Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before February 15, 2013

- **5230:** *Proposed by Kenneth Korbin, New York, NY*

  Given positive numbers x, y, z such that

$$\begin{aligned} x^2+xy+\frac{y^2}{3} &= 41, \\ \frac{y^2}{3}+z^2 &= 16, \\ x^2+xz+z^2 &= 25. \end{aligned}$$

  Find the value of $xy+2yz+3xz$.

- **5231:** *Proposed by Panagiote Ligouras, "Leonardo da Vinci" High School, Noci, Italy*

  The lengths of the sides of the hexagon $ABCDEF$ satisfy $AB=BC, CD=DE$, and $EF=FA$. Prove that

$$\sqrt{\frac{AF}{CF}}+\sqrt{\frac{CB}{EB}}+\sqrt{\frac{ED}{AD}}>2.$$

- **5232:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

  Prove that: If $a, b, c > 0$, then,

$$2\sqrt{\frac{a^2+b^2+c^2}{3}}\cdot\frac{\sin x}{x}+\frac{a+b+c}{3}\cdot\frac{\tan x}{x}>a+b+c,$$

  for any $x\in\left(0,\frac{\pi}{2}\right)$.

- **5233:** *Proposed by Anastasios Kotronis, Athens, Greece*

  Let $x\geq\dfrac{1+\ln 2}{2}$ and let $f(x)$ be the function defined by the relations:

$$f^2(x)-\ln f(x) = x$$

$$f(x) \quad \geq \quad \frac{\sqrt{2}}{2}.$$

- 1. Calculate $\lim\limits_{x \to +\infty} \dfrac{f(x)}{\sqrt{x}}$, if it exists.
- 2. Find the values of $\alpha \in \Re$ for which the series $\sum\limits_{k=1}^{+\infty} k^{\alpha} \left(f(k) - \sqrt{k} \right)$ converges.
- 3. Calculate $\lim\limits_{x \to +\infty} \dfrac{\sqrt{x} f(x) - x}{\ln x}$, if it exists.

- **5234:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

  Let $a < b$ be positive real numbers and let $f_1 : [a, b] \to \Re$ $(i = 1, 2)$ be continuous functions in $[a, b]$ and differentiable in (a, b). If f_2 is strictly decreasing then prove that there exists an $\alpha \in (a, b)$ such that

$$f_2(b) < f_2(\alpha) + 2 \left(\frac{f_2'(\alpha)}{f_1'(\alpha)} \right) < f_2(a).$$

- **5235:** *Proposed by Albert Stadler, Herrliberg, Switzerland*

  On December 21, 2012 ("$12 - 21 - 12$") the Mayan Calendar's 13^{th} Baktun cycle will end. On this date the world as we know it will also change (see <http://www.mayan-calendar.org/2012/end-of-the-world.html>). Since every end is a new beginning we are looking for natural numbers n such that the decimal representation of 2^n starts and ends with the digit sequence 122112. Let S be the set of natural numbers n such that $2^n = 122112....122112$. Let $s(x)$ be the number of elements of S that are $\leq x$.

  Prove that $\lim\limits_{x \to \infty} \dfrac{s(x)}{x}$ exists and is positive. Calculate the limit.

Solutions

- **5212:** *Proposed by Kenneth Korbin, New York, NY*

  Solve the equation

$$2x + y - \sqrt{3x^2 + 3xy + y^2} = 2 + \sqrt{2}$$

  if x and y are of the form $a + b\sqrt{2}$ where a and b are positive integers.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

In a similar way to the published solution to SSM problem 5105, we let $x = a + b\sqrt{2}, y = c + d\sqrt{2}$ and $y = \alpha x$, where a, b, c and d are positive integers and α is a positive real number. Substituting into the given equation gives

$$2 + \sqrt{2} = 2x + \alpha x - \sqrt{3x^2 + 3\alpha x^2 + \alpha^2 x^2} = \left(2 + \alpha - \sqrt{3 + 3\alpha + \alpha^2} \right) x = \varphi(\alpha) \left(a + b\sqrt{2} \right),$$

where $\varphi(\alpha) = \dfrac{(2 + \alpha)^2 - \left(\sqrt{3 + 3\alpha + \alpha^2} \right)^2}{2 + \alpha + \sqrt{3 + 3\alpha + \alpha^2}}$ is increasing (*editor's note:*

$\varphi'(\alpha) = 1 - \dfrac{3+2\alpha}{2\sqrt{3+3\alpha+\alpha^2}} > 0$, for $3+3\alpha+\alpha^2 > (1.5+\alpha)^2$.) and such that

$$\lim_{\alpha\to+\infty}\varphi(\alpha) = \lim_{\alpha\to+\infty}\frac{1/\alpha+1}{2/\alpha+1+\sqrt{3/\alpha^2+3/\alpha+1}} = \frac{0+1}{0+1+\sqrt{0+0+1}} = \frac{1}{2}.$$

On the other hand,

$$\varphi(0) \quad = \quad 2-\sqrt{3}, \text{ so } \varphi(0) \le \varphi(\alpha) < \lim_{\alpha\to+\infty}\varphi(\alpha) \text{ and hence,}$$

$$4+2\sqrt{2} \quad < \quad \frac{2+\sqrt{2}}{\varphi(\alpha)} \le \frac{2+\sqrt{2}}{2-\sqrt{3}}, \text{ that is,}$$

$$4+2\sqrt{2} \quad < \quad a+b\sqrt{2} \le \frac{2+\sqrt{2}}{2-\sqrt{3}}.$$

From this it follows that $b \le 9$ and that

$$\begin{cases} \text{if } b{=}0 \text{ then } 7\le a \le 12, \\ \text{if } b{=}1 \text{ then } 6\le a \le 11, \\ \text{if } b{=}2 \text{ then } 5\le a \le 9, \\ \text{if } b{=}3 \text{ then } 3\le a \le 8, \\ \text{if } b{=}4 \text{ then } 2\le a \le 7, \\ \text{if } b{=}5 \text{ then } 0\le a \le 5, \\ \text{if } b{=}6 \text{ then } 0\le a \le 4, \\ \text{if } b{=}7 \text{ then } 0\le a \le 2, \\ \text{if } b{=}8 \text{ then } 0\le a \le 1, \text{and} \\ \text{if } b{=}9 \text{ then } a{=}0. \end{cases}$$

