Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before February 15, 2014

- **5277:** *Proposed by Kenneth Korbin, New York, NY*

  Find x and y if a triangle with sides $(2013, 2013, x)$ has the same area and the same perimeter as a triangle with sides $(2015, 2015, y)$.

- **5278:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  The triangular numbers $6 = (2)(3)$ and $10 = (2)(5)$ are each twice a prime number. Find all triangular numbers that are twice a prime.

- **5279:** *Proposed by D.M. Bătinetu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "Geroge Emil Palade" General School, Buzu, Romania*

  Let $f : \Re_+ \longrightarrow \Re_+$ be a convex function on $\Re_+$, where $\Re_+$ stands for the positive real numbers. Prove that

  $$3\left(f^2(x) + f^2(y)_+ f^2(z)\right) - 9f^2\left(\frac{x+y+z}{3}\right) \geq \left(f(x) - f(y)\right)^2 + \left(f(y) - f(z)\right)^2 + \left(f(z) - f(x)\right)^2.$$

- **5280:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $a \geq b \geq c$ be nonnegative real numbers. Prove that

  $$\frac{1}{3}\left(\frac{(a+b)(c+a)}{2+\sqrt{a+b}} + \frac{(c+a)(b+c)}{2+\sqrt{c+a}} + \frac{(b+c)(a+b)}{2+\sqrt{b+c}}\right) \leq \frac{(a+b)^2}{2+\sqrt{b+c}}.$$

- **5281:** *Proposed by Arkady Alt, San Jose, CA*

  For the sequence $\{a_n\}_{n\geq 1}$ defined recursively by $a_{n+1} = \dfrac{a_n}{1+a_n^p}$ for $n \in \mathcal{N}, a_1 = a > 0$, determine all positive real p for which the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

- **5282:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate
$$\int_0^1 x\ln\left(\sqrt{1+x}-\sqrt{1-x}\right)\ln\left(\sqrt{1+x}+\sqrt{1-x}\right)dx.$$

Solutions

- **5259:** *Proposed by Kenneth Korbin, New York, NY*

Find a, b, and c such that with $a<b<c$, $\begin{cases} ab+bc+ca & =-2 \\ a^2b^2+b^2c^2+c^2a^2 & = \quad 6 \\ a^3b^3+b^3c^3+c^3a^3 & =-11. \end{cases}$

Solution 1 by Arkady Alt, San Jose, CA

Let $s=a+b+c,\ p=ab+bc+ca$, and $q=abc$. Then a,b,c are the roots of the equation $x^3-sx^2+px-q=0$. Since,

$$\begin{aligned} 6 &= a^2b^2+b^2c^2+c^2a^2=p^2-2sq=4-2sq \text{ and} \\ -11 &= a^3b^3+b^3c^3+c^3a^3=3q^2+p^3-3spq=3q^2-8+6sq, \text{ then} \\ sq &= -1 \text{ and } q^2=1 \iff q=1 \text{ or } q=-1. \end{aligned}$$

Thus we obtain $(s,p,q)=(-1,-2,1),(1,-2,-1)$ and, respectively, the two equations

$$x^3+x^2-2x-1=0 \quad\text{and}\quad x^3-x^2-2x+1=0.$$

Since,

$$\begin{aligned} (-x)^3+(-x)^2-2(-x)-1=0 &\iff x^3-x^2-2x+1=0, \text{ and} \\ x^3+x^2-2x-1=0 &\iff x=1.2470,\ -0.44504,\ -1.8019, \end{aligned}$$

we see that,

$$(a,b,c) \quad=\quad (-1.8019,\ -0.44504,\ 1.2470),\ (-1.2470,\ 0.44504,\ 1.8019).$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

As in problem 5135, let $x=ab,\ y=bc$ and $z=ca$, so that $x+y+z=-2,\ x^2+y^2+z^2=6$, and $x^3+y^3+z^3=-1$. We have

$$abc(a+b+c)=xy+yz+zx=\frac{(x+y+z)^2-x^2-y^2-z^2}{2}=\frac{(-2)^3-6}{2}=-1, \text{ and}$$

$$(abc)^3=xyz=\frac{x^3+y^3+z^3-(x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right)}{3}=\frac{-11+2(6+1)}{3}=1.$$

