t^{-4/3} dt = 3$. Moreover it is negative by (1), $a = b$. For

$t \geq 0$, let $f(t) = \int_0^t \left(x - [x] - \frac{1}{2}\right) dx$. For any integer k, we have

$\int_k^{k+1} \left(x - [x] - \frac{1}{2}\right) dx = 0$, and so $f(t) = O(1)$. Integrating by parts, we see that the

integral in (1) equals $\frac{4}{3}\int_n^\infty f(t) t^{-7/3} dt = O\left(n^{-4/3}\right)$. Hence by (1), we have

$$\sqrt[3]{n} \lim_{n\to\infty} \left(\frac{-\frac{3}{2}\sqrt[3]{n^2} + \sum_{k=1}^{n} \frac{1}{\sqrt[3]{k}}}{a}\right) = \sqrt[3]{n} \ln\left(1 + \frac{1}{2a} n^{-1/3} + O\left(n^{-4/3}\right)\right) = \frac{1}{2a} + O\left(n^{-1/3}\right),$$

as $n \to \infty$. It follows that the limit of the problem equals $e^{1/2a}$.

Solution 2 by Nicusor Zlota "Traian Vuia" Technical College, Focsani, Romania

We have the case of 1^∞.

Denoting $a_n = -\frac{3}{2}\sqrt[3]{n}^2 + \sum_{k=1}^{n} \frac{1}{\sqrt[3]{k}}$, we may write the limit as:

$$l = \lim_{n\to\infty} \left(1 + \frac{a_n}{a}\right)^{\sqrt[3]{n}} = \lim_{n\to\infty} \left[\left(1 + \frac{a_n - a}{a}\right)^{\frac{a}{a_n - a}}\right]^{\frac{a_n - a}{a}\sqrt[3]{n}} = e^{\lim_{n\to\infty} \frac{a_n - a}{a}\sqrt[3]{n}}$$

For $l_1 = \lim_{n\to\infty} \frac{a_n - a}{a}\sqrt[3]{n} = \frac{1}{a}\lim_{n\to\infty} \frac{a_n - a}{\frac{1}{\sqrt[3]{n}}}$, and by the Cesaro -Stolz lemma, we have successively:

$$l_1 = \frac{1}{a}\lim_{n\to\infty} \frac{a_{n+1} - a_n}{\frac{1}{\sqrt[3]{n+1}} - \frac{1}{\sqrt[3]{n}}} = \frac{1}{a}\lim_{n\to\infty} \frac{-\frac{3}{2}\sqrt[3]{(n+1)^2} + \frac{1}{\sqrt[3]{n+1}} = \frac{3}{2}\sqrt[3]{n}}{\frac{1}{\sqrt[3]{n+1}} - \frac{1}{\sqrt[3]{n}}}$$

$$l_1 = \frac{1}{2a}\lim_{n\to\infty} \frac{\left(3n + 1 - 3\sqrt[3]{n^2(n+1)}\right)\sqrt[3]{n}}{\sqrt[3]{n+1} - \sqrt[3]{n}}$$

$$= \quad \frac{1}{2a}\lim_{n\to\infty}\frac{(9n+1)\left(\sqrt[3]{n(n+1)^2}+\sqrt[3]{n^2(n+1)}+n\right)}{(3n+1)^2+(9n+3)\sqrt[3]{n^2(n+1)}+9n\sqrt[3]{n(n+1)^2}}=\frac{1}{2a}.$$

Therefore the limit is $l = 2^{1/2a}$.

Generalization:

If $\lim\limits_{n\to\infty}\left(-\frac{p}{p-1}\sqrt[p]{n^{p-1}}+\sum_{k=1}^{n}\frac{1}{\sqrt[p]{k}}\right)=a>0$, and we wish to compute

$$\lim_{n\to\infty}\left(\frac{-\frac{p}{p-1}\sqrt[p]{n^{p-1}}+\sum_{k=1}^{n}\frac{1}{\sqrt[p]{k}}}{a}\right)^{\sqrt[p]{n}}\quad ,p\in N,\ p\geq 2$$

the answer is $e^{1/(p-1)a}$ and its proof is similar to the above.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Moti Levy, Rehovot, Israel; Corneliu Mănescu-Avram, Transportation High School Ploiesti, Romania, and the proposers.

