# Problems Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

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*Solutions to the problems stated in this issue should be posted before*
*February 15, 2016*

- **5373:** *Proposed by Kenneth Korbin, New York, NY*

  Given the equation $\dfrac{2\sqrt{2}}{\sqrt{343-147\sqrt{5}}-\sqrt{315-135\sqrt{5}}}=\sqrt{x+y\sqrt{5}}.$

  Find positive integers $x$ and $y$.

- **5374:** *Proposed by Roger Izard, Dallas TX*

  In a certain triangle, three circles are tangent to the incircle, and all of these circles are tangent to two sides of the triangle. Derive a formula which gives the radius of the incircle in terms of the radii of these three circles.

- **5375*:** *Proposed by Kenneth Korbin, New York, NY*

  Prove or disprove the following conjecture. Let $k$ be the product of $N$ different prime numbers each congruent to $1(mod\ 4)$. Let $a$ be any positive integer.

  Conjecture: The total number of different rectangles and trapezoids with integer length sides that can be inscribed in a circle with diameter $k$ is exactly $\dfrac{5^N-3^N}{2}$.

  *Editor′s comment*: The number for this problem carries with it an astrick. The astrick signifies that neither the proposer nor the editor are aware of a proof of this conjecture.

- **5376:** *Proposed by Arkady Alt , San Jose ,CA*

  Let $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$ be positive real numbers such that
  $b_1 < a_1 < b_2 < a_2 < ... < a_{n-1} < b_n < a_n$.
  Let
  $$F(x)=\frac{(x-b_1)(x-b_2)...(x-b_n)}{(x-a_1)(x-a_2)...(x-a_n)}.$$
  Prove that $F'(x) < 0$ for any $x \in Dom(F)$.

- **5377:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Show that if $A, B, C$ are the measures of the angles of any triangle $ABC$ and $a, b, c$ the measures of the length of its sides, then holds

$$\prod_{cyclic} \sin^{1/3}(|A-B|) \le \sum_{cyclic} \frac{a^2+b^2}{3ab} \sin(|A-B|).$$

- **5378:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $k \geq 1$ be an integer. Calculate

$$\int_0^\infty \ln^k\left(\frac{e^x+1}{e^x-1}\right) dx.$$

*Solutions*

- **5355:** *Proposed by Kenneth Korbin, New York, NY*

  Find the area of the convex cyclic pentagon with sides

$$(13,\ 13,\ 12\sqrt{3}+5,\ 20\sqrt{3},\ 12\sqrt{3}-5)$$

**Solution by Kee-Wai Lau, Hong Kong, China**

We show that the area of the pentagon equals $370\sqrt{3}$.

Let the pentagon be $ABCDE$ with
$\overline{AB} = \overline{BC} = 13,\ \overline{CD} = 12\sqrt{3}+5,\ \overline{DE} = 20\sqrt{3},\ \overline{EA} = 12\sqrt{3}-5$. Denote the center and radius of the circumcircle by $O$ and $R$ respectively.

We first consider the case when $O$ lies inside the pentagon. We have
$\angle AOB + \angle BOC + \angle COD + \angle DOE + \angle EOA = 2\pi$, so that

$$2\sin^{-1}\left(\frac{13}{2R}\right) + \sin^{-1}\left(\frac{12\sqrt{3}+5}{2R}\right) + \sin^{-1}\left(\frac{10\sqrt{3}}{R}\right) + \sin^{-1}\left(\frac{12\sqrt{3}-5}{2R}\right) = \pi. \qquad (1)$$

The left side of (1) is a decreasing function of $R$ for $R \geq 10\sqrt{3}$, so (1) has at most one real valued solution.

Using the addition formula for the inverse sine function, we have

$$2\sin^{-1}\left(\frac{13}{37}\right) = \sin^{-1}\left(\frac{520\sqrt{3}}{1369}\right),$$

$$\sin^{-1}\left(\frac{12\sqrt{3}+5}{37}\right) + \sin^{-1}\left(\frac{12\sqrt{3}-5}{37}\right) = \sin^{-1}\left(\frac{20\sqrt{3}}{37}\right), \text{ and that}$$

$$2\sin^{-1}\left(\frac{20\sqrt{3}}{37}\right) \quad = \quad \pi - \sin^{-1}\left(\frac{520\sqrt{3}}{1369}\right).$$

It follows that the unique solution to (1) is $R = \dfrac{37}{2}$.

Using Heron's formula, we obtain the area of the triangles $OAB,\ OBC,\ OCD,\ ODE,\ OEA$ as $65\sqrt{3},\ 65\sqrt{3},\ \dfrac{175\sqrt{3}+39}{2},\ 65\sqrt{3},\ \dfrac{175\sqrt{3}-39}{2}$, and so the area of the pentagon equals $370\sqrt{3}$.

We next consider the case when $O$ lies on or outside the pentagon.

In this case $\angle EOA + \angle AOB + \angle BOC + \angle COD = \angle DOE$, so that

$$\sin^{-1}\left(\frac{12\sqrt{3}-5}{2R}\right) + 2\sin^{-1}\left(\frac{13}{2R}\right) + \sin^{-1}\left(\frac{12\sqrt{3}+5}{2R}\right) - \sin^{-1}\left(\frac{10\sqrt{3}}{R}\right) = 0. \qquad (2)$$

For $R \geq 20$, let $f(R) = 4\sin^{-1}\left(\dfrac{13}{2R}\right) - \sin^{-1}\left(\dfrac{10\sqrt{3}}{R}\right)$. Since $f(20) > 0$, $\lim_{R\to\infty} f(R) = 0$ and that $f$ attains the maximum value of $0.29\cdots$ at $R = \dfrac{195\sqrt{470}}{188}$, so in fact $f(R) > 0$.