Hence, either $\begin{cases} a+b+c &= -1 \\ ab+bc+ca &= 2 \\ abc &= 1 \end{cases}$ or $\begin{cases} a+b+c &= 1 \\ ab+bc+ca &= 2 \\ abc &= -1. \end{cases}$

In the former case a, b, and c are the roots of the polynomial t^3+t^2-2t-1, and in the latter case, the roots of the polynomial t^3-t^2-2t+1. By the trigonometric method to find the roots of a cubic polynomial equation, we obtain respectively

$$\begin{aligned} a &= \frac{2\sqrt{7}}{3}\cos\left(\frac{\cos^{-1}\left(\frac{1}{2\sqrt{7}}\right)+2\pi}{3}\right)-\frac{1}{3}\approx -1.80194, \\ b &= \frac{2\sqrt{7}}{3}\cos\left(\frac{\cos^{-1}\left(\frac{1}{2\sqrt{7}}\right)+4\pi}{3}\right)-\frac{1}{3}\approx -0.445042, \text{and} \\ c &= \frac{2\sqrt{7}}{3}\cos\left(\frac{\cos^{-1}\left(\frac{1}{2\sqrt{7}}\right)}{3}\right)-\frac{1}{3}\approx 1.24698 \end{aligned}$$

$a \approx -1.24698$, $b \approx 0.445042$, and $c \approx 1.80194$.

Solution 3 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

To begin, label the equations as follows: $\begin{cases} ab+bc+ca &= -2 & (1) \\ a^2b^2+b^2c^2+c^2a^2 &= 6 & (2) \\ a^3b^3+b^3c^3+c^3a^3 &= -11. & (3) \end{cases}$

Then, by (1) and (2),

$$\begin{aligned} 4 &= (ab+bc+ca)^2 \\ &= a^2b^2+b^2c^2+c^2a^2+2\left(ab^2c+bc^2a+ca^2b\right) \\ &= 6+2abc\,(a+b+c) \text{ and hence,} \\ abc\,(a+b+c) &= -1. \quad (4) \end{aligned}$$

Next, use (1), (2), (3), and (4) to obtain

$$\begin{aligned} -12 &= (ab+bc+ca)\left(a^2b^2+b^2c^2+c^2a^2\right) \\ &= a^3b^3+b^3c^3+c^3a^3+ab^3c^2+a^3bc^2+a^2b^3c \\ &\quad +a^2bc^3+a^3b^2c+ab^2c^3 \\ &= -11+abc\,[ab\,(a+b)+bc\,(b+c)+ca\,(c+a)] \\ &= -11+abc\,[(ab+bc+ca)\,(a+b+c)-3abc] \end{aligned}$$

$$\begin{aligned} &= -11 + abc\left[-2\left(a+b+c\right)\right] - 3\left(abc\right)^2 \\ &= -9 - 3\left(abc\right)^2 \text{ or} \\ \left(abc\right)^2 &= 1. \qquad (5) \end{aligned}$$

It follows from (4) and (5) that either $abc = 1$ and $a + b + c = -1$ or $abc = -1$ and $a + b + c = 1$. Since

$$(x-a)(x-b)(x-c) = x^3 - (a+b+c)\,x^2 + (ab+bc+ca)\,x - abc,$$

a, b, c must be the solutions of either

$$x^3 + x^2 - 2x - 1 = 0 \qquad (6)$$

or

$$x^3 - x^2 - 2x + 1 = 0 \qquad (7)$$

We will utilize a method for solving (6) described on pg. 59 of [1]. The solutions of (7) can then be found by making an appropriate adjustment in this method. Let $R = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$. Then, as a 7^{th} root of unity, R has several useful properties:

- 1. Since $R^7 = 1$, we have

$$1 + R + R^2 + R^3 + R^4 + R^5 + R^6 = \frac{R^7 - 1}{R - 1} = 0.$$

- 2. For $k = 1, \ldots, 7$,

  a) $\frac{1}{R^k} = R^{7-k}$

  b) $R^k = R^{7+k}$

  c) $R^k + \frac{1}{R^k} = 2\mathrm{Re}\left(R^k\right)$.

