Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
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- **5325:** *Proposed by Kenneth Korbin, New York, NY*

  Given the sequence $x = (1, 7, 41, 239, 1393, 8119, \dots)$, with $x_n = 6x_{n-1} - x_{n-2}$.

  Let $y = \dfrac{x_{2n} + x_{2n-1}}{x_n}$. Find an explicit formula for y expressed in terms of n.

- **5326:** *Proposed by Armend Sh. Shabani, University of Prishtina, Republic of Kosova*

  Find all positive integer solutions to $m! + 2^{4k-1} = l^2$.

- **5327:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

  Show that in any triangle ABC, with the usual notations, that

  $$\left(\frac{ab}{a+b}\right)^2 + \left(\frac{bc}{b+c}\right)^2 + \left(\frac{ca}{c+a}\right)^2 \geq 9r^2.$$

- **5328:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Without the aid of a computer, find the positive solutions of the equation

  $$2^{x+1}\left(1 - \sqrt{1 + x^2 + 2^x}\right) = \left(x^2 + 2^x\right)\left(1 - \sqrt{1 + 2^{x+1}}\right).$$

- **5329:** *Proposed by Arkady Alt, San Jose, CA*

  Find the smallest value of $\dfrac{x^3}{x^2+y^2} + \dfrac{y^3}{y^2+z^2} + \dfrac{z^3}{z^2+x^2}$ where real $x, y, z > 0$ and $xy + yz + zx = 1$.

- **5330:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $B(x) = \begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix}$ and let $n \geq 2$ be an integer.

Calculate the matrix product

$$B(2)B(3)\cdots B(n).$$

Solutions

- **5307:** *Proposed by Haishen Yao and Howard Sporn, Queensborough Community College, Bayside, NY*

  Solve for x:

$$\sqrt{x^{15}} = \sqrt{x^{10}-1} + \sqrt{x^5-1}.$$

Solution 1 by Arkady Alt, San Jose, CA

Let $a = \sqrt{x^{10}-1}$ and $b = \sqrt{x^5-1}$ then

$$\begin{aligned} x^5 &= b^2+1,\ x^{10} = a^2+1, \\ x^{15} &= x^{10}\cdot x^5 = \left(a^2+1\right)\left(b^2+1\right) \text{ and therefore,} \\ \sqrt{\left(a^2+1\right)\left(b^2+1\right)} &= a+b \iff \\ \left(a^2+1\right)\left(b^2+1\right) &= (a+b)^2 \iff \\ (ab-1)^2 &= 0 \iff \\ ab = 1. \end{aligned}$$

Also we have

$$x^{10} = \left(x^5\right)^2 \implies a^2+1 = \left(b^2+1\right)^2 \iff b^4+2b^2 = a^2 \iff b^6+2b^4 = a^2b^2.$$

Since $ab = 1$ then

$$\begin{aligned} b^6+2b^4-1 &= 0 \iff \\ \left(b^2+1\right)\left(b^4+b^2-1\right) &= 0 \iff \\ b^4+b^2-1 &= 0 \iff \\ b^2 &= \frac{-1+\sqrt{5}}{2}. \text{ Hence,} \\ x^5 &= b^2+1 \\ &= \frac{-1+\sqrt{5}}{2}+1 \\ &= \frac{1+\sqrt{5}}{2} \iff x = \sqrt[5]{\frac{1+\sqrt{5}}{2}}. \end{aligned}$$

Solution 2 by Charles McCracken, Dayton, OH

Let $\mu = x^5$ then $\sqrt{\mu^2} = \sqrt{\mu^2 - 1} + \sqrt{\mu - 1}$.

It is readily seen that $1 < x < 2$. A few successive approximations give $\mu \approx 1.618$. So we try $\mu = \phi = \dfrac{1+\sqrt{5}}{2}$, also known as, the Golden Ratio.

The equation then becomes

$$\begin{aligned}
\sqrt{\phi}^3 &= \sqrt{\phi^2 - 1} + \sqrt{\phi - 1} \\
\phi\sqrt{\phi} &= \sqrt{\phi + 1 - 1} + \sqrt{\frac{1}{\phi}} \\
\phi\sqrt{\phi} &= \sqrt{\phi} + \sqrt{\frac{1}{\phi}} \\
\phi^2 &= \phi + 1. \text{ A well known identity.}
\end{aligned}$$

Since $\phi = \mu,\ x = \sqrt[5]{\phi}\ \approx 1.101025882$.