The given equation is equivalent to

$$\left[2x+y-\left(2+\sqrt{2}\right)\right]^2 = \left(\sqrt{3x^2+3xy+y^2}\right)^2, \text{ that is,}$$

$$4x^2+4xy+y^2-\left(8+4\sqrt{2}\right)x-\left(4+2\sqrt{2}\right)y+4+4\sqrt{2}+2 = 3x^2+3xy+y^2.$$

So,

$$\begin{aligned} c+d\sqrt{2} &= y = \frac{x^2-\left(8+4\sqrt{2}\right)x+6+4\sqrt{2}}{4-x+2\sqrt{2}} \\ &= \frac{a^2+2b^2-8a-8b+6+(2ab-4a-8b+4)\sqrt{2}}{(4-a)+(2-b)\sqrt{2}} \\ &= \frac{\left[a^2+2b^2-8a-8b+6+(2ab-4a-8b+4)\sqrt{2}\right]\left[4-a+(b-2)\sqrt{2}\right]}{\left[4-a+(2-b)\sqrt{2}\right]\left[4-a+(b-2)\sqrt{2}\right]} \\ &= \frac{-a^3+2ab^2+12a^2-8b^2-8ab-22a+8b+8}{(4-a)^2-2(2-b)^2} + \end{aligned}$$

$$\frac{2b^3 - a^2b + 8ab + 2a^2 - 12b^2 - 4a - 10b + 4}{(4-a)^2 - 2(2-b)^2}\sqrt{2}.$$

So,

$$\begin{aligned} c &= \frac{-a^3 + 2ab^2 + 12a^2 - 8b^2 - 8ab - 22a + 8b + 8}{(4-a)^2 - 2(2-b)^2} \text{ and} \\ d &= \frac{2b^3 - a^2b + 8ab + 2a^2 - 12b^2 - 4a - 10b + 4}{(4-a)^2 - 2(2-b)^2}, \text{ where } c \text{ and } d \text{ are positive integers.} \end{aligned}$$

Restricting $a, b, c,$ and d to be positive integers we see that there are eleven solutions (x, y) to the problem. These are obtained by letting $x = a + b\sqrt{2}$ and $y = c + d\sqrt{2}$, where

$(a,b) \in \Big\{(6,1),(5,2),(6,2),(7,2),(3,3),(4,3),(5,3),(6,3),(2,4)(6,4),(1,5)\Big\}$ and respectively,

$(c,d) \in \Big\{(28,22),(17,12),(7,6),(3,4),(43,29),(12,8),(5,5),(4,2),(23,13),(1,1),(17,7)\Big\}.$

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

If we re-write the equation in the form

$$2x + y - \left(2 + \sqrt{2}\right) = \sqrt{3x^2 + 3xy + y^2}$$

and then square both sides and simplify, we get successively

$$\begin{aligned} x^2 - 4\left(2+\sqrt{2}\right)x + xy - 2\left(2+\sqrt{2}\right)y + 2\left(3+2\sqrt{2}\right) &= 0 \text{ and} \\ \left[x - 2\left(2+\sqrt{2}\right)\right]^2 + \left[x - 2\left(2+\sqrt{2}\right)\right]y &= 6\left(3+2\sqrt{2}\right). \end{aligned}$$

To simplify further, substitute $w = x - 2\left(2+\sqrt{2}\right)$ to obtain

$$w^2 + wy = 6\left(3+2\sqrt{2}\right). \qquad (1)$$

From the given instructions for x and y, we have

$$w = a_1 + b_1\sqrt{2} \text{ and } y = a_2 + b_2\sqrt{2},$$

where a_1, b_1, a_2, b_2 are integers with $a_2, b_2 \geq 1$, $a_1 \geq -3$, and $b_1 \geq -1$. If these are substituted into (1) and we use the fact that for integers a, b, c, d, $a + b\sqrt{2} = c + d\sqrt{2}$ if and only if $a = c$ and $b = d$ we obtain the following system:

$$(b_1 + b_2)\,a_1 + (a_1 + a_2)\,b_1 = 12 \qquad (2)$$

$$(a_1+a_2)\,a_1+2\,(b_1+b_2)\,b_1 \quad = \quad 18. \qquad (3)$$

Note that from the above information about a_1, b_1, a_2, b_2, it follows that $a_1+a_2 \geq -2$ and $b_1+b_2 \geq 0$.

If $b_1+b_2=0$, then we must have $b_1=-1$ and $b_2=1$. Equation (2) becomes $a_1+a_2=-12$, which is clearly impossible. If $b_1+b_2=1$, then either $b_1=-1$ and $b_2=2$ or $b_1=0$ and $b_2=1$. When $b_1=-1$ and $b_1+b_2=1$, equation (2) reduces to $-a_2=12$, which is impossible. When $b_1=0$ and $b_1+b_2=1$, (2) yields $a_1=12$ and (3) becomes $12\,(a_1+a_2)=18$, which is also impossible. Therefore, we will assume hereafter that $b_1+b_2 \geq 2$.

