This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before April 15, 2012

- **5194**: Proposed by Kenneth Korbin, New York, NY
  Find two pairs of positive integers \((a, b)\) such that,
  \[
  \frac{14}{a} + \frac{a}{b} + \frac{b}{14} = 41.
  \]

- **5195**: Proposed by Kenneth Korbin, New York, NY
  If \(N\) is a prime number or a power of primes congruent to 1 \((\text{mod } 6)\), then there are positive integers \(a\) and \(b\) such that \(3a^2 + 3ab + b^2 = N\) with \((a, b) = 1\).
  Find \(a\) and \(b\) if \(N = 2011\), and if \(N = 2011^2\), and if \(N = 2011^3\).

- **5196**: Proposed by Neculai Stanciu, Buzău, Romania
  Determine the last six digits of the product \((2010)(5^{2014})\).

- **5197**: Proposed by Pedro H. O. Pantoja, UFRN, Brazil
  Let \(x, y, z\) be positive real numbers such that \(x^2 + y^2 + z^2 = 4\). Prove that,
  \[
  \frac{1}{6 - x^2} + \frac{1}{6 - y^2} + \frac{1}{6 - z^2} \leq \frac{1}{xyz}.
  \]

- **5198**: Proposed by José Luis Díaz-Barrero, Barcelona, Spain
  Let \(m, n\) be positive integers. Calculate,
  \[
  \sum_{k=1}^{2n} \prod_{i=0}^{m} \left(\left\lfloor \frac{k}{2} \right\rfloor + a + i \right)^{-1},
  \]
  where \(a\) is a nonnegative number and \(\lfloor x \rfloor\) represents the greatest integer less than or equal to \(x\).

- **5199**: Proposed by Ovidiu Furdui, Cluj, Romania
Let \( k > 0 \) and \( n \geq 0 \) be real numbers. Calculate,

\[
\int_0^1 x^n \ln \left( \sqrt{1 + x^k} - \sqrt{1 - x^k} \right) \, dx.
\]

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Solutions

• 5176: Proposed by Kenneth Korbin, New York, NY

Solve:

\[
\begin{align*}
x^2 + xy + y^2 &= 3^2 \\
y^2 + yz + z^2 &= 4^2 \\
z^2 + xz + x^2 &= 5^2.
\end{align*}
\]

Solution 1 by Albert Stadler, Herrliberg, Switzerland

Let

\[
\begin{align*}
A &= x^2 + xy + y^2 - 9 \\
B &= y^2 + yz + z^2 - 16 \\
C &= z^2 + xz + x^2 - 25
\end{align*}
\]

By assumption \( A = B = C = 0 \). So, \( 0 = A + B - C = xy + yz - xz + 2y^2 \) or equivalently \( z(x - y) = y(x + 2y) \). Obviously \( x \neq y \), since if \( x = y \) then \( 0 = B = x^2 + xz + z^2 - 16 \) and \( 0 = C = z^2 + xz + x^2 - 25 \) which is a contradiction. So,

\[
z = \frac{y(x + 2y)}{x - y}.
\]

We insert this value of \( z \) into the equation \( B = 0 \) and obtain

\[
16 = y^2 + y \cdot \frac{y(x + 2y)}{x - y} + \left( \frac{y(x + 2y)}{x - y} \right)^2
\]

\[
= y^2 \cdot \frac{(x - y)^2 + (x - y)(x + 2y) + (x + 2y)^2}{(x - y)^2}
\]

\[
= y^2 \cdot \frac{x^2 - 2xy + y^2 + x^2 + xy - 2y^2 + x^2 + 4xy + 4y^2}{(x - y)^2}
\]

\[
= y^2 \cdot \frac{3x^2 + 3xy + 3y^2}{(x - y)^2} = \frac{27y^2}{(x - y)^2}.
\]

So,

\[
4(x - y) = \pm 3\sqrt{3}y \quad \text{or equivalently,}
\]

\[
x = \left( 1 + \frac{3\sqrt{3}}{4} \right) y \quad \text{or} \quad x = \left( 1 - \frac{3\sqrt{3}}{4} \right) y.
\]
\( A = 0 \) then implies
\[
\left\{ \left( 1 \pm \frac{3\sqrt{3}}{4} \right)^2 + \left( 1 \pm \frac{3\sqrt{3}}{4} \right) + 1 \right\} y^2 = 9.
\]