Thus the left side of (2) is always positive and so (2) has no solutions.

This completes the solution.

*Comments by Editor*

**1.** A sticky point with this problem was in showing that the center of the circle had to lie in the interior of the pentagon. **David Stone and John Hawkins of Georgia Southern University** argued it like this: Assume that the points $E, A, B, C$, and $D$ are arranged on the circumference of the circumscribing circle such that
$\overline{EA} = 12\sqrt{3}-5, \overline{AB} = 13, \overline{BC} = 13, \overline{CD} = 12\sqrt{3}+5,\ \overline{DE} = 20\sqrt{3}$. And suppose that the center of the circumscribing circle lies in the exterior of the pentagon. If it lies on the longest side of the pentagon, then it lies on $\overline{DE}$ and this would make $\overline{DE}$ a diameter of the circumscribing circle, so the radius $R$ of the circumscribing circle must be $10\sqrt{3}$ and the length of $arcDE = 1/2$ the circumference of the circle. I.e., $arcDE = \pi(10\sqrt{3})$.

The length of each arc of the circle is greater than the length of its corresponding chord. So,

$$\begin{aligned}
arc\ DE &= arc\ DC + arc\ CB + arc\ BA + arc\ AE,\\
arc\ DE &> \overline{DC} + \overline{CB} + \overline{BA} + \overline{AE},\\
\pi(10\sqrt{3}) &> \left(12\sqrt{3}+5\right) + 13 + 13 + \left(12\sqrt{3}-5\right),\\
\pi(10\sqrt{3}) &> 24\sqrt{3}+26;\ \ \pi(10\sqrt{3}) \approx 54.4,\ \text{and}\ 24\sqrt{3}+26 \approx 67.57;\ \text{So},\\
54.4 &> 67.57\ ?\ \text{No}.
\end{aligned}$$

Therefore the center of the circumscribing circle cannot lie on the longest side of the pentagon.

But as the center of the circle moves into the exterior of the pentagon, the radius of the circumcircle increases and $arcDE$ decreases. I.e., $arcDE \leq 10\pi\sqrt{3}$. So again we have

$$54.4 > arcDE = arcDE + arcDE + arcDE + arcDE = 67.57$$

Hence, the center of the circumscribing circle must be in interior of the pentagon.

**2. Bruno Salgueiro Fanego of Viveiro, Spain** mentioned in his solution that he was applying an algebraic approach that was developed in a paper by David P. Robbins (see: *Areas of Polygons Inscribed in a Circle,* **The American Mathematical Monthly,** 102(6)(June-July, 1995)). The background to this approach is that the area of a convex polygon with more than three sides is not uniquely determined by the length of its sides. But adding the restriction that the polygon must also be cyclic, circumvents this problem and allows us to extend Heron's formula for finding the area $K$ of a triangle (with side lengths $a, b, c$ and semiperimeter $s$ to Brahmagupta's formula for finding the area of a quadrilateral, $K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$.

Robbins' paper presents formulas for finding the areas of the cyclic pentagon and cyclic hexagon. He wrote: "We shall see that the calculations leading to the discovery of the pentagon formula are so complex that it would have been quite difficult to carry them out without the aid of a computer. In fact after some study of the problem I thought it likely that, even if I were to discover the formula, its complexity would make it of little interest to write down. However, it is possible to write the formulas for the areas of the cyclic pentagon and the cyclic hexagon in a compact form which is related to the formula of the discriminant of a cubic polynomial in one variable."

Using Robbins' method the formula for finding the area K of a triangle with sides $a, b$, and $c$ is

$$16K^2 = 2a^{2b}2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4,$$

while the formula for finding the area $K$ of a cyclic quadrilateral with sides $a, b, c$, and $d$ is

$$16K^2 = 2a^2b^2 + \cdots + 2c^2d^2 - a^4 - b^4 - c^4 - d^4 + 8abcd.$$

The formulas for finding the areas of cyclic pentagons and hexagons are spelled out in Robbins' paper, and although they are formidable, his method works.

**Also solved by Bruno Salgueiro Fanego of Viveiro, Spain; Ed Gray, Highland Beach, FL; Toshihiro Shimizu, Kawasaki, Japan; David Stone and John Hawkins, Southern Georgia University, and the proposer.**

- **5356:** *Proposed by Kenneth Korbin, New York, NY*

For every prime number $p$ there is a circle with diameter $4p^4 + 1$. In each of these circles, it is possible to inscribe a triangle with integer length sides and with area

$$(8p^3)(p+1)(p-1)(2p^2-1).$$

Find the sides of the triangles if $p = 2$ and if $p = 3$.