If $a_1+a_2=-2$, then since $a_1 \geq -3$ and $a_2 \geq 1$, we get $a_1=-3$ and $a_2=1$. Equation (3) becomes $(b_1+b_2)\,b_1=6$. Since $b_1+b_2 \geq 2$, it follows that $b_1 \geq 1$. Then, (2) is of the form

$$-3\,(b_1+b_2)-2b_1=12,$$

which is clearly impossible with $b_1 \geq 1$ and $b_1+b_2 \geq 2$.

If $a_1+a_2=-1$, then since $a_2 \geq 1$, it follows that $a_1<0$. However, equations (2) and (3) are

$$\begin{aligned}(b_1+b_2)\,a_1-b_1 &= 12\\ -a_1+2\,(b_1+b_2)\,b_1 &= 18\end{aligned}$$

and we get

$$a_1=6\frac{4\,(b_1+b_2)+3}{2\,(b_1+b_2)^2-1}>0$$

(since $b_1+b_2 \geq 2$). Hence, this case is impossible .

If $a_1+a_2=0$, (2) and (3) reduce to

$$\begin{aligned}(b_1+b_2)\,a_1 &= 12\\ (b_1+b_2)\,b_1 &= 9.\end{aligned}$$

Since $b_1+b_2 \geq 2$, this makes $a_1>0$, which is inconsistent with $a_1+a_2=0$.

If $a_1+a_2=1$, then $a_2 \geq 1$ implies that $a_1 \leq 0$. However, (2) and (3) become

$$\begin{aligned}(b_1+b_2)\,a_1+b_1 &= 12\\ a_1+2\,(b_1+b_2)\,b_1 &= 18\end{aligned}$$

and hence,

$$a_1=6\frac{4\,(b_1+b_2)-3}{2\,(b_1+b_2)^2-1}>0$$

(since $b_1+b_2 \geq 2$). Therefore, this case is also impossible and we may assume in the remainder of this solution that $a_1+a_2 \geq 2$.

In (2) and (3), if we treat a_1 and b_1 as coefficients and use Cramer's Rule, we obtain

$$a_1+a_2=6\frac{4b_1-3a_1}{2b_1^2-a_1^2}, \qquad b_1+b_2=6\frac{3b_1-2a_1}{2b_1^2-a_1^2} \quad \text{or}$$

$$a_2 = 6\frac{4b_1 - 3a_1}{2b_1^2 - a_1^2} - a_1, \qquad b_2 = 6\frac{3b_1 - 2a_1}{2b_1^2 - a_1^2} - b_1. \quad (4)$$

If $a_1 = -3$, then

$$a_2 = 6\frac{4b_1 + 9}{2b_1^2 - 9} + 3 = 3\left(2\frac{4b_1 + 9}{2b_1^2 - 9} + 1\right)$$

and

$$b_2 = 18\frac{b_1 + 2}{2b_1^2 - 9} - b_1 = \frac{-2b_1^3 + 27b_1 + 36}{2b_1^2 - 9}.$$

Using elementary calculus, it is straightforward to show that when $b_1 \geq 5$, $-2b_1^3 + 27b_1 + 36 < 0$ and $2b_1^2 - 9 > 0$, and hence, $b_2 < 0$. Also, by direct substitution, $b_2 < 0$ when $b_1 = 0, \pm 1$, or 2. Therefore, we are left with $b_1 = 3$ or 4. Of these, $b_1 = 4$ yields fractional values for a_2 and b_2, while $b_1 = 3$ gives the solution $a_1 = -3, b_1 = 3, a_2 = 17, b_2 = 7$. Therefore, our first solution is $w = -3 + 3\sqrt{2}$, $x = w + 2\left(2 + \sqrt{2}\right) = 1 + 5\sqrt{2}$, $y = 17 + 7\sqrt{2}$.

If $a_1 = -2$, (4) becomes

$$a_2 = 2\left(3\frac{2b_1 + 3}{b_1^2 - 2} + 1\right) \text{ and } b_2 = \frac{3b_1 + 4}{2b_1^2 - 4} - b_1 = \frac{-b_1^3 + 11b_1 + 12}{b_1^2 - 2}.$$

Proceeding as before, we see that $b_2 < 0$ for $b_1 \geq 4$ and $a_2 < 0$ for $b_1 = 0$ or ± 1. If $b_1 = 3$, then a_2 is a fraction. However, $b_1 = 2$ yields the solution $a_1 = -2, b_1 = 2, a_2 = 23, b_2 = 13$. Therefore, our next solution is $w = -2 + 2\sqrt{2}$, $x = 2 + 4\sqrt{2}$, $y = 23 + 13\sqrt{2}$.

If $a_1 = -1$, (4) reduces to

$$a_2 = 6\frac{4b_1 + 3}{2b_1^2 - 1} + 1 \text{ and } b_2 = 6\frac{3b_1 + 2}{2b_1^2 - 1} - b_1 = \frac{-2b_1^3 + 19b_1 + 12}{2b_1^2 - 1}.$$

If $b_1 \geq 4$, then $b_2 < 0$. Also, if $b_1 = -1$ or 0, $a_2 < 0$. Of the remaining choices, $b_1 = 2$ or 3 give fractional answers for a_2. When $b_1 = 1$, we get the solution $a_1 = -1, b_1 = 1, a_2 = 43, b_2 = 29$. This contributes $w = -1 + \sqrt{2}$, $x = 3 + 3\sqrt{2}$, $y = 43 + 29\sqrt{2}$ to our list of solutions.