Taking into account (1) and (2) we conclude that
\[
(x, y, z) \in \begin{cases} 
\left( \frac{9 + 4\sqrt{3}}{\sqrt{25 + 12\sqrt{3}}}, \frac{4\sqrt{3}}{\sqrt{25 + 12\sqrt{3}}}, \frac{4(4 + \sqrt{3})}{\sqrt{25 + 12\sqrt{3}}} \right), \\
\left( \frac{9 + 4\sqrt{3}}{\sqrt{25 + 12\sqrt{3}}}, \frac{-4\sqrt{3}}{\sqrt{25 + 12\sqrt{3}}}, \frac{4(-4 + \sqrt{3})}{\sqrt{25 + 12\sqrt{3}}} \right), \\
\left( \frac{9 - 4\sqrt{3}}{\sqrt{25 - 12\sqrt{3}}}, \frac{-4\sqrt{3}}{\sqrt{25 - 12\sqrt{3}}}, \frac{4(4 - \sqrt{3})}{\sqrt{25 - 12\sqrt{3}}} \right) 
\end{cases}
\]

The system of equations in the statement of the problem has an interesting geometric interpretation. Let \( ABC \) be a triangle all of whose angles are smaller than 120°. The Fermat point (or Torricelli point) of the triangle \( ABC \) is a point \( P \) such that the total distance from the three vertices of the triangle to the point is the minimum possible (see http://en.wikipedia.org/wiki/Fermat_point).

Let \( AB = c, BC = a, CA = b, AP = x, BP = y, CP = z \). Then
\[
\angle APB = \angle APC = \angle BPC = 120^\circ \quad \text{and} \quad x^2 + xy + y^2 = c^2,
\]
\[
y^2 + yz + z^2 = a^2,
\]
\[
z^2 + xz + x^2 = b^2,
\]
by the law of cosines. So \( x, y \) and \( z \) are the distances from the three vertices of the triangle to the Fermat point of the triangle.

• **Solution 2 by José Luis Díaz-Barrero, Barcelona, Spain**

Subtracting the equations term by term, we obtain
\[
(x^2 - y^2) + z(x - y) = 9, \quad \Leftrightarrow \quad (x - y)(x + y + z) = 9,
\]
\[
(x^2 - z^2) + y(x - z) = -7, \quad \Leftrightarrow \quad (x - z)(x + y + z) = -7.
\]

Putting \( u = x + y + z \), then we obtain \((x - y)u = 9\) and \((x - z)u = -7\). Adding both equations yields \((3x - (x + y + z))u = 2\) from which follows \( x = \frac{u^2 + 2}{3u} \). Likewise, we
obtain \( y = \frac{u^2 - 25}{3u} \), and \( z = \frac{u^2 + 23}{3u} \). Substituting the values of \( x, y, z \) into one of the equations of the given system, yields

\[
\left( \frac{u^2 + 2}{3u} \right)^2 + \left( \frac{u^2 + 2}{3u} \right) \left( \frac{u^2 - 25}{3u} \right) + \left( \frac{u^2 - 25}{3u} \right)^2 = 3^2
\]

or equivalently,

\[
3u^4 - 150u^2 + 579 = 0.
\]

Solving the preceding equation, we have the solutions:

\[
\pm \sqrt{25 - 12\sqrt{3}}, \quad \pm \sqrt{25 + 12\sqrt{3}}.
\]

Substituting these values in the expressions of \( x, y, z \) yields four triplets of solutions for the system. Namely,

\[
(x_1, y_1, z_1) = \left( \frac{27 - 12\sqrt{3}}{3\sqrt{25 - 12\sqrt{3}}}, \frac{-4\sqrt{3}}{\sqrt{25 - 12\sqrt{3}}}, \frac{48 - 12\sqrt{3}}{3\sqrt{25 - 12\sqrt{3}}} \right) = (1.009086173, -3.374440097, 4.418495493)
\]

\[
(x_2, y_2, z_2) = \left( \frac{-27 - 12\sqrt{3}}{3\sqrt{25 - 12\sqrt{3}}}, \frac{4\sqrt{3}}{\sqrt{25 - 12\sqrt{3}}}, \frac{-48 - 12\sqrt{3}}{3\sqrt{25 - 12\sqrt{3}}} \right) = (-1.009086173, 3.374440097, -4.418495493)
\]