**Solution by Brian D. Beasley, Presbyterian College, Clinton, SC**

We designate the side lengths of the triangle by $a$, $b$, and $c$. We also let $A$ be the area of the triangle and $r$ be the radius of the circle that circumscribes it. Then the formula for the circumradius and Heron's formula yield

$$abc = 4Ar = 16p^3(p+1)(p-1)(2p^2-1)(4p^4+1)$$

and

$$(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = 16A^2 = 1024p^6(p+1)^2(p-1)^2(2p^2-1)^2.$$

Inspired by the factorization $4p^4+1 = (2p^2+2p+1)(2p^2-2p+1)$, we let $a = 4p(2p^2-1)$, $b = 2p(p-1)(2p^2+2p+1)$, and $c = 2p(p+1)(2p^2-2p+1)$. Then

$$abc = 16p^3(p+1)(p-1)(2p^2-1)(4p^4+1)$$

as needed, so to complete the argument, it suffices to verify the second formula above. Letting $P = (a+b+c)(-a+b+c)(a-b+c)(a+b-c)$, we calculate

$$\begin{aligned} P &= (2p)^4(4p^3+4p^2-2p-2)(4p^3-4p^2-2p+2)(4p^2)(4p^2-4) \\ &= 1024p^6(p+1)(2p^2-1)(p-1)(2p^2-1)(p+1)(p-1) \\ &= 1024p^6(p+1)^2(p-1)^2(2p^2-1)^2. \end{aligned}$$

Hence the result holds for any integer $p > 1$. In particular, when $p = 2$, the triangle side lengths are 56, 52, and 60; when $p = 3$, the triangle side lengths are 204, 300, and 312.

*Addendum:* The sides of every Heronian triangle have the form $d(m+n)(mn-k-^2)$, $dm(n^2+k^2)$, and $dn(m^2+k^2)$, where $m, n$ and $k$ are positive integers with $\gcd(m,n,k) = 1$ and where $d$ is a proportionality factor; see [1] for more details. Given any integer $p > 1$, we may take $m = p^2, n = p^2-1, k = p(p-1)$, and $d = \dfrac{2}{k}$ to produce the values of $a, b$, and $c$ given above.

[1] https://en.wikipedia.org/wiki/Heronian-triangle

**Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego (two solutions), Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; David E. Manes, SUNY College at Oneonta, NY; Toshihiro Shimizu, Kawasaki Japan; David Stone and John Hawkins, Southern Georgia University; Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, Bazău, Romania, and the proposer.**

- **5357:** *Proposed by Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania*

Determine all triangles whose side-lengths are positive integers (of which at least one is a prime number), whose semiperimeter is a positive integer, and whose area is equal to its perimeter.

**Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain**

Let $a, b, c$ be the positive integer side-lengths of a triangle, $p = \dfrac{a+b+c}{2}$ its semiperimeter and let us suppose that the area of that triangle, given by Heron's formula $\sqrt{p(p-a)(p-b)(p-c)}$ is equal to its perimeter $2p$.

Let $x = p-b,\ y = p-c,\ z = p-a$; then $xyz = (p-a)(p-b)(p-c) = 4p = 4(x+y+z)$ so $x = \dfrac{4(x+y)}{xy-4}$. By the triangle inequalities, $x, y, z$ are positive integers so $xy = 4$ must be a positive integer as well. Without loss of generality, suppose that $a \leq b \leq c$; since $a = x+y, b = y+z,\ c = z+x$, and this is equivalent to $y \leq x \leq z$, so

$x+y \leq 2x \leq 2z = \dfrac{8(x+y)}{xy-4}$, from where $xy-4 \leq 8$; hence $y \leq \dfrac{12}{x} \leq \dfrac{12}{y}$ which implies $y \leq 3$, that is $y \in \{1,2,3\}$.

If $y=1$, then $x \leq \frac{1}{2}y = 12$ or equivalently $x \in \{1,2,3,4,5,6,7,8,9,10,11,12\}$. If $y=2$, then $2 = y \leq 12y = 6$, or what is the same $x \in \{2,3,4,5,6\}$ and if $y=3$, then $3 = y \leq x \leq 12y = 4$, or equivalently, $x \in \{3,4\}$.

From these possibilities the only ones that give positive integers for $z = \dfrac{4(x+y)}{xy-4}$ are $(x,y) = \{(5,1),\ (6,1),\ (8,1),\ (9,1),\ (3,2),\ (4,2),\ (6,2)\}$, which give

$$(a,b,c) = (x{+}y,\ y{+}z,\ z{+}x\} \in \{6,25,29),\ (7,15,20),\ (9,10,17),\ (10,9,17),\ (5,12,13),\ (6,8,10),\ (8,6,10)\}.$$

Thus, the triples of positive integer side-lengths of triangles whose area is equal to its perimeter are $(6,25,29), (7,15,20), (9,10,17), (5,12,13), (6,8,10)$ and since at least one of $a,b,c$ is a prime number, we exclude the triple $(6,8,10)$ and since in all the other four cases the semiperimeter $p = \dfrac{a+b+c}{2}$ is a positive integer, the triangles we are looking for are those whose side lengths are $(6,25,29), (7,15,20), (9,10,17)$, or $(5,12,13)$. (Note also that only the last of them corresponds to a right triangle.)

**Solution 2 and Comment by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA**

The $A = P$ problem has a long history. In [2], Markov tells us that Dickson [1] attributes the solution to Whitworth and Biddle in 1904, then lists the only triangles with Area = Perimeter:

$$\begin{array}{c}(6,8,10)\\(5,12,13)\\(6,25,29)\\(7,15,20)\\(9,10,17).\end{array}$$

Because our problem requires that one side be a prime, we see that the only solutions to the stated problem are the last four triangles above (note that each has an integral semiperimeter).

(The above result can probably now be considered as "common knowledge": it even appeared recently online on answers Yahoo.com [3]).