Pair the powers of R as follows:

$$\begin{aligned} x_1 &= R + R^6 = R + \frac{1}{R} = 2\cos\frac{2\pi}{7}, \\ x_2 &= R^2 + R^5 = R^2 + \frac{1}{R^2} = 2\cos\frac{4\pi}{7} = -2\cos\frac{3\pi}{7}, \\ x_3 &= R^3 + R^4 = R^3 + \frac{1}{R^3} = 2\cos\frac{6\pi}{7} = -2\cos\frac{\pi}{7}. \end{aligned}$$

Then, since

$$x_1 + x_2 + x_3 = R + R^2 + R^3 + R^4 + R^5 + R^6 = -1,$$

$$x_1x_2 + x_2x_3 + x_3x_1 = \left(R^3 + R^6 + R^8 + R^{11}\right) + \left(R^5 + R^6 + R^8 + R^9\right)$$

$$
\begin{aligned}
& + \left(R^4 + R^9 + R^5 + R^{10}\right) \\
= \; & \left(R^3 + R^6 + R + R^4\right) + \left(R^5 + R^6 + R + R^2\right) \\
& + \left(R^4 + R^2 + R^5 + R^3\right) \\
= \; & 2\left(R + R^2 + R^3 + R^4 + R^5 + R^6\right) \\
= \; & -2, \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
x_1x_2x_3 &= \left(R + R^6\right)\left(R^5 + R^6 + R + R^2\right) \\
&= R^6 + R^7 + R^2 + R^3 + R^{11} + R^{12} + R^7 + R^8 \\
&= 2 + R + R^2 + R^3 + R^4 + R^5 + R^6 \\
&= 1,
\end{aligned}
$$

x_1, x_2, x_3 must be the solutions of (6). The condition $a < b < c$ then implies that one possible solution of our system is $a = -2\cos\frac{\pi}{7}$, $b = -2\cos\frac{3\pi}{7}$, and $c = 2\cos\frac{2\pi}{7}$.

Similarly, if

$$
\begin{aligned}
y_1 &= -x_1 = -2\cos\frac{2\pi}{7}, \\
y_2 &= -x_2 = 2\cos\frac{3\pi}{7}, \text{ and} \\
y_3 &= -x_3 = 2\cos\frac{\pi}{7}, \text{ then,} \\
y_1 + y_2 + y_3 &= -\left(x_1 + x_2 + x_3\right) = 1, \\
y_1y_2 + y_2y_3 + y_3y_1 &= x_1x_2 + x_2x_3 + x_3x_1 = -2, \text{ and} \\
y_1y_2y_3 &= -x_1x_2x_3 = -1.
\end{aligned}
$$

Therefore, y_1, y_2, y_3 are the solutions of (7). Again, since $a < b < c$, the remaining possible solution of our system is $a = -2\cos\frac{2\pi}{7}$, $b = 2\cos\frac{3\pi}{7}$, and $c = 2\cos\frac{\pi}{7}$.

To show that neither solution is extraneous, we note first that since

$$y_1y_2 + y_2y_3 + y_3y_1 = x_1x_2 + x_2x_3 + x_3x_1 = -2,$$

we have

$$ab + bc + ca = -2$$

in both cases. Further, the conditions

$$x_1 + x_2 + x_3 = -1, \quad x_1x_2x_3 = 1$$

and

$$y_1 + y_2 + y_3 = 1, \quad y_1 y_2 y_3 = -1$$

imply that

$$(abc)^2 = 1 \quad \text{and} \quad abc\,(a+b+c) = -1$$

in both cases. It follows that both solutions also satisfy

$$\begin{aligned} a^2b^2 + b^2c^2 + c^2a^2 &= (ab+bc+ca)^2 - 2abc\,(a+b+c) \\ &= 4+2 \\ &= 6 \end{aligned}$$

and

$$\begin{aligned} a^3b^3 + b^3c^3 + c^3a^3 &= (ab+bc+ca)\left(a^2b^2+b^2c^2+c^2a^2\right) \\ &\quad -abc\,(ab+bc+ca)\,(a+b+c) + 3\,(abc)^2 \\ &= (-2)\,(6) - (-1)\,(-2) + 3 \\ &= -11. \end{aligned}$$