\[
(x_3, y_3, z_3) = \left( \frac{27 + 12\sqrt{3}}{3\sqrt{25 + 12\sqrt{3}}}, \frac{4\sqrt{3}}{\sqrt{25 + 12\sqrt{3}}}, \frac{48 + 12\sqrt{3}}{3\sqrt{25 + 12\sqrt{3}}} \right) = (2.354003099, 1.023907822, 3.388521646)
\]

\[
(x_4, y_4, z_4) = \left( \frac{-27 + 12\sqrt{3}}{3\sqrt{25 + 12\sqrt{3}}}, \frac{-4\sqrt{3}}{\sqrt{25 + 12\sqrt{3}}}, \frac{-48 + 12\sqrt{3}}{3\sqrt{25 + 12\sqrt{3}}} \right) = (-2.354003099, -1.023907822, -3.388521646)
\]

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, University “Tor Vergata,” Rome, Italy; Boris Rays, Brooklyn, NY; Titu Zvonaru, Comănești, Romania jointly with Neculai Stanciu, Buzău, Romania, and the proposer.
Proposed by Kenneth Korbin, New York, NY

A regular nonagon \( ABCDEFGHI \) has side 1.

Find the area of \( \triangle ACF \).

Solution 1 by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX

We begin with the following known facts:

1. Each angle in a regular nonagon is 140°.
2. \( \cos 140° = \cos(180° - 40°) = -\cos 40°. \)
3. \( \cos 100° = -\cos 80°. \)
4. \( 1 + \cos 2\theta = 2\cos^2 \theta. \)
5. \( A = \frac{1}{2}ab\sin C \) in \( \triangle ABC \).

Hence, \( \triangle ABC \cong \triangle HIA \cong \triangle HGF \) by SAS. Using Fact 1, since \( \angle B = \angle I = \angle G = 140° \), it follows that \( \angle BAC = \angle IAH = \angle IHA = \angle GHF = \angle GFH = 20° \). Thus, \( \angle AHF = 100° \). Since \( \triangle AHF \) is an isosceles triangle, \( \angle HAF = \angle HFA = 40° \). Therefore, \( \angle CAF = 60° \). In \( \triangle ABC \), using the Law of Cosines and Facts 2 and 4,

\[
AC^2 = 1 + 1 - 2\cos 140° = 2(1 - \cos 140°) = 2(1 + \cos 40°) = 4\cos^2 20°.
\]

Then,

\[
AC = 2\cos 20°.
\]

Similarly, since \( AC = HA = HF = 2\cos 20° \), using the Law of Cosines and Facts 3 and 4 in \( \triangle HAF \),

\[
AF^2 = (2\cos 20°)^2 + (2\cos 20°)^2 - 2(2\cos 20°)^2 \cos 100° = 8\cos^2 20°(1 - \cos 100°) = 8\cos^2 20°(1 + \cos 80°) = 16\cos^2 20°\cos^2 40°.
\]

Thus,

\[
AF = 4\cos 20°\cos 40°.
\]

In \( \triangle ACF \), using Fact 5,

\[
A = \frac{1}{2}(AC)(AF)\sin 60° = \frac{1}{2}(2\cos 20°)(4\cos 20°\cos 40°)\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}\cos^2 20°\cos 40° \approx 2.343237.
\]

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Denote the circumcenter and the circumradius of the nonagon by \( O \) and \( r \), respectively.
The nonagon can be oriented within the Cartesian plane so that its vertices are

\[ A (r \cos 0^\circ, r \sin 0^\circ) \quad B (r \cos 40^\circ, r \sin 40^\circ) \quad C (r \cos 80^\circ, r \sin 80^\circ) \]

\[ D (r \cos 120^\circ, r \sin 120^\circ) \quad E (r \cos 160^\circ, r \sin 160^\circ) \quad F (r \cos 200^\circ, r \sin 200^\circ) \]

\[ G (r \cos 240^\circ, r \sin 240^\circ) \quad H (r \cos 280^\circ, r \sin 280^\circ) \quad I (r \cos 320^\circ, r \sin 320^\circ) . \]