1. L. Dickson, History of the Theory of Numbers, Vol II, Dover Publications, Inc, New York, 2005 (reprint from the 1923 edition), p. 199.

2. L. P. Markov, Pythagorean Triples and the Problem A = mP for Triangles, Mathematics Magazine 79(2006) 114−121

3. From Dan, answers.Yahoo.com/question/index?qid=2081130185149AAua2RD, 7 years ago.


**Also solved by Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo State University San Angelo TX; Jerry Chu (student, Saint George's School), Spokane, WA; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton,**

**KS; David E. Manes SUNY College at Oneonta, Oneonta, NY; Ken Korbin, NewYork, NY; Kee-Wai Lau, Hong Kong, China; Toshihiro Shimizu, Kawasaki, Japan; Albert Stadler, Herrliberg, Switzerland, and the proposers.**

- **5358:** *Proposed by Arkady Alt, San Jose, CA*

Prove the identity $\sum_{k=1}^{m} k\binom{m+1}{k+1}r^{k+1} = (r+1)^m(mr-1)+1.$

**Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain**

$$\begin{aligned}(r+1)^m(mr-1)+1 &= \sum_{k=0}^{m} m\binom{m}{k}r^{k+1} - \sum_{k=1}^{m}\binom{m}{k}r^k\\ &= \sum_{k=1}^{m} m\binom{m}{k}r^{k+1} - \sum_{k=1}^{m-1}\binom{m}{k+1}r^{k+1}\\ &= mr^{m+1} + \sum_{k=1}^{m-1}\left(m\binom{m}{k} - \binom{m}{k+1}\right)r^{k+1}\\ &= mr^{m+1} + \sum_{k=1}^{m-1} k\binom{m+1}{k+1}r^{k+1}\\ &= \sum_{k=1}^{m} k\binom{m+1}{k+1}r^{k+1}\end{aligned}$$

where we have used that $m\binom{m}{k} - \binom{m}{k+1} = k\binom{m+1}{k+1}$.

**Solution 2 by Anastasios Kotronis, Athens, Greece**

We have

$$(1+r)^m = \sum_{k=0}^{m}\binom{m}{k}r^k \tag{1}$$

and differentiating

$$mr(1+r)^{m-1} = \sum_{k=0}^{m} k\binom{m}{k}r^k. \tag{2}$$

Now

$$\begin{aligned}\sum_{k=1}^{m} k\binom{m+1}{k+1}r^{k+1} &= \sum_{k=2}^{m+1}(k-1)\binom{m+1}{k}r^k = \sum_{k=2}^{m+1} k\binom{m+1}{k}r^k - \sum_{k=2}^{m+1}\binom{m+1}{k}r^k\\ &\overset{(2),(1)}{=\!=} (m+1)r(1+r)^m - (m+1)r - (1+r)^{m+1} + 1 + (m+1)r\\ &= (r+1)^m(mr-1)+1.\end{aligned}$$

**Solution 3 by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy**

*Proof* Induction. Let $m = 1$. We have

$$\binom{2}{2} r^2 = (r+1)(r-1) + 1$$

which clearly holds.

Let's suppose it is true for $2 \le m \le n-1$. For $m = n$ we have

$$\sum_{k=1}^{m+1} k \binom{m+2}{k+1} r^{k+1} = (m+1)r^{m+2} + \sum_{k=1}^{m} k \left[ \binom{m+1}{k+1} + \binom{m+1}{k} \right] r^{k+1} =$$
$$= (m+1)r^{m+2} + (r+1)^m(mr-1) + 1 \sum_{k=1}^{m} k \binom{m+1}{k} r^{k+1} \qquad (1)$$

$$\binom{m+2}{k+1} = \binom{m+1}{k+1} + \binom{m+1}{k}$$

and the induction hypothesis have been used. Moreover

$$\sum_{k=1}^{m} k \binom{m+1}{k} r^{k+1} \underbrace{=}_{k+1=p} r \sum_{p=0}^{m-1} (p+1) \binom{m+1}{p+1} r^{p+1} =$$
$$= r \sum_{p=1}^{m-1} p \binom{m+1}{p+1} r^{p+1} + r \sum_{p=0}^{m-1} \binom{m+1}{p+1} r^{p+1} =$$
$$= r \sum_{p=1}^{m} p \binom{m+1}{p+1} r^{p+1} - mr^{m+2} \underbrace{+}_{p+1=q} r \sum_{q=0}^{m+1} \binom{m+1}{q} r^q - r - r^{m+2}$$

The induction hypotheses and the Newton–binomial yield that it is equal to

$$r\left((r+1)^m(mr-1)+1\right) - mr^{m+2} + r(1+r)^{m+1} - r - r^{m+2}.$$

By inserting in (1) we get

$$(m+1)r^{m+2} + \left((r+1)^m(mr-1)+1\right)(r+1) - (m+1)r^{m+2} + r(1+r)^{m+1} - r =$$
$$= (r+1)^{m+1}(mr-1) + (r+1) + r(1+r)^{m+1} - r =$$
$$= (r+1)^{m+1}((m+1)r-1) - r(1+r)^{m+1} + (r+1) + r(1+r)^{m+1} - r =$$
$$= (r+1)^{m+1}((m+1)r-1) + 1.$$

and the proof is complete.

**Solution 4 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA**

Here we differentiate the given sum to get the Binomial Theorem, then integrate to get the desired sum.