Hence, our solutions for (6) and (7) both satisfy the original system as well.

Reference:

[1] Benjamin Bold, *Famous Problems of Geometry and How to Solve Them*, Dover Publications, Inc., 1969.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Ed Gray, Highland Beach, Fl; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5260:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

Find all primes p and q such that $a^{pq-1} \equiv a \pmod{pq}$, for all a relatively prime to pq.

Solution 1 by Ken Korbin, New York, NY

Let $p = 2$ and q be any odd prime.

$$\begin{aligned} \phi(pq) &= \phi(2q) = q-1 \\ (a, pq) &= 1, \text{ therefore} \end{aligned}$$

$$\begin{aligned}
a^{\phi(pq)} &\equiv 1\,(\text{mod } pq)\\
a^{q-1} &\equiv 1\,(\text{mod } pq)\\
\left[a^{q-1}\right]\cdot\left[a^{q-1}\right] &\equiv 1\cdot 1\,(\text{mod } pq)\\
a^{2q-2} &\equiv 1\,(\text{mod } pq)\\
a\cdot a^{2q-2} &\equiv a\cdot 1\,(\text{mod } pq)\\
a^{2q-1} &\equiv a\,(\text{mod } pq)\,,\text{therefore}\\
a^{pq-1} &\equiv a\,(\text{mod } pq)\,,\ \text{if } p\ = 2 \text{ and } q \text{ is any odd prime.}
\end{aligned}$$

Solution 2 by Kee-Wai Lau,Hong Kong, China

We show that primes p and q satisfy $a^{pq-1} \equiv a \pmod{pq}$ for all a relatively prime to pq, if and only if at least one of them is 2.

We need only that

I. For any prime q, $a^{2q-1} \equiv a \pmod{2q}$, for all a relatively prime to $2q$.

II. If $p \leq q$ are odd primes, then $a^{pq-1} \not\equiv a \pmod{pq}$ if $a > 1$ is a primitive root modulo q.

If $(a, 2q) = 1$, then $a^{q-1} + 1$ is even and by Fermat's little theorem, we have $a^{q-1} - 1 \equiv 0 \pmod{2q}$. Hence

$$a^{2q-1} - a = a(a^{q-1} + 1)(a^{q-1} - 1) \equiv 0 \pmod{2q}.$$

This proves I. We now prove II.

Suppose, on the contrary, that $a > 1$ is a primitive root modulo q such that

$$a^{pq-1} \equiv a \pmod{pq}. \qquad (1)$$

By Fermat's little theorem we have

$$\begin{aligned}
a^{pq-1} &= a^{p-1}(a^{q-1})^p\\
&= a^{p-1}(1+kq)^p\\
&= a^{p-1}\sum_{j=0}^{p}\binom{p}{j}(kq)^j \text{ for some positive integer } k.
\end{aligned} \qquad (3)$$

It is well known that p divides $\binom{p}{j}$ for $j = 1, 2, \cdots, p-1$. Hence

$$a^{pq-1} \equiv a^{p-1}(1 + k^p q^p) \pmod{pq}. \qquad (2)$$

From (1) and (2), we see that

$$a^{p-1} \equiv a \pmod{q}. \qquad (3)$$

Since a is a primitive root modulo q, so $a^r \not\equiv a \pmod{q}$ for $r = 2, 3, \cdots, q-1$.