If $a_1 = 0$, (4) becomes $a_2 = \dfrac{12}{b_1}$ and $b_2 = \dfrac{9}{b_1} - b_1 = \dfrac{9 - b_1^2}{b_1}$. In this case, $b_1 \neq 0$ and we get $a_2 < 0$ when $b_1 = -1$ and $b_2 \leq 0$ when $b_1 \geq 3$. Also, $b_1 = 2$ yields a fractional value for b_2. Hence, we are left with $b_1 = 1$, which gives the solution $a_1 = 0, b_1 = 1, a_2 = 12, b_2 = 8$ and we add $w = \sqrt{2}$, $x = 4 + 3\sqrt{2}$, $y = 12 + 8\sqrt{2}$ to our solution set.

We can now assume that $a_1 \geq 1$ in the remainder of this solution.

If $b_1 = -1$, (4) is of the form

$$a_2 = 6\frac{3a_1 + 4}{a_1^2 - 2} - a_1 = \frac{-a_1^3 + 20a_1 + 24}{a_1^2 - 2} \text{ and } b_2 = 6\frac{2a_1 + 3}{a_1^2 - 2} + 1.$$

As before, we get $a_2 < 0$ if $a_1 \geq 5$ and $b_2 < 0$ if $a_1 = 1$. When $a_1 = 3$ or 4, we get fractional values for b_2. Finally, $a_1 = 2$ gives the solution $a_1 = 2, b_1 = -1, a_2 = 28, b_2 = 22$, which yields $w = 2 - \sqrt{2}$, $x = 6 + \sqrt{2}$, $y = 28 + 22\sqrt{2}$.

If $b_1 = 0$, (4) becomes

$$a_2 = \frac{18}{a_1} - a_1 = \frac{18 - a_1^2}{a_1} \quad \text{and} \quad b_2 = \frac{12}{a_1}.$$

If $a_1 \geq 5$, we get $a_2 < 0$ and $a_1 = 4$ gives a fractional value for a_2. The remaining values $a_1 = 1, 2, 3$ produce the solutions listed below.

a_1	b_1	a_2	b_2	w	x	y
1	0	17	12	1	$5+2\sqrt{2}$	$17+12\sqrt{2}$
2	0	7	6	2	$6+2\sqrt{2}$	$7+6\sqrt{2}$
3	0	3	4	3	$7+2\sqrt{2}$	$3+4\sqrt{2}$

Finally, we are down to the situation where $a_1 \geq 1$, $b_1 \geq 1$, $a_1 + a_2 \geq 2$, and $b_1 + b_2 \geq 2$. Then, (2) implies that $1 \leq a_1 \leq 6$ and $1 \leq b_1 \leq 6$. By trying the 36 possibilities this presents for the system consisting of (2) and (3), we find that the remaining solutions are as follows:

a_1	b_1	a_2	b_2	w	x	y
1	1	5	5	$1+\sqrt{2}$	$5+3\sqrt{2}$	$5+5\sqrt{2}$
2	1	4	2	$2+\sqrt{2}$	$6+3\sqrt{2}$	$4+2\sqrt{2}$
2	2	1	1	$2+2\sqrt{2}$	$6+4\sqrt{2}$	$1+\sqrt{2}$

Our conclusion is that the full solution set for this problem is displayed below. With some algebraic fortitude, it can be checked that all are solutions to the original equation.

x	y
$1+5\sqrt{2}$	$17+7\sqrt{2}$
$2+4\sqrt{2}$	$23+13\sqrt{2}$
$3+3\sqrt{2}$	$43+29\sqrt{2}$
$4+3\sqrt{2}$	$12+8\sqrt{2}$
$5+2\sqrt{2}$	$17+12\sqrt{2}$
$5+3\sqrt{2}$	$5+5\sqrt{2}$
$6+\sqrt{2}$	$28+22\sqrt{2}$
$6+2\sqrt{2}$	$7+6\sqrt{2}$
$6+3\sqrt{2}$	$4+2\sqrt{2}$
$6+4\sqrt{2}$	$1+\sqrt{2}$
$7+2\sqrt{2}$	$3+4\sqrt{2}$

Comments: **David Stone and John Hawkins of Statesboro GA** noted that the solutions (x, y) lie on the hyperbola

$$y = \frac{x^2 - 4\left(2+\sqrt{2}\right)x + \left(2+\sqrt{2}\right)^2}{x - 2\left(2+\sqrt{2}\right)}$$

and it is not evident that there should be only finitely many solutions. However, imposing the specific form $a+b\sqrt{2}$ on x and y forces this to be the case.

And **Ken Korbin** (the proposer of the problem) characterized the solutions as follows:
Letting

$$(c,d)=(1,2+\sqrt{2}),(2+\sqrt{2},1),(\sqrt{2},1+\sqrt{2}),(1+\sqrt{2},\sqrt{2})$$

then

$$\begin{cases} x=c(2d+1) \\ y=c(3d^2-1) \quad \text{with } x<y, \end{cases} \quad and \quad \begin{cases} x=c(2d+3) \\ y=c(d^2-3) \quad \text{with } \ x<y. \end{cases}$$

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Titu Zvonaru and Comănesti Romania, Neculai Stanciu, Buzău, Romania (jointly); David Stone and John Hawkins of Georgia Southern University, Statesboro, GA (jointly), and the proposer.

5213: *Proposed by Tom Moore, Bridgewater, MA*

The triangular numbers T_n begin $1,3,6,10,\ldots$ and, in general, $T_n=\dfrac{n(n+1)}{2},\ n=1,2,3,\ldots$.

For every positive integer $n>1$, prove that n^4 is a sum of four triangular numbers.

