Problems
Ted Eisenberg, Section Editor

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>. Solutions to the problems stated in this issue should be posted before February 15, 2013

• 5230: Proposed by Kenneth Korbin, New York, NY

Given positive numbers $x, y, z$ such that

\[
x^2 + xy + \frac{y^2}{3} = 41,
\]

\[
\frac{y^2}{3} + z^2 = 16,
\]

\[
x^2 + xz + z^2 = 25.
\]

Find the value of $xy + 2yz + 3xz$.

• 5231: Proposed by Panagiote Ligouras, “Leonardo da Vinci” High School, Noci, Italy

The lengths of the sides of the hexagon $ABCDEF$ satisfy $AB = BC, CD = DE$, and $EF = FA$. Prove that

\[
\sqrt{\frac{AF}{CF}} + \sqrt{\frac{CB}{EB}} + \sqrt{\frac{ED}{AD}} > 2.
\]


Prove that: If $a, b, c > 0$, then,

\[
2\sqrt{\frac{a^2 + b^2 + c^2}{3}} \cdot \frac{\sin x}{x} + \frac{a + b + c}{3} \cdot \frac{\tan x}{x} > a + b + c,
\]

for any $x \in \left(0, \frac{\pi}{2}\right)$.

• 5233: Proposed by Anastasios Kotonis, Athens, Greece

Let $x \geq \frac{1 + \ln 2}{2}$ and let $f(x)$ be the function defined by the relations:

\[
f^2(x) - \ln f(x) = x
\]
\[ f(x) \geq \frac{\sqrt{2}}{2}. \]

- 1. Calculate \( \lim_{x \to +\infty} \frac{f(x)}{\sqrt{x}} \), if it exists.

- 2. Find the values of \( \alpha \in \mathbb{R} \) for which the series \( \sum_{k=1}^{+\infty} k^\alpha \left( f(k) - \sqrt{k} \right) \) converges.

- 3. Calculate \( \lim_{x \to +\infty} \frac{\sqrt{x}f(x) - x}{\ln x} \), if it exists.

- **5234**: Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain

Let \( a < b \) be positive real numbers and let \( f_i : [a, b] \to \mathbb{R} \) \((i = 1, 2)\) be continuous functions in \([a, b]\) and differentiable in \((a, b)\). If \( f_2 \) is strictly decreasing then prove that there exists an \( \alpha \in (a, b) \) such that

\[ f_2(b) < f_2(\alpha) + 2 \left( \frac{f'_2(\alpha)}{f'_1(\alpha)} \right) < f_2(a). \]

- **5235**: Proposed by Albert Stadler, Herrliberg, Switzerland

On December 21, 2012 (“12 - 21 - 12”) the Mayan Calendar’s 13\(^{th}\) Baktun cycle will end. On this date the world as we know it will also change (see <http://www.mayan-calendar.org/2012/end-of-the-world.html>). Since every end is a new beginning we are looking for natural numbers \( n \) such that the decimal representation of \( 2^n \) starts and ends with the digit sequence 122112. Let \( S \) be the set of natural numbers \( n \) such that \( 2^n = 122112...122112 \). Let \( s(x) \) be the number of elements of \( S \) that are \( \leq x \).

Prove that \( \lim_{x \to \infty} \frac{s(x)}{x} \) exists and is positive. Calculate the limit.

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**Solutions**

- **5212**: Proposed by Kenneth Korbin, New York, NY

Solve the equation

\[ 2x + y - \sqrt{3x^2 + 3xy + y^2} = 2 + \sqrt{2} \]

if \( x \) and \( y \) are of the form \( a + b\sqrt{2} \) where \( a \) and \( b \) are positive integers.

**Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain**

In a similar way to the published solution to SSM problem 5105, we let \( x = a + b\sqrt{2}, y = c + d\sqrt{2} \) and \( y = \alpha x \), where \( a, b, c \) and \( d \) are positive integers and \( \alpha \) is a positive real number. Substituting into the given equation gives

\[ 2 + \sqrt{2} = 2x + \alpha x - \sqrt{3x^2 + 3\alpha x^2 + \alpha^2 x^2} = \left( 2 + \alpha - \sqrt{3 + 3\alpha + \alpha^2} \right) x = \varphi(\alpha) \left( a + b\sqrt{2} \right), \]

where \( \varphi(\alpha) = \frac{(2 + \alpha)^2 - \left( \sqrt{3 + 3\alpha + \alpha^2} \right)^2}{2 + \alpha + \sqrt{3 + 3\alpha + \alpha^2}} \) is increasing (editor’s note:...
\[ \varphi'(\alpha) = 1 - \frac{3 + 2\alpha}{2\sqrt{3} + 3\alpha + \alpha^2} > 0, \text{ for } 3 + 3\alpha + \alpha^2 > (1.5 + \alpha)^2. \] and such that

\[ \lim_{\alpha \to +\infty} \varphi(\alpha) = \lim_{\alpha \to +\infty} \frac{1/\alpha + 1}{2/\alpha + 1 + \sqrt{3/\alpha^2 + 3/\alpha + 1}} = \frac{0 + 1}{0 + 1 + \sqrt{0 + 0 + 1}} = \frac{1}{2}. \]

On the other hand,

\[ \varphi(0) = 2 - \sqrt{3}, \text{ so } \varphi(0) \leq \varphi(\alpha) < \lim_{\alpha \to +\infty} \varphi(\alpha) \text{ and hence,} \]

\[ 4 + 2\sqrt{2} < \frac{2 + \sqrt{2}}{\varphi(\alpha)} \leq \frac{2 + \sqrt{2}}{2 - \sqrt{3}}, \text{ that is,} \]

\[ 4 + 2\sqrt{2} < a + b\sqrt{2} \leq \frac{2 + \sqrt{2}}{2 - \sqrt{3}}. \]