Then,

\[ AB^2 = (r \cos 40^\circ - r \cos 0^\circ)^2 + (r \sin 40^\circ - r \sin 0^\circ)^2 \]

\[ = r^2 (\cos^2 40^\circ - 2 \cos 40^\circ + 1 + \sin^2 40^\circ)^2 \]

\[ = 2r^2 (1 - \cos 40^\circ) \Rightarrow r^2 = \frac{1}{2 (1 - \cos 40^\circ)} . \]

The area of \( \triangle ACF \) is

\[ [\triangle ACF] = \frac{1}{2} \left| \begin{vmatrix} 1 & 1 & 1 \\ \hline r & r \cos 80^\circ & r \cos 200^\circ \\ 0 & r \sin 80^\circ & r \sin 200^\circ \end{vmatrix} \right| \]

\[ = \frac{1}{2} \left| r^2 \cos 80^\circ \sin 200^\circ + r^2 \sin 80^\circ - r^2 \cos 200^\circ \sin 80^\circ - r^2 \sin 200^\circ \right| \]

\[ = \frac{1}{2} \left| r^2 (\cos 80^\circ \sin 200^\circ - \sin 80^\circ \cos 200^\circ) + r^2 \sin 80^\circ - r^2 \sin 200^\circ \right| \]

\[ = \frac{r^2}{2} \left| (\sin(200^\circ - 80^\circ) + \sin 80^\circ - \sin 200^\circ \right| \]

\[ = \frac{1}{4 (1 - \cos 40^\circ)} \left| \sin 120^\circ + \sin 80^\circ - \sin 200^\circ \right| \]

\[ \approx 2.343237 . \]

**Solution 3 by Kee-Wai Lau, Hong Kong, China**

It is easy to check that \( \angle BAC = 20^\circ, \angle IAF = \angle FAC = 60^\circ \) and \( AC = 2 \cos 20^\circ \). Suppose that the perpendicular from \( I \) to \( AF \) meets \( AF \) at \( J \), the perpendicular from \( H \) to \( AF \) meets \( AF \) at \( K \), and the perpendicular from \( I \) to \( HK \) meets \( HK \) at \( L \). Then \( \angle HIL = 20^\circ \) and

\[ AF = 2(AJ + JK) = 2(AJ + IL) = 2(\cos 60^\circ + \cos 20^\circ) = 1 + 2 \cos 20^\circ . \]

Hence the area of \( \triangle ACF \) equals

\[
\frac{(AC)(AF) \sin \angle FAC}{2}
\]
\[
\begin{align*}
\cos 20^\circ (1 + 2 \cos 20^\circ \sqrt{3}) & \quad = \quad \frac{\cos 20^\circ (1 + 2 \cos 20^\circ \sqrt{3})}{2} \\
\left(1 + \cos 20^\circ + \cos 40^\circ \right) \sqrt{3} & \quad = \quad \frac{\left(1 + \cos 20^\circ + \cos 40^\circ \right) \sqrt{3}}{2} \\
\sqrt{3}(1 + \sqrt{3} \cos 10^\circ) & \quad = \quad \frac{\sqrt{3}(1 + \sqrt{3} \cos 10^\circ)}{2} \\
\approx \quad 2.343237.
\end{align*}
\]

Solution 4 by proposer

\[
\text{Area of } \triangle ACF = \frac{\sin 40^\circ \cdot \sin 60^\circ \cdot \sin 80^\circ}{2 \sin^2 20^\circ}
\]

\[
= \quad \frac{\sqrt{3}}{16} \left[3 \tan^2 70^\circ - 1\right]
\]

\[
\approx \quad 2.343237.
\]

Comment by editor: Sines and cosines of angles of \(10^\circ, 20^\circ, 40^\circ\) and their complements often appear in the above solutions. David Stone and John Hawkins of Statesboro, GA noted in their solution that: “It may be possible to express the result \((\sqrt{3} \cos 40^\circ (1 + \cos 40^\circ))\) in terms of radicals, even though \(\cos 40^\circ\) itself cannot be expressed in terms of surds; it (along with \(\sin 10^\circ\) and \(-\sin 70^\circ\)) is a zero of the famous \textit{casus irreducibilis} cubic \(8x^3 - 6x + 1 = 0\).”

Also solved by Scott H. Brown, Montgomery, AL; Brian D. Beasley, Clinton, SC; Kenneth Day and Michael Thew (jointly, students at Saint George’s School), Spokane, WA; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; John Nord, Spokane, WA; Boris Rays, Brooklyn, NY, and Albert Stadler, Herrliberg, Switzerland.