Solution 3 by Becca Rousseau, Ellie Erehart, and David Weerheim (jointly, students at Taylor University), Upland, IN

The common domain of definition for $\sqrt{x^{15}}, \sqrt{x^{10} - 1}$, and $\sqrt{x^5 - 1}$ is $x \geq 1$. We now solve for x:

$$\begin{aligned}
x^{15} &= (x^{10} - 1) + 2\sqrt{(x^{10} - 1)(x^5 - 1)} + (x^5 - 1) \\
x^{15} - x^{10} - x^5 + 2 &= 2\sqrt{x^{15} - x^{10} - x^5 + 1} \\
x^5\left(x^{10} - x^5 - 1\right) + 2 &= 2\sqrt{x^5\left(x^{10} - x^5 - 1\right) + 1}.
\end{aligned}$$

Letting $u = x^5\left(x^{10} - x^5 - 1\right)$, we obtain

$$\begin{aligned}
u + 2 &= 2\sqrt{u + 1} \\
u^2 + 4u + 4 &= 4(u + 1) \\
u^2 + 4u + 4 &= 4u + 4 \\
u^2 &= 4u + 4 - 4u - 4 \\
u^2 &= 0,\ u = 0.
\end{aligned}$$

Substituting $x^5(x^{10} - x^5 - 1)$for u see that

$$x^5\left(x^{10} - x^5 - 1\right) = 0, \text{ so}$$

$$x^5 = 0 \quad \text{or} \qquad x^{10} - x^5 - 1 = 0.$$

$$x^5 = 0 \qquad x^5 = \frac{1 \pm \sqrt{5}}{2}.$$

Therefore, $x = 0,\ x = \sqrt[5]{\dfrac{1-\sqrt{5}}{2}}$, or $x = \sqrt[5]{\dfrac{1+\sqrt{5}}{2}}$.

The first two roots must be discarded, because they are outside the domain of definition of x, as noted above.

Solution 4 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

It is not specified whether x is real or not, so let's assume $x \in R$. The domain of x is $[1, \infty)$ since both $x^{10} - 1$ and $x^5 - 1$ are the product of $(x-1)$ for a positive polynomial respectively of order 9 and 4. Let $x^5 = y$. Squaring we get

$$y^3 = y^2 - 1 + y - 1 + 2(y-1)\sqrt{y+1} \iff y^3 - y^2 + 1 - y + 1 = 2(y-1)\sqrt{y+1}.$$

The r.h.s. is nonnegative for $y \geq 1$. Moreover for $y \geq 0$

$$\frac{y^3}{2} + \frac{y^3}{2} + \frac{1}{2} \geq \frac{3}{2}y^2, \qquad \frac{1}{2}y^2 + \frac{1}{2} \geq y$$

and then

$$y^3 - y^2 - y + 2 \geq 1 + y^2 + y > y^2 + y.$$

We square both sides again getting

$$y^2(y^2 - y - 1)^2 = 0 \iff y = (1 + \sqrt{5})/2$$

and then $x = \big((1+\sqrt{5})/2\big)^{1/5}$.

Comment: Brian D. Beasley, Presbyterian College, Clinton, SC, Moti Levy of Rehovot, Israel, Michael Thew (student at Saint George's School), Spokane, WA, Neculai Stanciu, Buzău, Romania and Titu Zvonaru, Comănesti, Romania, and David Stone and John Hawkins of Georgia Southern University, Statesboro, Georgia noted in their solutions that if complex roots are allowed, the full set of roots is:

$$x = 0, x_k = \left((1+\sqrt{5})/2\right)^{1/5}\left(\cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}\right), k = 0,1,2,3,4, \text{and}$$

$$x_m = \left((1-\sqrt{5})/2\right)^{1/5}\left(\cos\frac{2m\pi}{5} + i\sin\frac{2m\pi}{5}\right), m = 0,1,2,3,4.$$

David Stone and John Hawkins also noted that if we let $y_1 = \sqrt{x^{15}}$ and $y_2 = \sqrt{x^{10}-1} + \sqrt{x^5 - 1}$, the graphs of these two functions intersect at the real root, and at this point the graphs are tangent to one another.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain;

Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; Kelley McKaig, Madison Thompson, and Melanie Schmocker, (Students at Taylor University), Upland, IN; Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain, and the proposers.

- **5308:** *Proposed by Kenneth Korbin, New York, NY*

Given the sequence

$$t = (1, 7, 41, 239, \ldots)$$

with $t_n = 6t_{n-1} - t_{n-2}$. Let (x, y, z) be a triple of consecutive terms in this sequence with $x < y < z$.

Part 1) Express the value of x in terms of y and express the value of y in terms of x.