From this it follows that \( b \leq 9 \) and that

\[
\begin{align*}
\text{if } b &= 0 \text{ then } 7 \leq a \leq 12, \\
\text{if } b &= 1 \text{ then } 6 \leq a \leq 11, \\
\text{if } b &= 2 \text{ then } 5 \leq a \leq 9, \\
\text{if } b &= 3 \text{ then } 3 \leq a \leq 8, \\
\text{if } b &= 4 \text{ then } 2 \leq a \leq 7, \\
\text{if } b &= 5 \text{ then } 0 \leq a \leq 5, \\
\text{if } b &= 6 \text{ then } 0 \leq a \leq 4, \\
\text{if } b &= 7 \text{ then } 0 \leq a \leq 2, \\
\text{if } b &= 8 \text{ then } 0 \leq a \leq 1, \text{ and} \\
\text{if } b &= 9 \text{ then } a = 0.
\end{align*}
\]

The given equation is equivalent to

\[
\left[ 2x + y - \left( 2 + \sqrt{2} \right) \right]^2 = \left( \sqrt{3x^2 + 3xy + y^2} \right)^2, \text{ that is,} \]

\[ 4x^2 + 4xy + y^2 - \left( 8 + 4\sqrt{2} \right) x - \left( 4 + 2\sqrt{2} \right) y + 4 + 4\sqrt{2} + 2 = 3x^2 + 3xy + y^2. \]

So,

\[
c + d\sqrt{2} = y = \frac{x^2 - \left( 8 + 4\sqrt{2} \right) x + 6 + 4\sqrt{2}}{4 - x + 2\sqrt{2}} = \frac{a^2 + 2b^2 - 8a - 8b + 6 + (2ab - 4a - 8b + 4)\sqrt{2}}{(4 - a) + (2 - b)\sqrt{2}}
\]

\[
= \frac{\left[ a^2 + 2b^2 - 8a - 8b + 6 + (2ab - 4a - 8b + 4)\sqrt{2} \right]}{(4 - a) + (2 - b)\sqrt{2}} \frac{\left[ 4 - a + (2 - b)\sqrt{2} \right]}{\left[ 4 - a + (2 - b)\sqrt{2} \right]} = \frac{-a^3 + 2ab^2 + 12a^2 - 8b^2 - 8ab - 22a + 8b + 8}{(4 - a)^2 - 2(2 - b)^2} +
\]
\[
\frac{2b^3 - a^2 b + 8ab + 2a^2 - 12b^2 - 4a - 10b + 4}{(4 - a)^2 - 2(2 - b)^2}\sqrt{2}.
\]

So,
\[
c = \frac{-a^3 + 2ab^2 + 12a^2 - 8b^2 - 8ab - 22a + 8b + 8}{(4 - a)^2 - 2(2 - b)^2}
\]
and
\[
d = \frac{2b^3 - a^2 b + 8ab + 2a^2 - 12b^2 - 4a - 10b + 4}{(4 - a)^2 - 2(2 - b)^2},
\]
where \(c\) and \(d\) are positive integers.

Restricting \(a, b, c,\) and \(d\) to be positive integers we see that there are eleven solutions \((x, y)\) to the problem. These are obtained by letting \(x = a + b\sqrt{2}\) and \(y = c + d\sqrt{2},\) where
\[
(a, b) \in \{ (6, 1), (5, 2), (6, 2), (7, 2), (3, 3), (4, 3), (5, 3), (6, 3), (2, 4)(6, 4), (1, 5) \}
\]
and respectively,
\[
(c, d) \in \{ (28, 22), (17, 12), (7, 6), (3, 4), (43, 29), (12, 8), (5, 5), (4, 2), (23, 13), (1, 1), (17, 7) \}.
\]

**Solution 2** by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

If we re-write the equation in the form
\[2x + y - (2 + \sqrt{2}) = \sqrt{3x^2 + 3xy + y^2}\]
and then square both sides and simplify, we get successively
\[
x^2 - 4\left(2 + \sqrt{2}\right)x + xy - 2\left(2 + \sqrt{2}\right)y + 2\left(3 + 2\sqrt{2}\right) = 0
\]
and
\[
\left[x - 2\left(2 + \sqrt{2}\right)\right]^2 + \left[x - 2\left(2 + \sqrt{2}\right)\right]y = 6\left(3 + 2\sqrt{2}\right).
\]
To simplify further, substitute \(w = x - 2\left(2 + \sqrt{2}\right)\) to obtain
\[w^2 + wy = 6\left(3 + 2\sqrt{2}\right).\tag{1}\]

From the given instructions for \(x\) and \(y,\) we have
\[w = a_1 + b_1\sqrt{2}\] and \(y = a_2 + b_2\sqrt{2},\]
where \(a_1, b_1, a_2, b_2\) are integers with \(a_2, b_2 \geq 1, a_1 \geq -3,\) and \(b_1 \geq -1.\) If these are substituted into (1) and we use the fact that for integers \(a, b, c, d, a + b\sqrt{2} = c + d\sqrt{2}\) if and only if \(a = c\) and \(b = d\) we obtain the following system:
\[(b_1 + b_2)a_1 + (a_1 + a_2)b_1 = 12\] (2)
\[(a_1 + a_2) a_1 + 2 (b_1 + b_2) b_1 = 18. \quad (3)\]

Note that from the above information about \(a_1, b_1, a_2, b_2\), it follows that \(a_1 + a_2 \geq -2\) and \(b_1 + b_2 \geq 0\).

If \(b_1 + b_2 = 0\), then we must have \(b_1 = -1\) and \(b_2 = 2\). Equation (2) becomes \(a_1 + a_2 = -12\), which is clearly impossible. If \(b_1 + b_2 = 1\), then either \(b_1 = -1\) and \(b_2 = 2\) or \(b_1 = 0\) and \(b_2 = 1\). When \(b_1 = -1\) and \(b_1 + b_2 = 1\), equation (2) reduces to \(-a_2 = 12\), which is impossible. When \(b_1 = 0\) and \(b_1 + b_2 = 1\), (2) yields \(a_1 = 12\) and (3) becomes \(12 (a_1 + a_2) = 18\), which is also impossible. Therefore, we will assume hereafter that \(b_1 + b_2 \geq 2\).