- **5323:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be positive real numbers greater than or equal to one. Prove that

$$\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^{-2}+\left(\frac{1}{n^2}\prod_{k=1}^{n}a_k^{-2}\right)\left(\sum_{k=1}^{n}\left(a_k^2-1\right)^{1/2}\right)^2\leq 1.$$

Solution 1 by Moti Levy, Rehovot, Israel

Let $p(x)=(x-1)\left(x-\prod_{j=1}^{n}a_j^2\right)$. Then clearly $p(x)\leq 0$ for $1\leq x\leq\prod_{j=1}^{n}a_j^2$.

Every a_k^2, satisfies $1\leq a_k^2\leq\prod_{j=1}^{n}a_j^2$, hence

$$p\left(a_k^2\right)=\left(a_k^2-1\right)\left(a_k^2-\prod_{j=1}^{n}a_j^2\right)\leq 0,\quad 1\leq k\leq n. \tag{1}$$

Rearranging the terms in (1), we obtain,

$$\frac{1}{a_k^2}+\left(\prod_{j=1}^{n}a_j^{-2}\right)\left(a_k^2-1\right)\leq 1,\quad 1\leq k\leq n. \tag{2}$$

Taking average of both sides of (2), we get

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k^2}+\left(\prod_{j=1}^{n}a_j^{-2}\right)\left(\frac{1}{n}\sum_{k=1}^{n}\left(a_k^2-1\right)\right)\leq 1. \tag{3}$$

The power mean $M_p(x_1,\ldots,x_n)$, is a mean of the form

$$M_p(x_1,\ldots,x_n)=\left(\frac{1}{n}\sum_{k=1}^{n}x_k^p\right)^{\frac{1}{p}},$$

$$M_0(x_1,\ldots,x_n)=\left(\prod_{k=1}^{n}x_k\right)^{\frac{1}{n}}.$$

The monotonicity property of the power mean is

$$\text{if}\quad p<q,\quad \text{then}\quad M_p((x_1,\ldots,x_n))\le M_q((x_1,\ldots,x_n)). \tag{4}$$

By this property $M_{\frac{1}{2}}\le M_1$, hence

$$\left(\frac{1}{n}\sum_{k=1}^{n}(a_k^2-1)^{\frac{1}{2}}\right)^2\le\frac{1}{n}\sum_{k=1}^{n}(a_k^2-1). \tag{5}$$

By (3) and (5),

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k^2}+\left(\prod_{j=1}^{n}a_j^{-2}\right)\left(\frac{1}{n}\sum_{k=1}^{n}(a_k^2-1)^{\frac{1}{2}}\right)^2\le 1. \tag{6}$$

Since the function $f(x)=\frac{1}{x^2}$ is convex for $x\ge 1$, then by Jensen's inequality

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k^2}\ge\frac{1}{\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^2}. \tag{7}$$

It follows from (6) and (7) that

$$\frac{1}{\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^2}+\left(\prod_{j=1}^{n}a_j^{-2}\right)\left(\frac{1}{n}\sum_{k=1}^{n}(a_k^2-1)^{\frac{1}{2}}\right)^2\le 1.$$

Solution 2 by Kee-Wai Lau, Hong Kong, China

For $k=1,2,\ldots,n$, let $a_k=\sec b_k$, where $0\le b_k<\dfrac{\pi}{2}$. Since the function $\sec x$ is convex for $0<\dfrac{\pi}{2}$, so $\dfrac{1}{n}\sum_{k=1}^{n}a_k\ge\sec\left(\dfrac{\sum_{k=1}^{n}b_k}{n}\right)$. By the concavity of the function $\sin x$ for $0\le x<\dfrac{\pi}{2}$, we have

$$\left(\frac{1}{n}\prod_{k=1}^{n}a_k^{-1}\right)\left(\sum_{k=1}^{n}(a_k^2-1)^{1/2}\right)=\left(\frac{1}{n}\prod_{k=1}^{n}\cos b_k\right)\left(\sum_{k=1}^{n}\frac{\sin b_k}{\cos b_k}\right)\le\frac{\sum_{k=1}^{n}\sin b_k}{n}\le\sin\left(\frac{\sum_{k=1}^{n}b_k}{n}\right).$$

It follows that

$$\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^{-2}+\left(\frac{1}{n^2}\prod_{k=1}^{n}a_k^{-2}\right)\left(\sum_{k=1}^{n}(a_k^2-1)^{1/2}\right)^2\le\cos^2\left(\frac{\sum\limits_{k=1}^{n}b_k}{n}\right)+\sin^2\left(\frac{\sum\limits_{k=1}^{n}b_k}{n}\right)=1,$$

as required.