Solution by Boris Rays, Brooklyn, NY

$$\begin{aligned} n^4 &= n^4-n^2+n^2=2\frac{n^4-n^2}{2}+2\frac{n^2}{2} \\ &= 2\frac{n^2}{2}+2\frac{n^4-n^2}{2} \\ &= \frac{n^2-n+n^2+n}{2}+2\frac{n^2(n^2-1)}{2} \\ &= \frac{(n-1)n}{2}+\frac{n(n+1)}{2}+\frac{(n^2-1)n^2}{2}+\frac{(n^2-1)n^2}{2} \\ &= T_{n-1}+T_n+T_{n^2-1}+T_{n^2-1}. \end{aligned}$$

Comments: **Albert Stadler of Herrliberg, Switzerland.** A.M. Legendre concluded from formulas in his treatise on elliptic functions [1] that the number of ways in which n is a sum of four triangular numbers equals the sum of the divisors of $2n+1$. As a result of this, every natural number can be represented as a sum of four triangular numbers. Reference: [1] Adrien Marie Legendre, Fonctions elliptiques et des intégrales Eulériennes: avec des tables pour en faciliter le calcul numérique; Vol 3 (1828), 133-134.

David Stone and John Hawkins of Statesboro, GA noted in their solution that n^4 is also the sum of *two* triangular numbers: $n^4=T_{n^2-1}+T_{n^2}$.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Elsie Campbell, Dionne Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Adrian Naco, Tirana, Albania; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Armend Sh. Shabani, (student, University of Prishtina), Republic of Kosova; Howard Sporn, Great Neck, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA (jointly); Titu Zvonaru, Comănesti, Romania and Neculai Stanciu Buzău, Romania (jointly), and the proposer.

5214: *Proposed by Pedro H. O. Pantoja, Natal-RN, Brazil*

Let a, b, c be positive real numbers. Prove that

$$\frac{a^3(b+c)^2+1}{1+a+2b}+\frac{b^3(c+a)^2+1}{1+b+2c}+\frac{c^3(a+b)^2+1}{1+c+2a}\geq\frac{4abc(ab+bc+ca)+3}{a+b+c+1}.$$

Solution 1 by David E. Manes, SUNY College at Oneonta, Oneonta, NY

Let $L=\dfrac{a^3(b+c)^2+1}{1+a+2b}+\dfrac{b^3(c+a)^2+1}{1+b+2c}+\dfrac{c^3(a+b)^2+1}{1+c+2a}$. Note that by the $AM-GM$ inequality, $(b+c)^2\geq 4bc, (c+a)\geq 4ca$, and $(a+b)^2\geq 4ab$ with equality if and only if $a=b=c$. Therefore,

$$\begin{aligned} L &\geq \frac{4a^3bc+1}{1+a+2b}+\frac{4b^3ca+1}{1+b+2c}+\frac{4c^3ab+1}{1+c+2a} \\ &= 4abc\left(\frac{a^2}{(1+a+2b)}+\frac{b^2}{(1+b+2c)}+\frac{c^2}{(1+c+2a)}\right)+\left(\frac{1^2}{(1+a+2b)}+\frac{1^2}{(1+b+2c)}+\frac{1^2}{(1+c+2a)}\right). \end{aligned}$$

The Cauchy-Schwarz inequality implies

$$\frac{a}{\sqrt{1+a+2b}}\cdot\sqrt{1+a+2b}+\frac{b}{\sqrt{1+b+2c}}\cdot\sqrt{1+b+2c}+\frac{c}{\sqrt{1+c+2a}}\cdot\sqrt{1+c+2a}^2\leq$$
$$\left(\frac{a^2}{1+a+2b}+\frac{b^2}{1+b+2c}+\frac{c^2}{1+c+2a}\right)(3a+3b+3c+3);$$

hence,

$$\frac{a^2}{1+a+2b}+\frac{b^2}{1+b+2c}+\frac{c^2}{1+c+2a}\geq\frac{(a+b+c)^2}{3\,(a+b+c+1)}.$$

Similarly,

$$\frac{1^2}{1+a+2b}+\frac{1^2}{1+b+2c}+\frac{1^2}{1+c+2a}\geq\frac{(1+1+1)^2}{3(a+b+c+1)}=\frac{3}{a+b+c+1}.$$

Therefore,

$$L\geq 4abc\left(\frac{(a+b+c)^2}{3(a+b+c+1)}\right)+\frac{3}{a+b+c+1}.$$

Furthermore, the Cauchy-Schwarz inequality also implies $a^2+b^2+c^2 \geq ab+bc+ca$ using vectors $\langle a,b,c\rangle$ and $\langle b,c,a\rangle$. Therefore,

$$\begin{aligned}(a+b+c)^2 &= a^2+b^2+c^2+2ab+2bc+2ca \\ &\geq ab+bc+ca+2(ab+bc+ca) \\ &= 3(ab+bc+ca).\end{aligned}$$

Hence,

$$\frac{(a+b+c)^2}{3(a+b+c+1)} \geq \frac{ab+bc+ca}{a+b+c+1}.$$

Accordingly,

$$\begin{aligned}L &\geq 4abc\left(\frac{(a+b+c)^2}{3(a+b+c+1)}\right)+\frac{3}{a+b+c+1} \\ &\geq \frac{4abc(ab+bc+ca)}{a+b+c+1}+\frac{3}{a+b+c+1} \\ &= \frac{4abc(ab+bc+ca)+3}{a+b+c+1},\end{aligned}$$

with equality if and only if $a=b=c$.

Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

We prove that

$$\frac{1}{1+a+2b}+\frac{1}{1+b+2c}+\frac{1}{1+c+2a} \geq \frac{3}{a+b+c+1}$$

and

$$\frac{a^3(b+c)^2}{1+a+2b}+\frac{b^3(c+a)^2}{1+b+2c}+\frac{c^3(a+b)^2}{1+c+2a} \geq \frac{4abc(ab+bc+ca)}{a+b+c+1}$$

By Cauchy–Schwarz

$$\frac{1}{1+a+2b}+\frac{1}{1+b+2c}+\frac{1}{1+c+2a} \geq \frac{(1+1+1)^2}{3+3(a+b+c)}$$

thus we prove

$$\frac{(1+1+1)^2}{3+3(a+b+c)} \geq \frac{3}{a+b+c+1}$$

which is actually an equality. As for the second inequality we have

$$\sum_{\text{cyc}} \frac{a^3(b+c)^2}{1+a+2b} \ge \sum_{\text{cyc}} \frac{a^3 4bc}{1+a+2b} \ge \frac{4abc(ab+bc+ca)}{a+b+c+1}$$

or

$$\sum_{\text{cyc}} \frac{a^2}{1+a+2b} \ge \frac{ab+bc+ca}{a+b+c+1}$$

Cauchy–Schwarz again yields

$$\sum_{\text{cyc}} \frac{a^2}{1+a+2b} \ge \frac{(a+b+c)^2}{3+3(a+b+c)} \ge \frac{ab+bc+ca}{a+b+c+1}$$

or

$$S^3 + S^2 \ge 3P + 3PS, \qquad S = a+b+c, \qquad P = ab+bc+ca$$

Now $S^2 \ge 3P$ since it is equivalent to

$$a^2+b^2+c^2 \ge ab+bc+ca$$

which is a well known inequality.

Solution 3 by Adrian Naco, Polytechnic University,Tirana, Albania.

Editor's comment: The following is a generalization of the stated problem.

Based on Cauchy-Schwarz inequality, for $a_i, b_i \in R^{*+}$ we have that

$$\begin{aligned}
\left(\sum_{i=1}^n a_i\right)^2 = \left[\sum_{i=1}^n \left(\frac{a_i}{\sqrt{b_i}}\right)(\sqrt{b_i})\right]^2 &\le \left[\sum_{i=1}^n \left(\frac{a_i}{\sqrt{b_i}}\right)^2\right]\left[\sum_{i=1}^n (\sqrt{b_i})^2\right] \\
&= \left(\sum_{i=1}^n \frac{{a_i}^2}{b_i}\right)\left(\sum_{i=1}^n b_i\right) \\
&\Rightarrow \quad \sum_{i=1}^n \frac{{a_i}^2}{b_i} \ge \frac{\left(\sum_{i=1}^n a_i\right)^2}{\sum_{i=1}^n b_i} \qquad (1)
\end{aligned}$$

where the equality holds for $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_n}{b_n}$.
Let us split the original inequality in two separate inequalities (2) and (3) as follows

$$\sum_{i=1}^{n}\frac{1}{1+x_i+2x_{i+1}}\geq\frac{n^2}{n+3\sum_{i=1}^{n}x_i}\qquad(2)$$

and

$$\sum_{i=1}^{n}\frac{{x_i}^3(x_{i+1}+x_{i+2})^2}{1+x_i+2x_{i+1})}\geq\frac{4\Big[\sum_{i=1}^{n}(x_ix_{i+1}x_{i+2}){x_i}^2+2\sum_{1\leq i<j\leq n}(x_ix_jx_{i+1}x_{j+1}x_{i+2}x_{j+2})^{\frac{1}{2}}x_ix_j\Big]}{n+3\sum_{i=1}^{n}x_i}\qquad(3)$$

Applying the above Cauchy-Schwarz inequality for each of the inequalities (2) and (3) we have that

$$\sum_{i=1}^{n}\frac{1^2}{1+x_i+2x_{i+1}}\geq\frac{\left(\sum_{i=1}^{n}1\right)^2}{\sum_{i=1}^{n}(1+x_i+2x_{i+1})}=\frac{(\underbrace{1+1++1}_{ntimes})^2}{\sum_{i=1}^{n}1+\sum_{i=1}^{n}x_i+2\sum_{i=1}^{n}x_{i+1}}$$

$$=\frac{n^2}{n+3\sum_{i=1}^{n}x_i}\qquad(4)$$

where the equality holds for $x_1=x_2=\ldots=x_n$. Thus we prove (2). Analogously we have that

$$\begin{aligned}\sum_{i=1}^{n}\frac{{x_i}^3(x_{i+1}+x_{i+2})^2}{1+x_i+2x_{i+1})} &= \sum_{i=1}^{n}\frac{\Big[{x_i}^{\frac{3}{2}}(x_{i+1}+x_{i+2})\Big]^2}{1+x_i+2x_{i+1})}\\ &\geq \frac{\Big[\sum_{i=1}^{n}{x_i}^{\frac{3}{2}}\overbrace{(x_{i+1}+x_{i+2})}^{x_{i+1}+x_{i+2}\geq2\sqrt{x_{i+1}x_{i+2}}}\Big]^2}{\sum_{i=1}^{n}(1+x_i+2x_{i+1})}\\ &\geq \frac{4\Big[\sum_{i=1}^{n}x_i(x_ix_{i+1}x_{i+2})^{\frac{1}{2}}\Big]^2}{\sum_{i=1}^{n}1+\sum_{i=1}^{n}x_i+2\sum_{i=1}^{n}x_{i+1}}\\ &= \frac{4\Big[\sum_{i=1}^{n}{x_i}^2(x_ix_{i+1}x_{i+2})+2\sum_{1\leq i<j\leq n}^{n}x_ix_j(x_ix_jx_{i+1}x_{j+1}x_{i+2}x_{j+2})^{\frac{1}{2}}\Big]}{n+3\sum_{i=1}^{n}x_i}\end{aligned}$$

$$= \frac{4\left[S + 3\sum_{1\le i<j\le n}^{n} x_i x_j (x_i x_j x_{i+1} x_{j+1} x_{i+2} x_{j+2})^{\frac{1}{2}}\right]}{n + 3\sum_{i=1}^{n} x_i}$$

where

$$S = \sum_{i=1}^{n} {x_i}^2 (x_i x_{i+1} x_{i+2}) - \sum_{1\le i<j\le n}^{n} x_i x_j (x_i x_j x_{i+1} x_{j+1} x_{i+2} x_{j+2})^{\frac{1}{2}}$$