If \(a_1 + a_2 = -2\), then since \(a_1 \geq -3\) and \(a_2 \geq 1\), we get \(a_1 = -3\) and \(a_2 = 1\). Equation (3) becomes \((b_1 + b_2) b_1 = 6\). Since \(b_1 + b_2 \geq 2\), it follows that \(b_1 \geq 1\). Then, (2) is of the form
\[-3 (b_1 + b_2) - 2b_1 = 12,
\]
which is clearly impossible with \(b_1 \geq 1\) and \(b_1 + b_2 \geq 2\).

If \(a_1 + a_2 = -1\), then since \(a_2 \geq 1\), it follows that \(a_1 < 0\). However, equations (2) and (3) are
\[
(b_1 + b_2) a_1 - b_1 = 12
\]
\[
-a_1 + 2 (b_1 + b_2) b_1 = 18
\]
and we get
\[
a_1 = 6 \frac{4(b_1 + b_2) + 3}{2(b_1 + b_2)^2 - 1} > 0
\]
(since \(b_1 + b_2 \geq 2\)). Hence, this case is impossible.

If \(a_1 + a_2 = 0\), (2) and (3) reduce to
\[
(b_1 + b_2) a_1 = 12
\]
\[
(b_1 + b_2) b_1 = 9.
\]
Since \(b_1 + b_2 \geq 2\), this makes \(a_1 > 0\), which is inconsistent with \(a_1 + a_2 = 0\).

If \(a_1 + a_2 = 1\), then \(a_2 \geq 1\) implies that \(a_1 \leq 0\). However, (2) and (3) become
\[
(b_1 + b_2) a_1 + b_1 = 12
\]
\[
a_1 + 2 (b_1 + b_2) b_1 = 18
\]
and hence,
\[
a_1 = 6 \frac{4(b_1 + b_2) - 3}{2(b_1 + b_2)^2 - 1} > 0
\]
(since \(b_1 + b_2 \geq 2\)). Therefore, this case is also impossible and we may assume in the remainder of this solution that \(a_1 + a_2 \geq 2\).

In (2) and (3), if we treat \(a_1\) and \(b_1\) as coefficients and use Cramer’s Rule, we obtain
\[
a_1 + a_2 = \frac{4b_1 - 3a_1}{2b_1^2 - a_1^2}, \quad b_1 + b_2 = \frac{3b_1 - 2a_1}{2b_1^2 - a_1^2}
\]
or
Therefore, our first solution is

\[
a_2 = \frac{4b_1 - 3a_1}{2b_1^2 - a_1^2} - a_1, \quad b_2 = \frac{3b_1 - 2a_1}{2b_1^2 - a_1^2} - b_1.
\]  

(4)

If \(a_1 = -3\), then

\[
a_2 = \frac{4b_1 + 9}{2b_1^2 - 9} + 3 = 3 \left( \frac{4b_1 + 9}{2b_1^2 - 9} + 1 \right)
\]

and

\[
b_2 = \frac{b_1 + 2}{2b_1^2 - 9} - b_1 = \frac{-2b_1^3 + 27b_1 + 36}{2b_1^2 - 9}.
\]

Using elementary calculus, it is straightforward to show that when \(b_1 \geq 5\), \(-2b_1^3 + 27b_1 + 36 < 0\) and \(2b_1^2 - 9 > 0\), and hence, \(b_2 < 0\). Also, by direct substitution, \(b_2 < 0\) when \(b_1 = 0, \pm 1, \) or 2. Therefore, we are left with \(b_1 = 3\) or 4. Of these, \(b_1 = 4\) yields fractional values for \(a_2\) and \(b_2\), while \(b_1 = 3\) gives the solution \(a_1 = -3, b_1 = 3, a_2 = 17, b_2 = 7\). Therefore, our first solution is \(w = -3 + 3\sqrt{2}, x = w + 2 \left(2 + \sqrt{2}\right) = 1 + 5\sqrt{2}, y = 17 + 7\sqrt{2}\).

If \(a_1 = -2\), (4) becomes

\[
a_2 = 2 \left( \frac{2b_1 + 3}{b_1^2 - 2} + 1 \right) \quad \text{and} \quad b_2 = \frac{3b_1 + 4}{2b_1^2 - 4} - b_1 = \frac{-b_1^3 + 11b_1 + 12}{b_1^2 - 2}.
\]

Proceeding as before, we see that \(b_2 < 0\) for \(b_1 \geq 4\) and \(a_2 < 0\) for \(b_1 = 0\) or \(\pm 1\). If \(b_1 = 3\), then \(a_2\) is a fraction. However, \(b_1 = 2\) yields the solution \(a_1 = -2, b_1 = 2, a_2 = 23, b_2 = 13\). Therefore, our next solution is \(w = -2 + 2\sqrt{2}, x = 2 + 4\sqrt{2}, y = 23 + 13\sqrt{2}\).

If \(a_1 = -1\), (4) reduces to

\[
a_2 = \frac{4b_1 + 3}{2b_1^2 - 1} + 1 \quad \text{and} \quad b_2 = \frac{3b_1 + 2}{2b_1^2 - 1} - b_1 = \frac{-2b_1^3 + 19b_1 + 12}{2b_1^2 - 1}.
\]

If \(b_1 \geq 4\), then \(b_2 < 0\). Also, if \(b_1 = -1\) or 0, \(a_2 < 0\). Of the remaining choices, \(b_1 = 2\) or 3 give fractional answers for \(a_2\). When \(b_1 = 1\), we get the solution \(a_1 = -1, b_1 = 1, a_2 = 43, b_2 = 29\). This contributes \(w = -1 + \sqrt{2}, x = 3 + 3\sqrt{2}, y = 43 + 29\sqrt{2}\) to our list of solutions.