\* \textbf{5178:} Proposed by Neculai Stanciu, Buzău, Romania

Prove: If \(x, y\) and \(z\) are positive real numbers such that \(xyz \geq 7 + 5\sqrt{2}\), then

\[
x^2 + y^2 + z^2 - 2(x + y + z) \geq 3.
\]

Solution 1 by Albert Stadler, Herrliberg, Switzerland

By the AM-GM inequality, \(\frac{x + y + z}{3} \geq \sqrt[3]{xyz} \geq \sqrt[3]{7 + 5\sqrt{2}} = 1 + \sqrt{2}\). Let

\[
f(x) = x^2 - 2x - 1.\ f(x)\ \text{is a convex function that is monotonically increasing for } x \geq 1.
\]

By Jensen’s inequality,

\[
x^3 + y^3 + z^3 - 2(x + y + z) - 3 = f(x) + f(y) + f(z) \geq 3f \left( \frac{x + y + z}{3} \right) \geq 3f \left( 1 + \sqrt{2} \right) = 0.
\]
Solution 2 by David E. Manes, Oneonta, NY

Note that for positive real numbers if \( x \geq 1 + \sqrt{2} \), then \( (x - 1)^2 \geq 2 \) with equality if and only if \( x = 1 + \sqrt{2} \). Therefore, if \( x, y, z \geq 1 + \sqrt{2} \), then \( xyz \geq 7 + 5\sqrt{2} \) and \( (x - 1)^2 + (y - 1)^2 + (z - 1)^2 \geq 6 \). Expanding this inequality yields \( x^2 + y^2 + z^2 - 2(x + y + z) \geq 3 \) with equality if and only if \( x = y = z = 1 + \sqrt{2} \).

Solution 3 by Paolo Perfetti, Department of Mathematics, University “Tor Vergata,” Rome, Italy

We know that \( x^2 + y^2 + z^2 \geq (x + y + z)^2 \) thus the inequality is implied by \( S^2 - 6S - 9 \geq 0 \), \( S = x + y + z \) yielding \( S \geq 3(1 + \sqrt{2}) \). Moreover by the AGM we have \( S \geq 3(xyz)^{1/3} \geq 3(7 + 5\sqrt{2})^{1/3} \), thus we need to check that \( 3(7 + 5\sqrt{2})^{1/3} \geq 3(1 + \sqrt{2}) \) or \( 7 + 5\sqrt{2} \geq (1 + \sqrt{2})^3 \) which is actually an equality, and we are done.

Also solved by Arkady Alt, San Jose California; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Boris Rays, Brooklyn, NY, and the proposer.

• 5179: Proposed by José Luis Díaz-Barrero, Barcelona, Spain

Find all positive real solutions \((x_1, x_2, \ldots, x_n)\) of the system

\[
\begin{align*}
    x_1 + \sqrt{x_2 + 11} &= \sqrt{x_2 + 76}, \\
    x_2 + \sqrt{x_3 + 11} &= \sqrt{x_3 + 76}, \\
    &\cdots \cdots \\
    x_{n-1} + \sqrt{x_n + 11} &= \sqrt{x_n + 76}, \\
    x_n + \sqrt{x_1 + 11} &= \sqrt{x_1 + 76}.
\end{align*}
\]

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

If \( f(t) = \sqrt{t + 76} - \sqrt{t + 11} \) on \((0, \infty)\), then

\[
f'(t) = \frac{1}{2} \left( \frac{1}{\sqrt{t + 76}} - \frac{1}{\sqrt{t + 11}} \right),
\]

and hence,

\[
|f'(t)| = \frac{1}{2} \left( \frac{1}{\sqrt{t + 11}} - \frac{1}{\sqrt{t + 76}} \right) < \frac{1}{2} \frac{1}{\sqrt{t + 11}} < \frac{1}{2} \frac{1}{\sqrt{11}} < 1
\]

for \( t > 0 \). It follows that \( f(t) \) is a contraction mapping on \((0, \infty)\) and therefore, \( f(t) \) has a unique fixed point \( t^* \in (0, \infty) \). Further, it is well-known that for any \( t \in (0, \infty) \), the
sequence defined recursively by \( t_1 = 7 \) and \( t_{k+1} = f(t_k) \) for \( k \geq 1 \) must converge to \( t^* \).