Since $p > 2$, so by (3) we have $p - 1 \geq q$, which contradicts the fact tht $p \leq q$. This proves II and completes the solution.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; David E. Manes, SUNY College at Oneonta, Oneonta, NY, and the proposer.

- **5261:** *Proposed by Michael Brozinsky, Central Islip, NY*

Show without calculus or trigonometric functions that the shortest focal chord of an ellipse is the latus rectum.

Solution 1 by Paul M. Harms, North Newton, KS

Any ellipse can be placed on a coordinate system so that the equation of the ellipse is $\dfrac{(x+c)^2}{a^2} + \dfrac{y^2}{b^2} = 1$ where $a > b$. One focal point is at $(0,0)$. I will consider the focal chords through $(0,0)$.

Focal chords with slope m are on the line $y = mx$. The x values of the points of intersection of the ellipse and the line $y = mx$ come from the equation $\dfrac{(x+c)^2}{a^2} + \dfrac{m^2x^2}{b^2} = 1$ which yields the quadratic equation $\left(a^2m^2 + b^2\right)x^2 + 2b^2cx - b^4 = 0$, where $b^4 = b^2\left(a^2 - c^2\right)$.

If $H = \sqrt{b^4c^2 + (a^2m^2 + b^2)b^4}$, the x solutions are $\dfrac{-b^2c + H}{a^2m^2 + b^2}$ and $\dfrac{-b^2c - H}{a^2m^2 + b^2}$.

Let the intersection points of the focal chord and the ellipse be (x_1, y_1) and (x_2, y_2). To determine the shortest focal chord, I will look for the minimum of the square of the distance L between (x_1, y_1) and (x_2, y_2).

Here $L = (y_2 - y_1)^2 + (x_2 - x_1)^2$. Since the points are on $y = mx$ we have $y_2 - y_1 = m(x_2 - x_1)$ and $L = (x_2 - x_1)^2(m^2 + 1)$. The points x_1 and x_2 are the two solutions of the quadratic equation given above.

We have

$$\begin{aligned}
(x_2 - x_1)^2 &= \left(\frac{2H}{a^2m^2 + b^2}\right)^2 \text{and } L = (x_2 - x_1)^2(m^2 + 1) \\
&= \frac{4b^4(c^2 + a^2m^2 + b^2)}{(a^2m^2 + b^2)^2}(m^2 + 1) \\
&> \frac{4b^4(a^2m^2 + b^2)(m^2 + 1)}{(a^2m^2 + b^2)^2}
\end{aligned}$$

$$= \frac{\frac{4b^4}{a^2}\left(m^2+1\right)}{m^2+\left(\frac{b}{a}\right)^2}$$

$$> \frac{4b^4}{a^2}(1).$$

The last inequality occurs since $0 < \frac{b}{a} < 1$.

Thus any focal chord with slope m has the square of its length greater than $\frac{4b^4}{a^2}$, which is the square of the length of the vertical chord and the latus rectum. The conclusion of the problem follows.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Let F be one of the foci, d the directrix closest to F, e the eccentricity, and M, N, L points on the ellipse such that MN is a focal chord (that is, $F \in MN$) and L is one of the endpoints of the latus rectum ($LF||d$) and M', N', L', F' the respective projections of M, N, L, on d.

We want to prove that the length of the focal chord MN is greater or equal to the length of the latus rectum that is, that $MN \geq 2LF$.

Since the distance of any point on the ellipse to F is equal to e times its distance to d, we have that $MN = MF + FN = eMM' + eNN' = e(MM' + NN')$ and $LF = eLL'$, so we want to prove that $MM' + NN' \geq 2LL'$.

By Thales' theorem $\frac{MM'}{FF'} = \frac{FF'}{NN'}$ that is $MM' \cdot NN' = (FF')^2$. So by the arithmetic mean-geometric mean inequality

$$MM' + NN' \geq 2\sqrt{MM' \cdot NN'} = 2FF'$$

with equality if, and only if, $MM' = NN'$, that is if, and only if, MN coincides with the latus rectum, as we wanted to prove.

Also solved by Ed Gray, Highland Beach, FL, and the proposer.