If \(a_1 = 0\), (4) becomes \(a_2 = \frac{12}{b_1} \) and \(b_2 = \frac{9}{b_1} - b_1 = \frac{9 - b_1^2}{b_1} \). In this case, \(b_1 \neq 0\) and we get \(a_2 < 0\) when \(b_1 = -1\) and \(b_2 \leq 0\) when \(b_1 \geq 3\). Also, \(b_1 = 2\) yields a fractional value for \(b_2\). Hence, we are left with \(b_1 = 1\), which gives the solution \(a_1 = 0, b_1 = 1, a_2 = 12, b_2 = 8\) and we add \(w = \sqrt{2}, x = 4 + 3\sqrt{2}, y = 12 + 8\sqrt{2}\) to our solution set.

We can now assume that \(a_1 \geq 1\) in the remainder of this solution.

If \(b_1 = -1\), (4) is of the form

\[
a_2 = \frac{3a_1 + 4}{a_1^2 - 2} - a_1 = \frac{-a_1^3 + 20a_1 + 24}{a_1^2 - 2} \quad \text{and} \quad b_2 = \frac{2a_1 + 3}{a_1^2 - 2} + 1.
\]

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As before, we get $a_2 < 0$ if $a_1 \geq 5$ and $b_2 < 0$ if $a_1 = 1$. When $a_1 = 3$ or $4$, we get fractional values for $b_2$. Finally, $a_1 = 2$ gives the solution $a_1 = 2, b_1 = -1, a_2 = 28, b_2 = 22$, which yields $w = 2 - \sqrt{2}, x = 6 + \sqrt{2}, y = 28 + 22\sqrt{2}$.

If $b_1 = 0$, (4) becomes

$$a_2 = \frac{18}{a_1} - a_1 = \frac{18 - a_1^2}{a_1} \quad \text{and} \quad b_2 = \frac{12}{a_1}.$$ 

If $a_1 \geq 5$, we get $a_2 < 0$ and $a_1 = 4$ gives a fractional value for $a_2$. The remaining values $a_1 = 1, 2, 3$ produce the solutions listed below.

\[
\begin{array}{ccccccc}
a_1 & b_1 & a_2 & b_2 & w & x & y \\
1 & 0 & 17 & 12 & 1 & 5 + 2\sqrt{2} & 17 + 12\sqrt{2} \\
2 & 0 & 7 & 6 & 2 & 6 + 2\sqrt{2} & 7 + 6\sqrt{2} \\
3 & 0 & 3 & 4 & 3 & 7 + 2\sqrt{2} & 3 + 4\sqrt{2}
\end{array}
\]

Finally, we are down to the situation where $a_1 \geq 1, b_1 \geq 1, a_1 + a_2 \geq 2$, and $b_1 + b_2 \geq 2$. Then, (2) implies that $1 \leq a_1 \leq 6$ and $1 \leq b_1 \leq 6$. By trying the 36 possibilities this presents for the system consisting of (2) and (3), we find that the remaining solutions are as follows:

\[
\begin{array}{ccccccc}
a_1 & b_1 & a_2 & b_2 & w & x & y \\
1 & 1 & 5 & 1 & 5 + 3\sqrt{2} & 5 + 5\sqrt{2} \\
2 & 1 & 4 & 2 & 6 + 3\sqrt{2} & 4 + 2\sqrt{2} \\
2 & 2 & 1 & 1 & 2 + 2\sqrt{2} & 1 + \sqrt{2}
\end{array}
\]

Our conclusion is that the full solution set for this problem is displayed below. With some algebraic fortitude, it can be checked that all are solutions to the original equation.

\[
\begin{array}{ccccccc}
x & y \\
1 + 5\sqrt{2} & 17 + 7\sqrt{2} \\
2 + 4\sqrt{2} & 23 + 13\sqrt{2} \\
3 + 3\sqrt{2} & 43 + 29\sqrt{2} \\
4 + 3\sqrt{2} & 12 + 8\sqrt{2} \\
5 + 2\sqrt{2} & 17 + 12\sqrt{2} \\
5 + 3\sqrt{2} & 5 + 5\sqrt{2} \\
6 + \sqrt{2} & 28 + 22\sqrt{2} \\
6 + 2\sqrt{2} & 7 + 6\sqrt{2} \\
6 + 3\sqrt{2} & 4 + 2\sqrt{2} \\
6 + 4\sqrt{2} & 1 + \sqrt{2} \\
7 + 2\sqrt{2} & 3 + 4\sqrt{2}
\end{array}
\]

Comments: David Stone and John Hawkins of Statesboro GA noted that the solutions $(x, y)$ lie on the hyperbola

\[
y = \frac{x^2 - 4 \left(2 + \sqrt{2}\right) x + \left(2 + \sqrt{2}\right)^2}{x - 2 \left(2 + \sqrt{2}\right)}
\]
and it is not evident that there should be only finitely many solutions. However, imposing the specific form \( a + b\sqrt{2} \) on \( x \) and \( y \) forces this to be the case.

And Ken Korbin (the proposer of the problem) characterized the solutions as follows: Letting 

\[
(c, d) = (1, 2 + \sqrt{2}), (2 + \sqrt{2}, 1), (\sqrt{2}, 1 + \sqrt{2}), (1 + \sqrt{2}, \sqrt{2})
\]

then

\[
\begin{cases}
  x = c(2d + 1) \\
  y = c(3d^2 - 1)
\end{cases}
\]

and

\[
\begin{cases}
  x = c(2d + 3) \\
  y = c(d^2 - 3)
\end{cases}
\]

with \( x < y \).

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Titu Zvonaru and Comănești Romania, Neculai Stanciu, Buzău, Romania (jointly); David Stone and John Hawkins of Georgia Southern University, Statesboro, GA (jointly), and the proposer.