Let $f(r) = \sum_{k=1}^{m} \binom{m+1}{k+1} r^{k+1} = \sum_{k=1}^{m} k \frac{(m+1)!}{(k+1)k(k-1)!(m+1-k-1)!} r^{k+1}$,

so,

$$\begin{aligned}
f'(r) &= \sum_{k=1}^{m} k \frac{(k+1)(m+1)!}{(k+1)k(k-1)!(m-k)!} r^{k} \\
&= \sum_{k=1}^{m} k \frac{(m+1)!}{(k-1)!(m-k)!} r^{k} \\
&= \sum_{k=0}^{m-1} mk \frac{(m+1)!}{(k-1)!(m-1-k)!} r^{k}, \text{by reindexing} \\
&= m(m+1) \sum_{k=0}^{m-1} \frac{(m-1)!}{k!(m-1-k)!} r^{k}, \\
&= m(m+1) r \sum_{k=0}^{m-1} \binom{m-1}{k} r^{k} \\
&= m(m+1) r (r+1)^{m-1} \text{ by the Binomial Theorem.}
\end{aligned}$$

Now we can integrate by parts to fine $f(r)$:

$$\begin{aligned}
f(r) &= \int m(m+1)(r(r+1)^{m-1} dr \\
&= m(m+1) \int r(r+1)^{m-1} dr \\
&= m(m+1) \left[ \frac{1}{m} r(r+1)^{m} - \int \frac{1}{m}(r+1) dr \right] \\
&= m(m+1) \left[ \frac{1}{m} r(r+1)^{m} - \frac{1}{m} \frac{(r+1)^{m+1}}{m+1} \right] + C \\
&= m(m+1) \left\{ \frac{(r+1)^{m}}{m} \frac{(mr-1)}{m+1} \right\} + C \\
&= (r+1)^{m}(mr-1) + C
\end{aligned}$$

Using the initial condition $f(0) = 0$ we find $C = 1$, so $f(r) = (r+1)^m(mr-1)+1$, as desired.

Editor's note: David and John also submitted a second solution to this problem that was similar to Solution 2 above.

**Also solved by Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo State University San Angelo TX; Charles Burnette (Graduate student, Drexel University), Philadelphia, PA; Jerry Chu (student, Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Carl Libis, Eastern Connecticut State University, Willimantic, CT David E. Manes SUNY College at Oneonta, Oneonta, NY; Toshihiro Shimizu, Kawasaki, Japan; Albert Stadler, Herrliberg, Switzerland, and the proposers.**

**5359:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Let $a, b, c$ be positive real numbers. Prove that

$$\sqrt[4]{15a^3b+1}+\sqrt[4]{15b^3c+1}+\sqrt[4]{15c^3a+1} \le \frac{63}{32}(a+b+c)+\frac{1}{32}\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right).$$

**Solution 1 by Arkady Alt, San Jose, CA**

Since $15a^3b+1$ can be represented as $(2a)^3\cdot\dfrac{15b+\dfrac{1}{a^3}}{8}$ then by AM-GM Inequality we obtain

$$\begin{aligned}\sum_{cyc}\sqrt[4]{15a^3b+1} = \sum_{cyc}\sqrt[4]{(2a)^3\cdot\frac{15b+\dfrac{1}{a^3}}{8}} \quad &\le \quad \sum_{cyc}\frac{3\cdot(2a)+\dfrac{15b+\dfrac{1}{a^3}}{8}}{4} \\ &= \quad \sum_{cyc}\frac{48a+15b+\dfrac{1}{a^3}}{32} \\ &\le \quad \frac{63}{32}(a+b+c)+\frac{1}{32}\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right).\end{aligned}$$

**Solution 2 by Albert Stadler, Herrliberg, Switzerland**

We first claim that

$$\sqrt[4]{11+15x^4} \le \frac{63}{32}x+\frac{1}{32x^3},\ x>0. \qquad (1)$$

Indeed,

$$\left(\frac{63}{32}x+\frac{1}{32x^3}\right)^4-\left(1+15x^4\right)=\frac{(x-1)^2\,(x+1)^2\left(x^2+1\right)^2\left(24321x^8+254x^4+1\right)}{2^{20}x^{12}}\geq 0$$

We replace $x$ by $\sqrt[4]{a^3b}$ in (1) and use the AM$-$GM inequality to obtain

$$\sqrt[4]{1+15a^3b}\leq\frac{63}{32}\sqrt[4]{b^3c}+\frac{1}{32\sqrt[4]{a^9b^3}}\leq\frac{63}{32}\left(\frac{3}{4}\cdot a+\frac{1}{4}\cdot b\right)+\frac{1}{32}\left(\frac{3}{4}\cdot\frac{1}{a^3}+\frac{1}{4}\cdot\frac{1}{b^3}\right).\tag{2}$$

Similarly,

$$\sqrt[4]{1+15b^3c}\leq\frac{63}{32}\sqrt[4]{b^3c}+\frac{1}{32\sqrt[4]{b^9c^3}}\leq\frac{63}{32}\left(\frac{3}{4}\cdot b+\frac{1}{4}\cdot c\right)+\frac{1}{32}\left(\frac{3}{4}\cdot\frac{1}{b^3}+\frac{1}{4}\cdot\frac{1}{c^3}\right).\tag{3}$$

$$\sqrt[4]{1+15c^3a}\leq\frac{63}{32}\sqrt[4]{cb^3a}+\frac{1}{32\sqrt[4]{c^9a^3}}\leq\frac{63}{32}\left(\frac{3}{4}\cdot c+\frac{1}{4}\cdot a\right)+\frac{1}{32}\left(\frac{3}{4}\cdot\frac{1}{c^3}+\frac{1}{4}\cdot\frac{1}{a^3}\right).\tag{4}$$

We complete the proof by adding (2), (3), and (4).