Also solved by, Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, and the proposer.

- **5324:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate

$$\sum_{n=1}^{\infty}\left(n\ln\left(1+\frac{1}{n}\right)-1+\frac{1}{2n}\right).$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We note that

$$\begin{aligned}\sum_{n=1}^{N}n\ln\left(1+\frac{1}{n}\right)&=\sum_{n=1}^{N}n\ln(n+1)-\sum_{n=1}^{N}n\ln(n)=\sum_{n=1}^{N+1}(n-1)\ln(n)-\sum_{n=1}^{N}n\ln(n)\\&=N\ln(N+1)-\sum_{n=1}^{N}\ln(n)=N\ln(N)+N\ln\left(1+\frac{1}{N}\right)-\ln(N!)\\&=N\ln(N)+1+O\left(\frac{1}{N}\right)-\ln\left(\sqrt{2\pi N}\right)-N\ln(N)+N+o(1)\\&=N+1-\frac{1}{2}\ln(N)-\frac{1}{2}\ln(2\pi)+o(1),\ \text{as}\ \ N\to\infty,\end{aligned}$$

where we have used Stirling's formula in the form $N!=\sqrt{2\pi N}N^Ne^{-N+o(1)}$, as $N\to\infty$.

$$\sum_{n=1}^{N}1=N$$

$$\sum_{n=1}^{N}\frac{1}{2n}=\frac{1}{2}\ln(N)+\frac{\gamma}{2}+o\left(\frac{1}{N}\right),\text{as } N\to\infty.$$

Collecting results we find that

$$\sum_{n=1}^{N}\left(n\ln\left(1+\frac{1}{n}\right)-1+\frac{1}{2n}\right)\ =\ N+1-\frac{1}{2}\ln(N)-\frac{1}{2}\ln(2\pi)-N+\frac{1}{2}\ln(N)+\frac{\gamma}{2}+o(1)$$

$$= 1-\frac{1}{2}\ln(2\pi)+\frac{\gamma}{2}+o(1),\ \text{as } N\to\infty,\ \text{and so}$$

$$\sum_{n=1}^{\infty}\left(n\ln\left(1+\frac{1}{n}\right)-1+\frac{1}{2n}\right) = 1-\frac{1}{2}\ln(2\pi)+\frac{\gamma}{2}.$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

$$\sum_{n=1}^{\infty}\left(n\ln\left(1+\frac{1}{n}\right)-1+\frac{1}{2n}\right)=\lim_{N\to\infty}\sum_{n=1}^{\infty}\left(n\ln\left(n+1\right)-n\ln n-1+\frac{1}{2n}\right)$$

$$= \lim_{N\to\infty}\sum_{n=1}^{N}\left(\left(\left(n+1\right)\ln\left(n+1\right)\right)-n\ln n-\ln(n+1)-1+\frac{1}{2n}\right)$$

$$= \lim_{N\to\infty}\sum_{n=1}^{N}\left(n+1\right)\ln(n+1)-n\ln n)-\lim_{N\to\infty}\sum_{n=1}^{N}\ln(n+1)-\lim_{N\to\infty}\sum_{n=1}^{N}1+\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{2n}$$

$$= \lim_{N\to\infty}(N+1)\ln(N+1)-\lim_{N\to\infty}\left(\ln(N+1)!\right)-\lim_{N\to\infty}\ln N+\frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n}$$

$$= \lim_{N\to\infty}\left(\ln(N+1)^N-\ln(N!)-\ln(e^N)\right)+\frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n}$$

$$= \lim_{N\to\infty}\ln\frac{(N+1)^N}{N!e^N}+\frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n}$$