5213: Proposed by Tom Moore, Bridgewater, MA

The triangular numbers \( T_n \) begin 1, 3, 6, 10, \ldots and, in general, \( T_n = \frac{n(n+1)}{2}, n = 1, 2, 3, \ldots \).

For every positive integer \( n > 1 \), prove that \( n^4 \) is a sum of four triangular numbers.

Solution by Boris Rays, Brooklyn, NY

\[
\begin{align*}
  n^4 & = n^4 - n^2 + n^2 = 2\frac{n^4 - n^2}{2} + 2\frac{n^2}{2} \\
    & = 2\frac{n^2}{2} + 2\frac{n^4 - n^2}{2} \\
    & = \frac{n^2 - n + n^2 + n}{2} + 2\frac{n^2(n^2 - 1)}{2} \\
    & = \frac{(n - 1)n}{2} + \frac{n(n + 1)}{2} + \frac{(n^2 - 1)n^2}{2} + \frac{(n^2 - 1)n^2}{2} \\
    & = T_{n-1} + T_n + T_{n^2-1} + T_{n^2-1}.
\end{align*}
\]

Comments: Albert Stadler of Herrliberg, Switzerland. A.M. Legendre concluded from formulas in his treatise on elliptic functions [1] that the number of ways in which \( n \) is a sum of four triangular numbers equals the sum of the divisors of \( 2n + 1 \). As a result of this, every natural number can be represented as a sum of four triangular numbers. Reference: [1] Adrien Marie Legendre, Fonctions elliptiques et des intégrales Eulériennes: avec des tables pour en faciliter le calcul numérique; Vol 3 (1828), 133-134.

David Stone and John Hawkins of Statesboro, GA noted in their solution that \( n^4 \) is also the sum of two triangular numbers: \( n^4 = T_{n^2-1} + T_{n^2} \).
Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Elsie
Campbell, Dionne Bailey, and Charles Diminnie, Angelo State University, San
Angelo, TX; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York,
NY; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at
Oneonta, Oneonta, NY; Adrian Naco, Tirana, Albania; Ángel Plaza, University of
Las Palmas de Gran Canaria, Spain; Armend Sh. Shabani, (student, University of
Prishtina), Republic of Kosova; Howard Sporn, Great Neck, NY; Albert Stadler,
Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern
University, Statesboro, GA (jointly); Titu Zvonaru, Comănești, Romania and
Neculai Stanciu Buzău, Romania (jointly), and the proposer.

5214: Proposed by Pedro H. O. Pantoja, Natal-RN, Brazil

Let $a, b, c$ be positive real numbers. Prove that

$$\frac{a^3(b + c)^2}{1 + a + 2b} + \frac{b^3(c + a)^2}{1 + b + 2c} + \frac{c^3(a + b)^2}{1 + c + 2a} \geq \frac{4abc(ab + bc + ca) + 3}{a + b + c + 1}.$$ 

Solution 1 by David E. Manes, SUNY College at Oneonta, Oneonta, NY

Let $L = \frac{a^3(b + c)^2}{1 + a + 2b} + \frac{b^3(c + a)^2}{1 + b + 2c} + \frac{c^3(a + b)^2}{1 + c + 2a}$. Note that by the AM − GM inequality, $(b + c)^2 \geq 4bc$, $(c + a)^2 \geq 4ca$, and $(a + b)^2 \geq 4ab$ with equality if and only if $a = b = c$. Therefore,

$$L \geq \frac{4a^3bc + 1}{1 + a + 2b} + \frac{4b^3ca + 1}{1 + b + 2c} + \frac{4c^3ab + 1}{1 + c + 2a} = 4abc \left( \frac{a^2}{1 + a + 2b} + \frac{b^2}{1 + b + 2c} + \frac{c^2}{1 + c + 2a} \right) + \left( \frac{1^2}{1 + a + 2b} + \frac{1^2}{1 + b + 2c} + \frac{1^2}{1 + c + 2a} \right).$$

The Cauchy-Schwarz inequality implies

$$\frac{a}{\sqrt{1 + a + 2b}} \cdot \sqrt{1 + a + 2b} + \frac{b}{\sqrt{1 + b + 2c}} \cdot \sqrt{1 + b + 2c} + \frac{c}{\sqrt{1 + c + 2a}} \cdot \sqrt{1 + c + 2a} \leq \left( \frac{a^2}{1 + a + 2b} + \frac{b^2}{1 + b + 2c} + \frac{c^2}{1 + c + 2a} \right)^{1/2} \leq 3(a + 3b + 3c + 3);$$

hence,

$$\frac{a^2}{1 + a + 2b} + \frac{b^2}{1 + b + 2c} + \frac{c^2}{1 + c + 2a} \geq \frac{(a + b + c)^2}{3(a + b + c + 1)}.$$  

Similarly,

$$\frac{1^2}{1 + a + 2b} + \frac{1^2}{1 + b + 2c} + \frac{1^2}{1 + c + 2a} \geq \frac{(1 + 1 + 1)^2}{3(a + b + c + 1)} = \frac{3}{a + b + c + 1}. $$

Therefore,

$$L \geq 4abc \left( \frac{(a + b + c)^2}{3(a + b + c + 1)} \right) + \frac{3}{a + b + c + 1}.$$

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Furthermore, the Cauchy-Schwarz inequality also implies \( a^2 + b^2 + c^2 \geq ab + bc + ca \) using vectors \( \langle a, b, c \rangle \) and \( \langle b, c, a \rangle \). Therefore,

\[
(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \\
\geq ab + bc + ca + 2(ab + bc + ca) \\
= 3(ab + bc + ca).
\]

Hence,

\[
\frac{(a + b + c)^2}{3(a + b + c + 1)} \geq \frac{ab + bc + ca}{a + b + c + 1}.
\]

Accordingly,

\[
L \geq 4abc \left( \frac{(a + b + c)^2}{3(a + b + c + 1)} \right) + \frac{3}{a + b + c + 1} \\
\geq \frac{4abc(ab + bc + ca)}{a + b + c + 1} + \frac{3}{a + b + c + 1} \\
= \frac{4abc(ab + bc + ca) + 3}{a + b + c + 1},
\]

with equality if and only if \( a = b = c \).

**Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy**

We prove that

\[
\frac{1}{1 + a + 2b} + \frac{1}{1 + b + 2c} + \frac{1}{1 + c + 2a} \geq \frac{3}{a + b + c + 1}
\]

and

\[
\frac{a^3(b + c)^2}{1 + a + 2b} + \frac{b^3(c + a)^2}{1 + b + 2c} + \frac{c^3(a + b)^2}{1 + c + 2a} \geq \frac{4abc(ab + bc + ca)}{a + b + c + 1}
\]

By Cauchy-Schwarz

\[
\frac{1}{1 + a + 2b} + \frac{1}{1 + b + 2c} + \frac{1}{1 + c + 2a} \geq \frac{(1 + 1 + 1)^2}{3 + 3(a + b + c)}
\]

thus we prove

\[
\frac{(1 + 1 + 1)^2}{3 + 3(a + b + c)} \geq \frac{3}{a + b + c + 1}
\]

which is actually an equality. As for the second inequality we have
\[
\sum_{\text{cyc}} \frac{a^3(b + c)^2}{1 + a + 2b} \geq \sum_{\text{cyc}} \frac{a^3bc}{1 + a + 2b} \geq \frac{4abc(ab + bc + ca)}{a + b + c + 1}
\]

or
\[
\sum_{\text{cyc}} \frac{a^2}{1 + a + 2b} \geq \frac{ab + bc + ca}{a + b + c + 1}
\]

Cauchy–Schwarz again yields
\[
\sum_{\text{cyc}} \frac{a^2}{1 + a + 2b} \geq \frac{(a + b + c)^2}{3 + 3(a + b + c)} \geq \frac{ab + bc + ca}{a + b + c + 1}
\]

or
\[
S^3 + S^2 \geq 3P + 3PS, \quad S = a + b + c, \quad P = ab + bc + ca
\]

Now \(S^2 \geq 3P\) since it is equivalent to
\[
a^2 + b^2 + c^2 \geq ab + bc + ca
\]

which is a well known inequality.

**Solution 3 by Adrian Naco, Polytechnic University, Tirana, Albania.**

*Editor’s comment: The following is a generalization of the stated problem.*

Based on Cauchy-Schwarz inequality, for \(a_i, b_i \in R^{+}\) we have that
\[
\left(\sum_{i=1}^{n} a_i\right)^2 \leq \left(\sum_{i=1}^{n} \left(\frac{a_i}{\sqrt{b_i}}\right)\right)^2 \leq \left[\sum_{i=1}^{n} \left(\frac{a_i}{\sqrt{b_i}}\right)^2\right]\left[\sum_{i=1}^{n} \frac{\sqrt{b_i}}{}\right]^2
\]

\[= \left(\sum_{i=1}^{n} \frac{a_i^2}{b_i}\right)\left(\sum_{i=1}^{n} b_i\right)
\]

\[\Rightarrow \sum_{i=1}^{n} \frac{a_i^2}{b_i} \geq \frac{\left(\sum_{i=1}^{n} a_i\right)^2}{\sum_{i=1}^{n} b_i} \quad (1)
\]

where the equality holds for \(\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n}\).

Let us split the original inequality in two separate inequalities (2) and (3) as follows
\[ \sum_{i=1}^{n} \frac{1}{1 + x_i + 2x_{i+1}} \geq \frac{n^2}{n + 3 \sum_{i=1}^{n} x_i} \]  \hspace{1cm} (2)

and

\[ \sum_{i=1}^{n} \frac{x_i^3(x_{i+1} + x_{i+2})^2}{1 + x_i + 2x_{i+1}} \geq \frac{4 \left[ \sum_{i=1}^{n} (x_ix_{i+1}x_{i+2})x_i^2 + 2 \sum_{1 \leq i < j \leq n} (x_ix_jx_{i+1}x_{j+1}x_{i+2}x_{j+2}) \frac{1}{2}x_ix_j \right]}{n + 3 \sum_{i=1}^{n} x_i} \]  \hspace{1cm} (3)

Applying the above Cauchy-Schwarz inequality for each of the inequalities (2) and (3) we have that

\[ \sum_{i=1}^{n} \frac{1}{1 + x_i + 2x_{i+1}} \geq \frac{\left( \sum_{i=1}^{n} 1 \right)^2}{\sum_{i=1}^{n} (1 + x_i + 2x_{i+1})} = \frac{(1 + 1 + 1)^2}{\sum_{i=1}^{n} 1 + \sum_{i=1}^{n} x_i + 2 \sum_{i=1}^{n} x_{i+1}} = \frac{n^2}{n + 3 \sum_{i=1}^{n} x_i} \]  \hspace{1cm} (4)

where the equality holds for \( x_1 = x_2 = \ldots = x_n \). Thus we prove (2). Analogously we have that

\[ \sum_{i=1}^{n} \frac{x_i^3(x_{i+1} + x_{i+2})}{1 + x_i + 2x_{i+1}} = \sum_{i=1}^{n} \left[ x_i^2 \left( x_{i+1} + x_{i+2} \right) \frac{1}{1 + x_i + 2x_{i+1}} \right]^2 \geq \frac{\left[ \sum_{i=1}^{n} x_i^2 \left( x_{i+1} + x_{i+2} \right) \frac{1}{1 + x_i + 2x_{i+1}} \right]^2}{\sum_{i=1}^{n} (1 + x_i + 2x_{i+1})} \]

\[ \quad \geq \frac{4 \left[ \sum_{i=1}^{n} x_i \left( x_ix_{i+1}x_{i+2} \right)^{\frac{1}{2}} \right]^2}{n + \sum_{i=1}^{n} x_i + 2 \sum_{i=1}^{n} x_{i+1}} \]