**Solution 3 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania**

Let $f:(0,\infty)\longrightarrow\Re$ and $f(x)=\sqrt[4]{x}$, which is concave on $(0,\infty)$. Therefore,

$$\sqrt[4]{x}=f(x)\leq f(t)+f'(t)(x-t)=\sqrt[4]{t}+\frac{1}{4\sqrt[4]{t^3}}(x-t),\forall x,t>0.$$

Let $x=15a^3b+1$ and $t=16a^4$. Then we have:

$$\sqrt[4]{15a^3b+1}\leq 2a+\frac{1}{32a^3}\left(15a^3b+1-16a^4\right)=2a+\frac{1}{32}\left(15b+\frac{1}{a^3}-16a\right).$$

Summing the analogous upper bounds on the other two terms, gives

$$\begin{aligned}\sqrt[4]{15a^3b+1}+\sqrt[4]{15b^3c+1}+\sqrt[4]{15c^3a+1}\quad&\leq\quad 2\sum a+\frac{15}{32}\sum a-\frac{1}{2}\sum a+\frac{1}{32}\sum\frac{1}{a^3}\\&=\quad\frac{63}{32}(a+b+c)+\frac{1}{32}\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right).\end{aligned}$$

**Charles Burnette (Graduate student, Drexel University), Philadelphia, PA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Toshihiro Shimizu, Kawasaki, Japan, and the proposer.**

- **5360:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $n\geq 1$ be an integer and let

$$I_n=\int_0^{\infty}\frac{\arctan x}{(1+x^2)^n}dx.$$

Prove that

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{I_n}{n}=\zeta(2);$

(b) $\displaystyle\int_0^{\infty}\arctan x\ln\left(1+\frac{1}{x^2}\right)dx=\zeta(2).$

**Solution 1 by Anastasios Kotronis, Athens, Greece**

- a) We have

$$I_n \overset{x=\tan y}{=\!=\!=} \int_0^{\frac{\pi}{2}} y\cos^{2n-2}y\,dy$$

and since the integrand doesn't change sign:

$$\begin{aligned}
\sum_{n\geq 1}\frac{I_n}{n} &= \sum_{n\geq 1}\frac{1}{n}\int_0^{\frac{\pi}{2}} y\cos^{2n-2}y\,dy=\sum_{n\geq 1}\int_0^{\frac{\pi}{2}} y\sum_{n\geq 1}\frac{(\cos^2 y)^{n-1}}{n}\,dy=-\int_0^{\frac{\pi}{2}} y\frac{\ln(1-\cos^2 y)}{\cos^2 y}\,dy\\
&=-2\int_0^{\frac{\pi}{2}} y\frac{\ln(\sin y)}{\cos^2 y}\,dy=-2\int_0^{\frac{\pi}{2}}(y\tan y+\ln(\cos y))'\ln(\sin y)\,dy\\
&=-2\left(y\tan y+\ln(\cos y)\right)\ln(\sin y)\Big|_0^{\frac{\pi}{2}}+2\int_0^{\frac{\pi}{2}}(y\tan y+\ln(\cos y))\cot y\,dy\\
&=\frac{\pi^2}{4}+2\int_0^{\frac{\pi}{2}}\cot y\ln(\cos y)\overset{\cos y=t}{=\!=\!=}\frac{\pi^2}{4}+2\int_0^1\frac{t\ln t}{1-t^2}\,dt=\frac{\pi^2}{4}+\int_0^1\frac{\ln t}{1-t}\,dt-\int_0^1\frac{\ln t}{1+t}\,dt\\
&=\frac{\pi^2}{4}+\int_0^1\sum_{n\geq 0}t^n\ln t-\int_0^1\sum_{n\geq 0}(-t)^n\ln t.
\end{aligned}$$

From Dominated Convergence Theorem, the order of integration and summation can change, so

$$\sum_{n\geq 1}\frac{I_n}{n}=\frac{\pi^2}{4}+\sum_{n\geq 0}\int_0^1 t^n\ln t-\sum_{n\geq 0}\int_0^1(-t)^n\ln t=\frac{\pi^2}{4}-\sum_{n\geq 1}\frac{1}{n^2}+\sum_{n\geq 1}\frac{(-1)^{n-1}}{n^2}=\frac{\pi^2}{6}=\zeta(2).$$

b)

$$\int_0^{\infty}\arctan x\ln\left(1+\frac{1}{x^2}\right)dx\overset{x=\tan y}{=\!=\!=}-2\int_0^{\frac{\pi}{2}} y\frac{\ln(\sin y)}{\cos^2 y}\,dy=\sum_{n\geq 1}\frac{I_n}{n},$$

from the first part of the problem, so the result is immediate.

**Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain**

We first prove (b) and then (a).

b)

$$\int_0^{\infty}\arctan x\ln\left(1+\frac{1}{x^2}\right)dx=\left[\begin{array}{c}u=\arctan x\Longrightarrow du=\dfrac{1}{1+x^2}dx\\ dy=\ln\left(1+\dfrac{1}{x^2}\right)dx\Longrightarrow v=2\arctan x+x\ln\left(1+\frac{1}{x^2}\right)\end{array}\right]$$

$$= \int_0^\infty u dv = uv]_0^\infty - \int_0^\infty v du$$

$$= \arctan x\left(2\arctan x + x\ln\left(1+\frac{1}{x^2}\right)\right)\Big]_0^\infty - \int_0^\infty \frac{2\arctan x + x\ln\left(1+\frac{1}{x^2}\right)}{1+x^2}dx$$

$$= \frac{\pi^2}{2}\left(2\frac{\pi}{2}+0\right) - 0\,(2\cdot 0+0) - 2\int_0^\infty \frac{\arctan x}{1+x^2}dx - \int_0^\infty \frac{x\ln\left(1a+\frac{1}{x^2}\right)}{1+x^2}dx$$

$$\underset{1}{=} \frac{\pi^2}{2} - \left(\arctan^2 x\Big]_0^\infty\right) - \frac{\pi^2}{12}$$

$$= \frac{\pi^2}{2} - \frac{\pi^2}{4} + 0 - \frac{\pi^2}{12} = \frac{\pi^2}{6} = \zeta(2).$$

(a)

$$\sum_{n=1}^\infty \frac{I_n}{n} = \sum_{n=1}^\infty \frac{1}{n}\int_0^\infty \frac{\arctan x}{(1+x^2)^n}dx = \int_0^\infty \arctan x \sum_{n=1}^\infty \frac{1}{n}\left(\frac{1}{1+x^2}\right)^n dx$$

$$= \int_0^\infty \arctan x\left(-\ln\left(1-\frac{1}{1+x^2}\right)\right)dx = \int_0^\infty \arctan x \ln\left(1+\frac{1}{x^2}\right)dx = \zeta(2), \text{ from part b.}$$

(1) Table of Integrals, Series and Products, Gradshteyn, I.S. and Ryzhik, I.M., Seventh Edition Elsevier Inc., 2007, 4,298(16) page 564.

**Solution 3 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy**

Part a)

$$I_n = \frac{x\arctan x}{(1+x^2)^n}\Big|_0^\infty - \int_0^\infty \frac{x}{(1+x^2)^{n+1}} + 2n\int_0^\infty \frac{x^2\arctan x}{(1+x^2)^{n+1}}$$

$$= \frac{1}{2n}\frac{1}{(1+x^2)^n}\Big|_0^\infty + 2n\int_0^\infty \frac{\arctan x}{(1+x^2)^n}dx - 2n\int_0^\infty \frac{\arctan x}{(1+x^2)^{n+1}}dx$$

$$= -\frac{1}{2n} + 2nI_n - 2nI_{n-1}.$$

We have obtained the recursive sequence

$$I_{n+1} = I_n\left(1 - \frac{1}{2n}\right) - \frac{1}{4n^2} \iff \frac{1}{2n}I_n = I_n - I_{n+1} - \frac{1}{4n^2}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{I_n}{n} = 2I_1 - \lim_{n\to\infty} I_n - \frac{\pi^2}{12}$$

$$I_1 = \int_0^{\infty} \frac{\arctan x}{1+x^2}dx \underbrace{=}_{x=\tan t} \int_0^{\pi/2} t dt = \frac{\pi^2}{8}$$

As for $\lim_{n\to\infty} I_n$, we break $I_n$ into two addends.

$$I_n = \int_0^1 \frac{\arctan x}{(1+x^2)^n}dx + \int_1^{\infty} \frac{\arctan x}{(1+x^2)^n}dx \doteq J_1 + J_2.$$

$J_1$ converges to zero for instance by the dominated convergence theorem of Lebesgue after observing that $\dfrac{\arctan x}{(1+x^2)^n} \to 0$.

As for $J_2$ we bound,

$$0 < \int_1^{\infty} \frac{\arctan x}{(1+x^2)^n}dx \le \frac{\pi}{2}\int_1^{\infty} \frac{1}{x^{2n}}dx = \frac{\pi}{2}\frac{1}{2n-1} \to 0.$$

We have obtained,

$$\sum_{n=1}^{\infty} \frac{I_n}{n} = \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6}.$$

Part (b). Let's define $I$ the integral. Integrating by parts,

$$I = x\arctan x \ln\left(1 + \frac{1}{x^2}\right)\Big|_0^{\infty} - \int_0^{\infty} \frac{x\ln\left(1+\frac{1}{x^2}\right)}{1+x^2}dx + \int_0^{\infty} \frac{2\arctan x}{1+x^2}dx \quad (1)$$

The first summand annihilates because

$$\lim_{x\to 0} x\arctan x \ln\left(1 + \frac{1}{x^2}\right) = \lim_{x\to 0} x\arctan x(\ln(1+x^2) - 2\ln x) = 0.$$

The third is equal to $(\arctan^2 x)\Big|_0^{\infty} = \dfrac{\pi^2}{4}$.