$$= \lim_{N\to\infty}\ln\frac{\frac{(N+1)^N}{N^N}N^N\sqrt{N}\frac{1}{\sqrt{N}}}{N!e^N}+\frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n}$$

$$= \lim_{N\to\infty}\ln\frac{\frac{\left(1+\frac{1}{N}\right)^N}{N^N}N^N\sqrt{N}\frac{1}{\sqrt{N}}}{N!e^N}+\frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n}$$

$$= \lim_{N\to\infty}\left(\ln\left(\left(1+\frac{1}{N}\right)^N\right)+\ln\frac{N^N\sqrt{N}}{N!e^N}+\ln\frac{1}{\sqrt{N}}\right)+\frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n}$$

$$= \ln\lim_{N\to\infty}\left(1+\frac{1}{N}\right)^N+\ln\lim_{N\to\infty}\frac{N^N\sqrt{N}}{N!e^N}+\frac{1}{2}\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n}-\ln N\right)$$

$$= \ln e + \ln\left(\frac{1}{\sqrt{2\pi}}\right) + \frac{\gamma}{2}$$

$$= \frac{1}{2}\left(2 - \ln(2\pi) + \gamma\right),$$

where we have used the Stirling approximation for $N!$ and where γ is the Euler-Mascheroni constant.

Also solved by Ed Gray, Highland Beach, FL; G.E. Greubel, Newport News, VA; Moti Levy, Rehovot, Israel; Kee-Wai Lau, Hong Kong, China; Corneliu Mănescu-Avram, Transportation High School Ploiesti, Romania; Paolo Perfetti, Department of Mathematics, Tor Vergata Roma University, Rome, Italy, and the proposer.

Late Solutions, Comments, and an Announcement

A late solution to problem #5316 was received from **Raymon M. Melone of Waynesburg University, Waynesburg, PA.**

Comment by **Titu Zvonaru, Comănesti, Romania.** Solution 4 of problem #5317 is incorrect, because inequality (2) in the solution does not hold. For example: If

$$n = 3,\ b_1^{s+2} = 6,\ b_2^{s+2} = 9,\ b_3^{s+2} = 3,\ a_1 = \frac{1}{6},\ a_2 = \frac{1}{2} \text{ and } a_3 = \frac{1}{3}$$

then the LHS$= 36 + 18 + 9+ = 63$, while the RHS$= \frac{1}{3}(6+2+3)(6+9+3) = 66$.

The Chebyschev inequality maybe applied only if the sequences are both ascending or both descending. Of course, we may assume that one of the sequences is ascending but this assumption does not imply that the second sequence is also ascending:

$$b_1 \geq b_2 \geq \cdots \geq b_n \not\Longrightarrow a_1 \geq a_2 \geq \cdots \geq a_n.$$

For example, the inequality

$$\sum_{k=1}^{n} b_k^{s+2} \geq \frac{1}{n}\left(\sum_{k=1}^{n} b_k\right)\left(\sum_{k=1}^{n} b_k^{s+1}\right)$$

is correct.

Announcement: Following is part of a letter that was received from **Don Allen of Brossard, Canada.** Don has agreed that I may distribute his pdf file and an accompanying article entitled "The verse problems of early American arithmetics" to anyone who is interested in receiving them. Please send your requests to me at <eisenbt@013.net>

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Dear Professor Eisenberg:

When we corresponded in late October, I related how SSM program 5314 had reminded me of the more challenging problems routinely posed in nineteenth-century school algebra and arithmetic texts, which I had searched through in a then-uncatalogued collection at the United States university when I was completing doctoral studies – Rutgers, in New Jersey. I copied hundreds of such early "word problems" (authors had been copying one another for decades), and used many of them as challenges for teachers and for abler students. When I was working in Canada's Eastern Arctic decades later, I assembled some of the more satisfying teacher columns that I had prepared for such problems and their suggested solutions, and shared them with able, interested students and their parents on an evening at the library/museum of an appropriate arctic community. I recently located the original of the 30-page handout, I would like to put them at your disposal. You may print any you wish, and use in SSM any you feel appropriate and desirable.

Cordially,

Don Allen
Brossard, Canada