- **5262:** *Proposed by Pedro H.O. Pantoja, IMPA, Rio de Janeiro, Brazil*

Prove that the equation $\varphi(10x^2) + \varphi(30x^3) + \varphi(34x^4) = y^2 + y^3 + y^4$ has infinitely many solutions for $x, y \in N$ where $\varphi(x)$ is the Euler-φ function.

Solution by Tom Moore, Bridgewater State University, Bridgewater, MA

Let $x = 2^k$. Then,

$$\varphi(10x^2) = \varphi\left(5 \cdot 2^{2k+1}\right) = \varphi(5)\varphi\left(2^{2k+1}\right) = 4 \cdot 2^{2k} = 2^{2k+2} = \left(2^{k+1}\right)^2.$$

$$\varphi(30x^3) \;=\; \varphi\left(2\cdot 5\cdot 6\cdot 2^{3k}\right) = \varphi(5)\varphi(3)\varphi\left(2^{3k}\right) = 8\cdot 2^{3k} = 2^{2k+3} = \left(2^{k+1}\right)^3.$$

$$\varphi(34x^4) \;=\; \varphi\left(2\cdot 17\cdot 2^{4k}\right) = \varphi(17)\varphi\left(2^{4k}\right) = 16\cdot 2^4 k = 2^{4k+4} = \left(2^{k+1}\right)^4.$$

So, we have infinitely many solutions$(x,y) = \left(2^k, 2^{k+1}\right),\ k \geq 0$.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Ed Gray, Highland Beach, FL; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Ken Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY, and the proposer.

- **5263:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

Let a, b, c be positive numbers lying in the interval $(0, 1]$. Prove that

$$a\cdot\sqrt{\frac{bc}{1+c+ab}} + b\cdot\sqrt{\frac{ca}{1+a+bc}} + c\cdot\sqrt{\frac{ab}{1+b+ca}} \leq \sqrt{3}.$$

Solution 1 by Ed Gray, Highland Beach, FL

Consider the function $f(x,y,z) = x\sqrt{\dfrac{y}{1+z+xy}}$. Each term in the problem is a representation of f by assigning a, b, c appropriately. Maximizing any term in the problem is equivalent to maximizing f.

Write f as $\sqrt{\dfrac{(x^2)\,yz}{1+z+xy}}$. Define $u = xy$ and f becomes $\sqrt{\dfrac{xuz}{1+z+u}}$. Note that u is in $(0, 1]$.

Since x appears alone in the numerator and we wish to maximize the function, we assign to x its largest value possible: that is, $x = 1$. The problem now becomes to maximize $\dfrac{uz}{1+z+u}$, for then its square root will attain its maximum.

Define $z + u = 2t$, where t is in $(0, 1]$. It is well know that the maximum of the product zu is t^2. Since if

$$r \;=\; zu = u(2t-u) = 2tu - u^2.$$

$$\frac{dr}{du} \;=\; 2t - 2u = 0 \Longrightarrow u = t,\ \text{and } z = t.$$

$$\frac{uz}{1+z+u} \text{ becomes } \frac{t^2}{1+2t}.$$

Since the derivative of this last term is greater than zero, it attains its maximum for $t = 1$ and is $\dfrac{1}{3}$.

Therefore the maximum of the left hand side of the statement of the problem is

$$3\sqrt{\frac{1}{3}} = 3\sqrt{\frac{3}{9}} = \frac{3}{3}\sqrt{3} \leq \sqrt{3}.\ \text{ Q.E.D.}$$

Solution 2 by Adrian Naco, Polytechnic University,Tirana, Albania.