\[ \quad = \frac{4 \left[ \sum_{i=1}^{n} x_i \left( x_ix_{i+1}x_{i+2} \right)^{\frac{1}{2}} + 2 \sum_{1 \leq i < j \leq n} x_ix_j \left( x_ix_jx_{i+1}x_{j+1}x_{i+2}x_{j+2} \right)^\frac{1}{2} \right]}{n + 3 \sum_{i=1}^{n} x_i} \]
\[
S = \sum_{i=1}^{n} x_i^2(x_{i+1}x_{i+2}) - \sum_{1 \leq i < j \leq n} x_i x_j (x_i x_{i+1} x_{j+1} x_{i+2} x_{j+2})^{1/2}
\]

where

\[
S = \sum_{i=1}^{n} x_i^2(x_{i+1}x_{i+2}) - \sum_{1 \leq i < j \leq n} x_i x_j (x_i x_{i+1} x_{j+1} x_{i+2} x_{j+2})^{1/2}
\]

The problem proposed is a special case of the above generalized problem for

\[
x_1 = a \quad x_2 = b \quad x_3 = c.
\]

Thus, we have that

\[
S = a b c a^2 + b c a b^2 + c a b c^2 - \left( (abccba)^{1/2} ab + (acbacb)^{1/2} ac + (bcbaab)^{1/2} bc \right)
\]

\[
= abc(a^2 + b^2 + c^2 - ab - ac - bc) = abc \frac{1}{2} \left[ (a - b)^2 + (b - c)^2 + (c - a)^2 \right] \geq 0
\]

with the equality holding for \(a = b = c\). The inequality proposed gets the simplified form

\[
\sum_{cyc} \frac{a^3(b + c)^2 + 1}{1 + a + 2b} \geq \frac{4 \left\{ S + 3 (abccca)^{1/2} ab + (acbacb)^{1/2} ac + (bcbaab)^{1/2} bc \right\} + 3^2}{3 + 3(a + b + c)}
\]

\[
= \frac{4 \left\{ S + 3abc (ab + ac + bc) \right\} + 3^2}{3 + 3(a + b + c)} \geq \frac{4 \left[ 0 + 3abc (ab + ac + bc) \right] + 3^2}{3 + 3(a + b + c)}
\]

\[
= \frac{4abc (ab + ac + bc) + 3}{1 + a + b + c}
\]

Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo TX; Andrea Fanchini, Cantù, Italy; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania and Neculai Stanciu Buzău, Romania (jointly); and the proposer.
Evaluate the integral
\[ \int_{-1}^{1} \frac{2x^{1004} + x^{3014} + x^{2008} \sin x^{2007}}{1 + x^{2010}} \, dx. \]

Solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

\[ \int_{-1}^{1} \frac{2x^{1004} + x^{3014} + x^{2008} \sin x^{2007}}{1 + x^{2010}} \, dx = \int_{-1}^{1} \frac{2x^{1004} + x^{3014}}{1 + x^{2010}} \, dx + \int_{-1}^{1} \frac{x^{2008} \sin x^{2007}}{1 + x^{2010}} \, dx \]

Note that \[ \int_{-1}^{1} \frac{x^{2008} \sin x^{2007}}{1 + x^{2010}} \, dx = 0, \]
since the integrand function is odd and the interval of integration is centered at the origin.

The remaining integral may be solved using the change of variable \( x^{1005} = t. \)

\[ \int_{-1}^{1} \frac{2x^{1004} + x^{3014}}{1 + x^{2010}} \, dx = \frac{1}{1005} \int_{-1}^{1} \frac{2 + t^2}{1 + t^2} \, dt \]

\[ = \frac{1}{1005} \left( 2 + \int_{-1}^{1} \frac{1}{1 + t^2} \, dt \right) \]

\[ = \frac{1}{1005} \left( 2 + \arctan(1) - \arctan(-1) \right) \]

\[ = \frac{1}{1005} \left( 2 + \frac{\pi}{2} \right) = \frac{4 + \pi}{2010}. \]

Also solved by Daniel Lopez Aguayo, Institute of Mathematics, UNAM, Morelia, Mexico; Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo State University, San Angelo, TX; Michael C. Faleski, University Center, MI; Paul M. Harms, North Newton, KS; Fotini Kotroni and Anastasios Kotronis, Athens, Greece; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Adrian Naco, Department of Mathematics, Polytechnic University of Tirana, Albania; Paolo Perfetti, Department of Mathematics, “Tor Vergata” University, Rome, Italy; Boris Rays, Brooklyn, NY; Armend Sh. Shabani, (student, University of Prishtina), Republic of Kosova; Albert Stadler, Herrliberg, Switzerland; Howard Sporn, Great Neck, NY, and the proposer.
Let \( f : \mathbb{R} \to \mathbb{R}^+ \) be a function such that for all \( a, b \in \mathbb{R} \)
\[ f(ab) = f(a)^b f(b)^a \]
and \( f(3) = 64 \). Find all real solutions to the equation
\[ f(x) + f(x + 1) - 3x - 2 = 0. \]

Solution by Armend Sh. Shabani, (Graduate Student) University of Prishtina, Republic of Kosova.

Since \( f(ab) = f(ba) \) we have
\[ f(a)^b \cdot f(b)^a = f(b)^a \cdot f(a)^b \iff f(a)^b \cdot f(b)^a \left[ f(b)^{a^2-a} - f(a)^{b^2-b} \right] = 0 \]
\[ \iff f(b)^{a^2-a} - f(a)^{b^2-b} = 0 \iff f(b)^{a^2-a} = f(a)^{b^2-b}. \]

Taking \( b = x; a = 3 \), one obtains:
\[ f(x)^{3^2-3} = f(3)^{x^2-x} \]
\[ f(x)^6 = (64)^{x^2-x} \]
\[ f(x)^6 = (4^3)^{x^2-x} \Rightarrow f(x) = 2^{x^2-x} = 2^{x(x-1)}. \]

Substituting into \( f(x) + f(x + 1) - 3x - 2 = 0 \) we obtain:
\[ 2^{x^2-x} + 2^{x^2+x} - (3x + 2) = 0. \]  
(1)

Clearly \( x = 0; x = 1 \) are solutions of equation (1).