Part 2) Express the value of x in terms of z and express the value of z in terms of x.

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

From the recursive formula

$$t_{n+2} = 6t_{n+1} - t_n \tag{1}$$

and the initial conditions $t_1 = 1$ and $t_2 = 7$, we can find a closed form expression for t_n by using the customary techniques for solving homogeneous linear difference equations. If we consider solutions of the form $t_n = \lambda^n$, with $\lambda \neq 0$, equation (1) provides us with the auxiliary equation

$$\lambda^2 = 6\lambda - 1$$

whose solutions are

$$\lambda = 3 \pm 2\sqrt{2}.$$

Then, there are constants c_1, c_2 such that

$$t_n = c_1 \left(3 + 2\sqrt{2}\right)^n + c_2 \left(3 - 2\sqrt{2}\right)^n$$

for all $n \geq 1$. The initial conditions $t_1 = 1$ and $t_2 = 7$ give

$$c_1 = \frac{\sqrt{2} - 1}{2} \quad \text{and} \quad c_2 = -\frac{\sqrt{2} + 1}{2}$$

and we have

$$t_n = \frac{\sqrt{2} - 1}{2} \left(3 + 2\sqrt{2}\right)^n - \frac{\sqrt{2} + 1}{2} \left(3 - 2\sqrt{2}\right)^n.$$

Finally, since

$$\left(3 + 2\sqrt{2}\right) = \left(\sqrt{2} + 1\right)^2 \quad \text{and} \quad \left(3 - 2\sqrt{2}\right) = \left(\sqrt{2} - 1\right)^2,$$

we conclude that

$$\begin{aligned} t_n &= \frac{\sqrt{2} - 1}{2} \left(\sqrt{2} + 1\right)^{2n} - \frac{\sqrt{2} + 1}{2} \left(\sqrt{2} - 1\right)^{2n} \\ &= \frac{\left(\sqrt{2} + 1\right)^{2n-1} - \left(\sqrt{2} - 1\right)^{2n-1}}{2} \end{aligned} \tag{2}$$

for all $n \geq 1$.

Equation (2) shows that $t_n > 0$ for all n and then an elementary Mathematical Induction argument using (1) establishes that $t_{n+1} > t_n$ for all n. Therefore, if (x, y, z) is a triple of consecutive terms in this sequence with $x < y < z$, we must have $x = t_n$, $y = t_{n+1}$, and $z = t_{n+2}$ for some $n \geq 1$.

For Part 1), we note that

$$\begin{aligned}
y &= t_{n+1} \\
&= \frac{1}{2}\left[\left(\sqrt{2}+1\right)^{2n+1} - \left(\sqrt{2}-1\right)^{2n+1}\right] \\
&= \frac{1}{2}\left[\left(3+2\sqrt{2}\right)\left(\sqrt{2}+1\right)^{2n-1} - \left(3-2\sqrt{2}\right)\left(\sqrt{2}-1\right)^{2n-1}\right] \\
&= \frac{3+2\sqrt{2}}{2}\left[\left(\sqrt{2}+1\right)^{2n-1} - \left(\sqrt{2}-1\right)^{2n-1}\right] + 2\sqrt{2}\left(\sqrt{2}-1\right)^{2n-1} \\
&= \left(3+2\sqrt{2}\right)t_n + 2\sqrt{2}\left(\sqrt{2}-1\right)^{2n-1} \\
&= \left(3+2\sqrt{2}\right)x + 2\sqrt{2}\left(\sqrt{2}-1\right)^{2n-1}.
\end{aligned}$$

Then,

$$\begin{aligned}
x &= \frac{1}{3+2\sqrt{2}}\left[y - 2\sqrt{2}\left(\sqrt{2}-1\right)^{2n-1}\right] \\
&= \left(3-2\sqrt{2}\right)\left[y - 2\sqrt{2}\left(\sqrt{2}-1\right)^{2n-1}\right] \\
&= \left(3-2\sqrt{2}\right)y - 2\sqrt{2}\left(\sqrt{2}-1\right)^{2}\left(\sqrt{2}-1\right)^{2n-1} \\
&= \left(3-2\sqrt{2}\right)y - 2\sqrt{2}\left(\sqrt{2}-1\right)^{2n+1}.
\end{aligned}$$