Considering the left side of the last inequality and applying the wellknown AM-GM inequality we have that

$$a\cdot\sqrt{\frac{bc}{1+c+ab}}+b\cdot\sqrt{\frac{ca}{1+a+bc}}+c\cdot\sqrt{\frac{ab}{1+b+ca}}=$$

$$=\sqrt{abc}\left[\frac{\sqrt{a}}{\sqrt{1+c+ab}}+\frac{\sqrt{b}}{\sqrt{1+a+bc}}+\frac{\sqrt{c}}{\sqrt{1+b+ca}}\right]\le$$

$$\le\sqrt{abc}\left[\frac{\sqrt{a}}{\sqrt{3}\sqrt[6]{abc}}+\frac{\sqrt{b}}{\sqrt{3}\sqrt[6]{abc}}+\frac{\sqrt{c}}{\sqrt{3}\sqrt[6]{abc}}\right]$$

$$=\frac{\sqrt[3]{abc}}{\sqrt{3}}\left[\sqrt{a}+\sqrt{b}+\sqrt{c}\right]\le\frac{\sqrt[3]{1}}{\sqrt{3}}\left[\sqrt{1}+\sqrt{1}+\sqrt{1}\right]=\sqrt{3}$$

since

$$1+c+ab\ge 3\sqrt[3]{1\cdot c\cdot ab}=3\sqrt[3]{abc}\qquad\Rightarrow\qquad\frac{1}{\sqrt{1+c+ab}}\le\frac{1}{\sqrt{3}\sqrt{\sqrt[3]{abc}}}$$
$$1+a+bc\ge 3\sqrt[3]{1\cdot a\cdot bc}=3\sqrt[3]{abc}\qquad\Rightarrow\qquad\frac{1}{\sqrt{1+a+bc}}\le\frac{1}{\sqrt{3}\sqrt{\sqrt[3]{abc}}}$$
$$1+b+ca\ge 3\sqrt[3]{1\cdot b\cdot ca}=3\sqrt[3]{abc}\qquad\Rightarrow\qquad\frac{1}{\sqrt{1+b+ca}}\le\frac{1}{\sqrt{3}\sqrt{\sqrt[3]{abc}}}$$

The equality holds for $a=b=c=1$

Solution 3 by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

By applying the Cauchy-Schwarz inequality we obtain

$$\left(\sum_{\text{cyclic}}a\cdot\sqrt{\frac{bc}{1+c+ab}}\right)^2\le\left(\sum_{\text{cyclic}}a^2\right)\left(\sum_{\text{cyclic}}\frac{bc}{1+c+ab}\right)$$
$$\le 3\left(\sum_{\text{cyclic}}\frac{bc}{ac+bc+ab}\right)=3.$$

Solution 4 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

The concavity of $\sqrt{x}$ yields

$$\sum_{\text{cyc}}a\sqrt{\frac{bc}{1+c+ab}}=(a+b+c)\sum_{\text{cyc}}\frac{a}{a+b+c}\sqrt{\frac{bc}{1+c+ab}}\le$$

$$\le (a+b+c)\sqrt{\sum_{\text{cyc}} \frac{a}{a+b+c}\frac{bc}{1+c+ab}} \le \sqrt{3}.$$

Squaring we get

$$(abc)(a+b+c)\sum_{\text{cyc}} \frac{1}{1+c+ab} \le 3.$$

Now define $x = 1/a \ge 1$, $y = 1/b \ge 1$, $z = 1/c \ge 1$. We have

$$\frac{xy+yz+zx}{xyz}\sum_{\text{cyc}} \frac{1}{z+xy+xyz} \le 3,$$

and moreover

$$\frac{xy+yz+zx}{xyz}\sum_{\text{cyc}} \frac{1}{z+xy+xyz} \le \frac{xy+yz+zx}{xyz}\sum_{\text{cyc}} \frac{1}{3} \le 3 \iff 3xyz \ge xy+yz+zx,$$

which follows by $x, y, z \ge 1$.