As for the second summand it is equal to

$$\lim_{a\to\infty} \int_0^a \frac{x\ln(1+x^2) - 2x\ln x}{1+x^2}dx = \lim_{a\to\infty} \frac{\ln^2(1+x^2)}{4}\Big|_0^a - \lim_{a\to\infty} 2\int_0^a \frac{x\ln x}{1+x^2}. \quad (2)$$

$$\lim_{a\to\infty} 2\int_0^a \frac{x\ln x}{1+x^2} = \lim_{a\to\infty} \ln x \ln(1+x^2)\Big|_0^a - \lim_{a\to\infty} \int_0^a \frac{\ln(1+x^2)}{x}\, dx\,;$$

$$\lim_{a\to\infty}\int_0^a \frac{\ln(1+x^2)}{x}\,dx = \int_0^1 \frac{\ln(1+x^2)}{x}\,dx + \lim_{a\to\infty}\int_1^a \frac{\ln(1+x^2)}{x}\,dx\,;$$

$$\int_0^1 \frac{\ln(1+x^2)}{x}\,dx = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\int_0^1 x^{2k-1}dx = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k^2}$$

$$\begin{aligned}
\lim_{a\to\infty}\int_1^a \frac{\ln(1+x^2)}{x}\,dx \quad &\underbrace{=}_{x=1/y} \quad \lim_{a\to\infty}\int_{1/a}^1 \frac{\ln(1+y^2)-2\ln y}{y}dy \\
&= \int_0^1 \frac{\ln(1+y^2)}{y}dy - \lim_{a\to\infty}\ln^2 y\Big|_{1/a}^{1} \\
&= \int_0^1 \frac{\ln(1+y^2)}{y}dy + \lim_{a\to\infty}\ln^2 a
\end{aligned}$$

Plugging in (2) we get

$$\lim_{a\to\infty}\frac{\ln^2(1+a^2)}{4} - \ln a\ln(1+a^2) + \ln^2 a + 2\int_0^1 \frac{\ln(1+x^2)}{x}dx = 2\int_0^1 \frac{\ln(1+x^2)}{x}dx.$$

and

$$\int_0^1 2\frac{\ln(1+x^2)}{x}dx = \sum_{k=1}^{\infty}\frac{2(-1)^k}{k}\int_0^1 x^{2k-1}dx = \sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} = \frac{\pi^2}{12},$$

and finally (1) is equal to

$$\frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6}.$$

**Solution 4 by G.C. Greubel, Newport News, VA**

**Part a)** Given the integral

$$I_n = \int_0^{\infty}\frac{\arctan x}{(1+x^2)^n}\,dx \tag{1}$$

make the change of variable $x = \tan t$ to obtain

$$I_n = \int_0^{\pi/2}\frac{t\,\sec^2 t}{(\sec^2 t)^n}\,dt = \int_0^{\pi/2} t\,\cos^{2n-2} t\,dt. \tag{2}$$

By considering the summation of $I_n$ in the desired manner leads to

$$S = \sum_{n=1}^{\infty}\frac{I_n}{n} = -\int_0^{\pi/2}\frac{t\,\ln(\sin^2 t)}{\cos^2 t}\,dt. \tag{3}$$

The integral in (3) may be evaluated by use of the Dilogarithm function as seen by the following.

$$
\begin{aligned}
J &= \int_0^{\pi/2} \frac{t\,\ln(\sin^2 t)}{\cos^2 t}\,dt &(4)\\
&= \Bigg[-Li_2\left(-\tan^2\left(\frac{t}{2}\right)\right) - 2Li_2\left(\frac{1}{2}\sec^2\left(\frac{t}{2}\right)\right) - Li_2\left(\cos t\,\sec^2\left(\frac{t}{2}\right)\right)\\
&\quad -t^2 - \ln^2\left(\sec^2\left(\frac{t}{2}\right)\right) + 2\ln 2\,\ln\left(\sec^2\left(\frac{t}{2}\right)\right) + t\,\tan t\,\ln\left(\sin^2 t\right)\\
&\quad -\ln\left(\sin^2 t\right)\,\ln\left(\sec^2\left(\frac{t}{2}\right)\right) + \ln\left(\sin^2 t\right)\,\ln\left(\cos t\,\sec^2\left(\frac{t}{2}\right)\right)\\
&\quad -\ln\left(\tan^2\left(\frac{t}{2}\right)\right)\,\ln\left(\cos t\,\sec^2\left(\frac{t}{2}\right)\right)\Bigg]_0^{\pi/2}\\
&= -Li_2(-1) - Li_2(1) - \frac{\pi^2}{4} + \ln^2 2 + 2Li_2\left(\frac{1}{2}\right)\\
&= -Li_2(1) = -\zeta(2). &(5)
\end{aligned}
$$

By using the resulting integral value of (5) in (3) the desired result is obtained, namely,

$$\sum_{n=1}^{\infty} \frac{I_n}{n} = \zeta(2). \tag{6}$$

**Part b)** The integral in question is given by

$$I = \int_0^{\infty} \tan^{-1} x\,\ln\left(1 + \frac{1}{x^2}\right)\,dx. \tag{7}$$

Making the change of variable $x = \tan t$ leads to the integral

$$I = -\int_0^{\pi/2} \frac{t\,\ln(\sin^2 t)}{\cos^2 t}\,dt. \tag{8}$$

This is the same integral defined as (4) and has the resulting value given by (5). By comparison of results the integral of this section is presented as

$$\int_0^{\infty} \tan^{-1} x\,\ln\left(1 + \frac{1}{x^2}\right)\,dx = \zeta(2) \tag{9}$$

which is the desired result.

**Also solved by Ed Gray, Highland Beach, FL; Kee Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Toshihiro Shimizu, Kawasaki, Japan; Albert Stadler, Herrliberg, Switzerland, and the proposer.**