By trial and error, we find that \( t^* = 5 \).

In this problem,

\[
\begin{align*}
x_1 &= f(x_2), \\
x_2 &= f(x_3), \\
&\vdots \\
x_{n-1} &= f(x_n), \\
x_n &= f(x_1).
\end{align*}
\]

If we let \( t_1 = x_1 \) and define \( t_{k+1} = f(t_k) \) for \( k \geq 1 \), then \((x_1, x_2, \ldots, x_3, x_2)\) is a cycle in the sequence \( \{t_k\} \). However, as described above, \( t_k \to 5 \) as \( k \to \infty \). These conditions force \( x_1 = x_2 = \cdots = x_n = 5 \) and therefore, this must be the unique solution for this system.

Also solved by Arkady Alt, San Jose, CA; Scott H. Brown, Montgomery, AL; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, “Tor Vergata” University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; Neculai Stanciu, Buzău Romania, jointly with Titu Zvonaru, Comănești, Romania, and the proposer.

\[ \bullet \text{5180: Paolo Perfetti, Department of Mathematics, “Tor Vergata” University, Rome, Italy} \]

Let \( a, b \) and \( c \) be positive real numbers such that \( a + b + c = 1 \). Prove that

\[
\frac{1 + a}{bc} + \frac{1 + b}{ac} + \frac{1 + c}{ab} \geq \frac{4}{\sqrt{a^2 + b^2 - ab}} + \frac{4}{\sqrt{b^2 + c^2 - bc}} + \frac{4}{\sqrt{a^2 + c^2 - ac}}.
\]

**Solution 1 by Kee-Wai Lau, Hong Kong, China**

Multiplying both sides of the desired inequality by \( abc \), we see that it is equivalent to

\[
1 + a^2 + b^2 + c^2 \geq 4abc \left( \frac{1}{\sqrt{a^2 + b^2 - ab}} + \frac{1}{\sqrt{b^2 + c^2 - bc}} + \frac{1}{\sqrt{a^2 + c^2 - ac}} \right).
\]

Since

\[
a^2 + b^2 - ab = (a - b)^2 + ab \geq ab, \quad b^2 + c^2 - bc \geq bc, \quad a^2 + c^2 - ac \geq ac,
\]

the right hand side of (1) is less than or equal to

\[
4 \left( \sqrt{abc} + \sqrt{bca} + \sqrt{cab} \right)
\]

\[
\leq 2 ((a + b)c + (b + c)a + (c + a)b)
\]

\[
= 4(ab + bc + ca)
\]

\[
= 2 \left( (a + b + c)^2 - a^2 - b^2 - c^2 \right)
\]
\[= 2 - 2 \left( a^2 + b^2 + c^2 \right).\]

Now
\[a^2 + b^2 + c^2 = (a - \frac{1}{3})^2 + (b - \frac{1}{3})^2 + (c - \frac{1}{3})^2 + \frac{2(a + b + c)}{3} - \frac{1}{3} \geq \frac{1}{3},\]
so that \[1 + a^2 + b^2 + c^2 \geq 2 - 2 \left( a^2 + b^2 + c^2 \right),\]
This proves (1) and completes the solution.

**Solution 2 by Albert Stadler, Herrliberg, Switzerland.**

By the AM-GM inequality,
\[
\frac{1 + a}{bc} + \frac{1 + b}{ca} + \frac{1 + c}{ab} = \frac{a + a^2 + b + b^2 + c + c^2}{abc}
\]
\[
= \frac{1 + a^2 + b^2 + c^2}{abc}
\]
\[
= \frac{(a + b + c)^2 + a^2 + b^2 + c^2}{abc}
\]
\[
= \frac{(2a^2 + 2bc) + (2b^2 + 2ac) + (2c^2 + 2ab)}{abc}
\]
\[
\geq 4\frac{a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}}{abc}
\]
\[
= \frac{4}{\sqrt{bc}} + \frac{4}{\sqrt{ca}} + \frac{4}{\sqrt{ab}}.
\]

The conclusion follows since
\[
\frac{1}{\sqrt{xy}} \geq \frac{1}{\sqrt{x^2 + y^2 - xy}}.
\]
(Note that this inequality is equivalent to \[x^2 + y^2 - xy \geq xy\] which is obviously true.)