Solution 5 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

Since $a, b, c > 0$, the Arithmetic - Geometric Mean Inequality implies that

$$1+c+ab \ge 3\sqrt[3]{abc}.$$

Then, because $0 < a, b, c \le 1$, we have

$$\begin{aligned} a \cdot \sqrt{\frac{bc}{1+c+ab}} &= \sqrt{a} \cdot \sqrt{\frac{abc}{1+c+ab}} \\ &\le \sqrt{a} \cdot \sqrt{\frac{abc}{3\sqrt[3]{abc}}} \\ &= \frac{\sqrt{a} \cdot \sqrt{(abc)^{\frac{2}{3}}}}{\sqrt{3}} \\ &= \frac{\sqrt{a}\sqrt[3]{abc}}{\sqrt{3}} \\ &\le \frac{1}{\sqrt{3}}, \end{aligned}$$

with equality if and only if $a = b = c = 1$.

Similarly,

$$b \cdot \sqrt{\frac{ca}{1+a+bc}} \le \frac{1}{\sqrt{3}} \quad \text{and} \quad c \cdot \sqrt{\frac{ab}{1+b+ca}} \le \frac{1}{\sqrt{3}},$$

with equality in each case if and only if $a = b = c = 1$.

Therefore,

$$\begin{aligned} a\cdot\sqrt{\frac{bc}{1+c+ab}}+b\cdot\sqrt{\frac{ca}{1+a+bc}}+c\cdot\sqrt{\frac{ab}{1+b+ca}} \\ \leq 1\sqrt{3}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}} \\ =\sqrt{3}, \end{aligned}$$

with equality if and only if $a=b=c=1$.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY, and the proposer.

- **5264:** *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia*

Let x, y, z, α be positive real numbers. Show that if

$$\sum_{cyclic}\frac{(n+1)x^3+nx}{x^2+1}=\alpha$$

then

$$\sum_{cyclic}\frac{1}{x}>\frac{3n}{\alpha}+\frac{(2n-1)\alpha}{3n}+\frac{3n\alpha}{9n^2+\alpha^2}$$

where n is a positive integer. Cyclic means the cyclic permutation of x, y, z (and not x, y, z and α).

Solution by proposer

Doing easy manipulations we have

$$\alpha=\sum_{cycl}\frac{(n+1)x^3+nx}{x^2+1}=\sum_{cycl}\frac{1}{x}+\sum_{cycl}\frac{-1+(n-1)x^2+(n+1)x^4}{x(x^2+1)}.$$

Let $f(x)=\dfrac{-1+(n-1)x^2+(n+1)x^4}{x(x^2+1)}$. One easily observes that

$$\begin{aligned} f'(x) &= \frac{1+(n+2)x^2+(2n+4)x^4+(n+1)x^6}{x^2(1+x^2)^2} \\ f''(x) &= -\frac{2(1+3x^2+2x^6)}{x^3(1+x^2)^3}. \end{aligned}$$

It is obvious that $f'(x)>0$ and $f''(x)<0$ for any x that is a positive real number, which implies that the function $f(x)$ is an increasing and concave function in the positive real domain. Applying Jensen's inequality we have

$$\sum_{cycl}\frac{-1+(n-1)x^2+(n+1)x^4}{x(x^2+1)}=\sum_{cycl}f(x)\leq 3f\left(\frac{\sum_{cycl}x}{3}\right).$$

Doing easy manipulations, one easily observes that

$$\alpha = \sum_{cycl} \frac{(n+1)x^3 + nx}{x^2+1} = \sum_{cycl} nx + \sum_{cycl} \frac{x^3}{x^2+1} > n\sum_{cycl} x \Longrightarrow \sum_{cycl} x < \frac{\alpha}{2n}.$$

Finally, using the above results we have

$$\begin{aligned}
\sum_{cycl} \frac{1}{x} &= \alpha - \sum_{cycl} \frac{-1+(n-1)x^2+(n+1)x^4}{x(x^2+1)} \\
&\geq \alpha - 3f\left(\frac{\displaystyle\sum_{cycl} x}{3}\right) \\
&> \alpha - 3f\left(\frac{\frac{\alpha}{n}}{3}\right) \\
&= \alpha - 3f\left(\frac{\alpha}{3n}\right) \\
&= \frac{3n}{\alpha} + \frac{(2n-1)\alpha}{3n} + \frac{3n\alpha}{9n^2+\alpha^2}
\end{aligned}$$

and this completes the proof.