We show that there are no other solutions.

Let \( g(x) = 2^{x^2-x} + 2^{x^2+x} \). One easily finds that
\[ g'(x) = \ln 2 \cdot \left( 2x + 1 \right) \cdot 2^{x^2+x} + \left( 2x - 1 \right) \cdot 2^{x^2-x} \]
and
\[ g''(x) = \ln 2 \cdot \left( 2 \cdot 2^{x^2+x+1} + 2^{x^2-x+1} \right) + (\ln 2)^2 \cdot \left( (2x + 1)^2 \cdot 2^{x^2+x} + (2x - 1)^2 \cdot 2^{x^2-x} \right). \]

So \( g''(x) > 0 \), and this means that \( g \) is a convex function. So the line \( h(x) = 3x + 2 \) can meet function \( g \) in at most 2 points. Therefore equation (1) has no other solutions. (Note that this can also be seen by drawing the graphs of functions \( g \) and \( h \).)

Also solved by Dionne Bailey, Elsie Campbell, and Charles Dominnie, Angelo State University, San Angelo, TX; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Adrian Naco, Department of Mathematics, Polytechnic University of Tirana, Albania; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University Statesboro, GA (jointly), and the proposer.
Proposed by Ovidiu Furdui, Cluj-Napoca, Romania

Find the value of:

$$\lim_{n \to \infty} \int_0^1 \int_0^1 \sqrt[n]{(x^n + y^n)^k} \, dx \, dy,$$

where $k$ is a positive real number.

Solution 1 by Anastasios Kotronis, Athens, Greece

It is easily shown that $\sqrt[n]{(x^n + y^n)^k} \to \begin{cases} x^k, & y \leq x \\ y^k, & x < y \end{cases}$ and since $0 \leq \sqrt[n]{(x^n + y^n)^k} \leq 2^k$, by the dominated convergence theorem we have

$$\lim_{n \to +\infty} \int_0^1 \int_0^1 \sqrt[n]{(x^n + y^n)^k} \, dy \, dx = \int_0^1 \int_0^1 \lim_{n \to +\infty} \sqrt[n]{(x^n + y^n)^k} \, dy \, dx$$

$$= \int_0^1 \int_0^x x^k \, dy \, dx + \int_0^1 \int_x^1 y^k \, dy \, dx$$

$$= \int_0^1 x^{k+1} \, dx + \int_0^1 \frac{1 - x^{k+1}}{k+1} \, dx$$

$$= \frac{2}{k+2}.$$

Solution 2 by Howard Sporn, Great Neck, NY

We use the fact that for $n$ going to $\infty$, when $x < y$, the $y^n$ term dominates over $x^n$, and when $x > y$, the $x^n$ term dominates over $y^n$.

We break up the inner integral into two integrals, like so:

$$\int_0^1 \sqrt[n]{(x^n + y^n)^k} \, dx = \int_0^y \sqrt[n]{(x^n + y^n)^k} \, dx + \int_y^1 \sqrt[n]{(x^n + y^n)^k} \, dx.$$

Note that for the first integral $x \leq y$, and for the second integral $x \geq y$. By factoring,

$$\int_0^1 \sqrt[n]{(x^n + y^n)^k} \, dx = \int_0^y \sqrt[n]{y^n \left( \left( \frac{x}{y} \right)^n + 1 \right)} \, dx + \int_y^1 \sqrt[n]{x^n \left( 1 + \left( \frac{y}{x} \right)^n \right)} \, dx.$$

For the first integral, in which $x$, we first consider the case $x < y$. In that case, $\left( \frac{x}{y} \right)^n \to 0$ for $n \to \infty$. Then the integrand becomes $\sqrt[n]{y^n (0 + 1)}^k = y^k$.
If, on the other hand, \( x = y \), then the integrand

\[
\sqrt[\sqrt{n}]{\left[ y^n \left( \frac{y}{y} \right)^n + 1 \right]}^k = \sqrt[\sqrt{n}]{\left[ y^n (1 + 1) \right]^k} = y^k \sqrt[2k]{2},
\]

which approaches \( y^k \) (once again) as \( n \to \infty \).

Similarly, the integrand in the second integral approaches \( x^k \).

The quantity we are seeking is now

\[
\int_0^1 \left( \int_0^y y^k \, dx + \int_y^1 x^k \, dx \right) \, dy
\]

which is straight-forward to compute.

The solution is

\[
\int_0^1 \left( \left. \left( y^k x \right) \right|_y^{x=0} + \frac{x^{k+1}}{k+1} \right|_y^{x=y} \right) \, dy
\]

\[
= \int_0^1 \left( y^{k+1} + \frac{1}{k+1} - \frac{y^{k+1}}{k+1} \right) \, dy
\]

\[
= \left. \left( \frac{y^{k+2}}{k+2} + \frac{y}{k+1} - \frac{y^{k+2}}{(k+1)(k+2)} \right) \right|_0^1
\]

\[
= \frac{1}{k+2} + \frac{1}{k+1} - \frac{1}{(k+1)(k+2)}
\]

\[
= \frac{(k+1) + (k+2) - 1}{(k+1)(k+2)}
\]

\[
= \frac{2k + 2}{(k+1)(k+2)}
\]

\[
= \frac{2}{k+2}.
\]

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China; Adrian Naco, Department of Mathematics, Polytechnic University of Tirana, Albania; Paolo Perfetti, Department of Mathematics, “Tor Vergata” University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland, and the proposer.