Also solved by Arkady Alt, San Jose, CA; Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie (jointly), San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; David E. Manes, Oneonta, NY; Titu Zvonaru, Comănești, Romania jointly with Neculai Stanciu, Buzău, Romania, and the proposer.

**5181: Proposed by Ovidiu Furdui, Cluj, Romania**

Calculate:
\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n \cdot m}{(n + m)!}.
\]
Solution 1 by Anastasios Kotronis, Athens, Greece

The summands being all positive we can sum by triangles:

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{nm}{(n+m)!} = \sum_{k, \ell, n \in \mathbb{N}} \frac{nm}{(n+m)!} = \sum_{n=2}^{\infty} \sum_{\ell=1}^{n-1} \frac{(n-\ell)\ell}{n!} = \sum_{n=2}^{\infty} \frac{(n-1)(n+1)}{n!} = \frac{1}{6} \sum_{n=2}^{\infty} \frac{(n+1)!}{n^2} = \frac{1}{6} \sum_{n=2}^{\infty} \frac{(n+1)!}{n^2} = \frac{1}{6} \sum_{n=2}^{\infty} \frac{(n+3)}{n!} = \frac{1}{6} \sum_{n=2}^{\infty} \frac{1}{n!} \frac{d}{dx} x^{n+3} \bigg|_{x=1} = \frac{1}{6} \frac{d}{dx} \left( \sum_{n=0}^{\infty} \frac{x^{n+3}}{n!} \right) \bigg|_{x=1} = \frac{1}{6} \frac{d}{dx} (x^3 e^x) \bigg|_{x=1} = 2e^3.
\]

Solution 2 by Arkady Alt, San Jose, CA

Let \( k = m + n \). Then \( m = k - n \) and domain of summation \( \{1 \leq n \leq k \mid 1 \leq m \} \) can be represented as \( \{2 \leq k, 1 \leq n \leq k - 1\} \). Hence,

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{nm}{(n+m)!} = \sum_{k=2}^{\infty} \sum_{n=1}^{k-1} \frac{n(k-n)}{k!} = \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{n=1}^{k-1} n(k-n) = \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{n=1}^{k-1} n(k-n).
\]

Since

\[
\sum_{n=1}^{k-1} n(k-n) = \frac{k^2(k-1)}{2} - \frac{(k-1)k(2k-1)}{6} = \frac{k}{6} \left(3k^2 - 3k - 2k^2 + 3k - 1\right) = \frac{k(k^2 - 1)}{6},
\]

then

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{nm}{(n+m)!} = \frac{1}{6} \sum_{k=2}^{\infty} \frac{k+1}{(k-2)!}.
\]
Solution 3 by the proposer

The series equals $\frac{2e}{3}$. First we note that for $m \geq 0$ and $n \geq 1$ one has that

$$\int_0^1 x^m (1 - x)^{n-1} dx = B(m + 1, n) = \frac{m! \cdot (n - 1)!}{(n + m)!}.$$

Thus,

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n \cdot m}{(n + m)!} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n}{(n - 1)!} \cdot \frac{1}{(m - 1)!} \int_0^1 x^m (1 - x)^{n-1} dx
\]

\[
= \int_0^1 \left( \sum_{n=1}^{\infty} \frac{n}{(n - 1)!} (1 - x)^{n-1} \right) \cdot \left( \sum_{m=1}^{\infty} \frac{x^m}{(m - 1)!} \right) dx
\]

\[
= \int_0^1 \left( 1 + \sum_{n=2}^{\infty} \frac{n}{(n - 1)!} (1 - x)^{n-1} \right) \cdot xe^x dx
\]

\[
= \int_0^1 \left( 1 + \sum_{n=2}^{\infty} \frac{(1 - x)^{n-1}}{(n - 2)!} + \sum_{n=2}^{\infty} \frac{(1 - x)^{n-1}}{(n - 1)!} \right) \cdot xe^x dx
\]

\[
= \int_0^1 \left( 1 + (1 - x)e^{1-x} + e^{1-x} - 1 \right) \cdot xe^x dx
\]

\[
= e \int_0^1 (2 - x)dx = \frac{2e}{3},
\]
and the problem is solved.

Also solved by Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, “Tor Vergata” University, Rome, Italy, and Albert Stadler, Herrliberg, Switzerland.