The problem proposed is a special case of the above generalized problem for

$$x_1 = a \qquad x_2 = b \qquad x_3 = c.$$

Thus, we have that

$$\begin{aligned} S &= abca^2 + bcab^2 + cabc^2 - \left[(abbcca)^{\frac{1}{2}} ab + (acbacb)^{\frac{1}{2}} ac + (bccaab)^{\frac{1}{2}} bc\right] \\ &= abc(a^2 + b^2 + c^2 - ab - ac - bc) = abc\frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] \ge 0 \end{aligned}$$

with the equality holding for $a = b = c$. The inequality proposed gets the simplified form

$$\begin{aligned} \sum_{cyc} \frac{a^3(b+c)^2 + 1}{1 + a + 2b)} &\ge \frac{4\left\{S + 3\left[(abbcca)^{\frac{1}{2}} ab + (acbacb)^{\frac{1}{2}} ac + (bccaab)^{\frac{1}{2}} bc\right]\right\} + 3^2}{3 + 3(a+b+c)} \\ &= \frac{4\left[S + 3abc\left(ab + ac + bc\right)\right] + 3^2}{3 + 3(a+b+c)} \ge \frac{4\left[0 + 3abc\left(ab + ac + bc\right)\right] + 3^2}{3 + 3(a+b+c)} \\ &= \frac{4abc\left(ab + ac + bc\right) + 3}{1 + a + b + c} \end{aligned}$$

Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo TX; Andrea Fanchini, Cantú, Italy; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănesti, Romania and Neculai Stanciu Buzău, Romania (jointly); and the proposer.

5215: *Proposed by Neculai Stanciu, Buzău, Romania*

Evaluate the integral

$$\int_{-1}^{1} \frac{2x^{1004} + x^{3014} + x^{2008}\sin x^{2007}}{1 + x^{2010}} dx.$$

Solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

$$\begin{aligned}
\int_{-1}^{1} \frac{2x^{1004} + x^{3014} + x^{2008}\sin x^{2007}}{1 + x^{2010}} dx &= \int_{-1}^{1} \frac{2x^{1004} + x^{3014}}{1 + x^{2010}} dx + \int_{-1}^{1} \frac{x^{2008}\sin x^{2007}}{1 + x^{2010}} dx \\
&= \int_{-1}^{1} \frac{2x^{1004} + x^{3014}}{1 + x^{2010}} dx
\end{aligned}$$

Note that $\displaystyle\int_{-1}^{1} \frac{x^{2008}\sin x^{2007}}{1 + x^{2010}} dx = 0$, since the integrand function is odd and the interval of integration is centered at the origin.

The remaining integral may be solved using the change of variable $x^{1005} = t$.

$$\begin{aligned}
\int_{-1}^{1} \frac{2x^{1004} + x^{3014}}{1 + x^{2010}} dx &= \frac{1}{1005} \int_{-1}^{1} \frac{2 + t^2}{1 + t^2} dt \\
&= \frac{1}{1005} \left(2 + \int_{-1}^{1} \frac{1}{1 + t^2} dt \right) \\
&= \frac{1}{1005} \left(2 + \arctan(1) - \arctan(-1) \right) \\
&= \frac{1}{1005} \left(2 + \frac{\pi}{2} \right) = \frac{4 + \pi}{2010}.
\end{aligned}$$

Also solved by Daniel Lopez Aguayo, Institute of Mathematics, UNAM, Morelia, Mexico; Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo Sate University, San Angelo, TX; Michael C. Faleski, University Center, MI; Paul M. Harms, North Newton, KS; Fotini Kotroni and Anastasios Kotronis, Athens, Greece; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Adrian Naco, Department of Mathematics, Polytechnic University of Tirana, Albania; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Boris Rays, Brooklyn, NY; Armend Sh. Shabani, (student, University of Prishtina), Republic of Kosova; Albert Stadler, Herrliberg, Switzerland; Howard Sporn, Great Neck, NY, and the proposer.

5216: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let $f : \Re \to \Re^+$ be a function such that for all $a, b \in \Re$

$$f(ab) = f(a)^b f(b)^{a^2}$$

and $f(3) = 64$. Find all real solutions to the equation

$$f(x) + f(x+1) - 3x - 2 = 0.$$

Solution by Armend Sh. Shabani, (Graduate Student) University of Prishtina, Republic of Kosova.

Since $f(a \cdot b) = f(b \cdot a)$ we have

$$f(a)^b \cdot f(b)^{a^2} = f(b)^a \cdot f(a)^{b^2} \Leftrightarrow f(a)^b \cdot f(b)^a \left[f(b)^{a^2-a} - f(a)^{b^2-b} \right] = 0$$

$$\Leftrightarrow f(b)^{a^2-a} - f(a)^{b^2-b} = 0 \Leftrightarrow f(b)^{a^2-a} = f(a)^{b^2-b}.$$

Taking $b = x; a = 3$, one obtains:

$$\begin{aligned} f\left(x\right)^{3^2-3} &= f(3)^{x^2-x} \\ f(x)^6 &= (64)^{x^2-x} \\ f(x)^6 &= (4^3)^{x^2-x} \Rightarrow f(x) = 2^{x^2-x} = 2^{x(x-1)}. \end{aligned}$$

Substituting into $f(x) + f(x+1) - 3x - 2 = 0$ we obtain:

$$2^{x^2-x} + 2^{x^2+x} - (3x+2) = 0. \qquad (1)$$

Clearly $x = 0$; $x = 1$ are solutions of equation (1).

We show that there are no other solutions.

Let $g(x) = 2^{x^2-x} + 2^{x^2+x}$. One easily finds that

$$g'(x) = \ln 2 \cdot \left((2x+1) \cdot 2^{x^2+x} + (2x-1) \cdot 2^{x^2-x} \right) \text{ and}$$

$$g''(x) = \ln 2 \cdot \left(2^{x^2+x+1} + 2^{x^2-x+1} \right) + (\ln 2)^2 \cdot \left((2x+1)^2 \cdot 2^{x^2+x} + (2x-1)^2 \cdot 2^{x^2-x} \right).$$

So $g''(x) > 0$, and this means that g is a convex function, So the line $h(x) = 3x + 2$ can meet function g in at most 2 points. Therefore equation (1) has no other solutions. (Note that this can also be seen by drawing the graphs of functions g and h.)

Also solved by Dionne Bailey, Elsie Campbell, and Charles Dominnie, Angelo State University, San Angelo, TX; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Adrian Naco, Department of Mathematics, Polytechnic University of Tirana, Albania; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University Statesboro, GA (jointly), and the proposer.

5217: *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

Find the value of:

$$\lim_{n\to\infty}\int_0^1\int_0^1 \sqrt[n]{(x^n+y^n)^k}\,dxdy,$$

where k is a positive real number.

Solution 1 by Anastasios Kotronis, Athens, Greece

It is easily shown that $\sqrt[n]{(x^n+y^n)^k} \to \begin{cases} x^k, & y \le x \\ y^k, & x < y \end{cases}$ and since $0 \le \sqrt[n]{(x^n+y^n)^k} \le 2^k$, by the dominated convergence theorem we have

$$\begin{aligned}
\lim_{n\to+\infty}\int_0^1\int_0^1 \sqrt[n]{(x^n+y^n)^k}\,dydx &= \int_0^1\int_0^1 \lim_{n\to+\infty}\sqrt[n]{(x^n+y^n)^k}\,dydx \\
&= \int_0^1\int_0^x x^k\,dy\,dx + \int_0^1\int_x^1 y^k\,dy\,dx \\
&= \int_0^1 x^{k+1}\,dx + \int_0^1 \frac{1-x^{k+1}}{k+1}\,dx \\
&= \frac{2}{k+2}.
\end{aligned}$$

Solution 2 by Howard Sporn, Great Neck, NY

We use the fact that for n going to ∞, when $x < y$, the y^n term dominates over x^n, and when $x > y$, the x^n term dominates over y^n.

We break up the inner integral into two integrals, like so:

$$\int_0^1 \sqrt[n]{(x^n+y^n)^k}\,dx = \int_0^y \sqrt[n]{(x^n+y^n)^k}\,dx + \int_y^1 \sqrt[n]{(x^n+y^n)^k}\,dx.$$

Note that for the first integral $x \le y$, and for the second integral $x \ge y$. By factoring,

$$\int_0^1 \sqrt[n]{(x^n+y^n)^k}\,dx = \int_0^y \sqrt[n]{\left[y^n\left(\left(\frac{x}{y}\right)^n+1\right)\right]^k}\,dx + \int_y^1 \sqrt[n]{\left[x^n\left(1+\left(\frac{y}{x}\right)^n\right)\right]^k}\,dx.$$

For the first integral, in which x, we first consider the case $x < y$. In that case, $\left(\dfrac{x}{y}\right)^n \to 0$ for n$\to\infty$. Then the integrand becomes $\sqrt[n]{[y^n\,(0+1)]^k} = y^k$.

If, on the other hand, $x = y$, then the integrand becomes $\sqrt[n]{(\left[y^n\left(\left(\frac{y}{y}\right)^n + 1\right)\right]^k} = \sqrt[n]{\left[y^n\,(1+1)\right]^k} = y^k\,\sqrt[n]{2^k}$, which approaches y^k (once again) as $n \to \infty$.

Similarly, the integrand in the second integral approaches x^k.

The quantity we are seeking is now

$$\int_0^1 \left(\int_0^y y^k dx + \int_y^1 x^k dx\right) dy$$

which is straight-forward to compute.

The solution is

$$\int_0^1 \left(\left(y^k x\right)\Big|_{x=0}^{y} + \frac{x^{k+1}}{k+1}\Big|_{x=y}^{1}\right) dy$$

$$= \int_0^1 \left(y^{k+1} + \frac{1}{k+1} - \frac{y^{k+1}}{k+1}\right) dy$$

$$= \left(\frac{y^{k+2}}{k+2} + \frac{y}{k+1} - \frac{y^{k+2}}{(k+1)(k+2)}\right)\Bigg|_0^1$$

$$= \frac{1}{k+2} + \frac{1}{k+1} - \frac{1}{(k+1)(k+2)}$$

$$= \frac{(k+1)+(k+2)-1}{(k+1)(k+2)}$$

$$= \frac{2k+2}{(k+1)(k+2)}$$

$$= \frac{2}{k+2}.$$

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China; Adrian Naco, Department of Mathematics, Polytechnic University of Tirana, Albania; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland, and the proposer.

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