For Part 2), equation (1) and Part 1) imply that

$$\begin{aligned}
z &= t_{n+2} \\
&= 6t_{n+1} - t_n \\
&= 6\left[\left(3+2\sqrt{2}\right)x + 2\sqrt{2}\left(\sqrt{2}-1\right)^{2n-1}\right] - x \\
&= \left(17+12\sqrt{2}\right)x + 12\sqrt{2}\left(\sqrt{2}-1\right)^{2n-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
x &= \frac{1}{17+12\sqrt{2}}\left[z - 12\sqrt{2}\left(\sqrt{2}-1\right)^{2n-1}\right] \\
&= \left(17-12\sqrt{2}\right)\left[z - 12\sqrt{2}\left(\sqrt{2}-1\right)^{2n-1}\right] \\
&= \left(17-12\sqrt{2}\right)z - 12\sqrt{2}\left(\sqrt{2}-1\right)^{4}\left(\sqrt{2}-1\right)^{2n-1} \\
&= \left(17-12\sqrt{2}\right)z - 12\sqrt{2}\left(\sqrt{2}-1\right)^{2n+3}.
\end{aligned}$$

Remark. On page 253 of Recreations in the Theory of Numbers by A. H. Beiler (Dover Publications, Inc., 1966), it is shown that the sequence $\{t_n\}$ provides the solutions for x in the Pell Equation $x^2 - 2y^2 = -1$. The corresponding y solutions satisfy the recursive formula $y_{n+2} = 6y_{n+1} - y_n$ with $y_1 = 1$ and $y_2 = 5$. This yields

$$y_n = \frac{\left(\sqrt{2}+1\right)^{2n-1} + \left(\sqrt{2}-1\right)^{2n-1}}{2\sqrt{2}}$$

for $n \geq 1$.

Solution 2 by Moti Levy, Rehovot, Israel

The solution of this type of recurrence formulas is

$$t_n = a\alpha^n + b\beta^n,$$

where α and β are the roots of $r^2 - 6r + 1$.

Here,

$$t_n = a\alpha^n - (a+1)\,\alpha^{-n}; \quad a = \left(\frac{1}{2}\sqrt{2} - \frac{1}{2}\right);\ \alpha = 3 + 2\sqrt{2}.$$

Part 1):

$$x = a\alpha^n - (a+1)\,\alpha^{-n}$$
$$y = a\alpha^{n+1} - (a+1)\,\alpha^{-n-1}$$

Solving for α^n in terms of x, we get,

$$\alpha^n = \left(\sqrt{2}+1\right)\left(x + \sqrt{x^2+1}\right),$$

$$y = 3x + 2\sqrt{2}\sqrt{x^2+1}.$$

Solving for α^n in terms of y, we get,

$$\alpha^n = \left(\sqrt{2}-1\right)\left(y + \sqrt{y^2+1}\right),$$

$$x = 3y - 2\sqrt{2}\sqrt{y^2+1}.$$

Part 2):

$$z = a\alpha^{n+2} - (a+1)\,\alpha^{-n-2},$$

$$z = 17x + 12\sqrt{2}\sqrt{x^2+1}.$$

Solving for α^n in terms of z, we get,

$$\alpha^n = \left(5\sqrt{2}-7\right)\left(z + \sqrt{z^2+1}\right),$$

$$x = 17z - 12\sqrt{2}\sqrt{z^2+1}.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Titu Zvonaru, Comănesti, Romania (jointly with) Neculai Stanciu, "Geroge Emil Palade School," Buzău, Romania, and the proposer.

- **5309:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

Consider the expression $3^n + n^2$ for positive integers n. It is divisible by 13 for $n = 18$ and $n = 19$. Prove, however, that it is never divisible by 13 for three consecutive values of n.

Solution 1 by Bruno Salgueiro Fanego Viveiro, Spain

Let n be an integer such that $n \geq 1$. We argue by contradiction. If, for the three consecutive values $n-1, n$, and $n+1$ the expressions $3^{n-1} + (n-1)^2, 3^n + n^2$, and $3^{n+1} + (n+1)^2$ are each divisible by 13, then their sum, $\left(1 + 3 + 3^2\right) \cdot 3^{n-1} + 3n^2 + 2$ is divisible by 13, or equivalently, the expression $3n^2 + 2$ is divisible by 13.

If we divide n by 13, we obtain an integer quotient c and remainder r, $0 \leq r < 13$, such that $n = 13c + r$, so $3n^2 + 2 = 3\left(13c + r\right)^2 + 2 = 13 \cdot \left(39c^2 + 2cr\right) + 3r^2 + 2$, which is divisible by 13, so $3r^2 + 2$ is also divisible by 13.

Since $0 \leq r \leq 12$, $3r^2 + 2 \in \{5, 14, 29, 50, 77, 110, 149, 194, 245, 302, 365, 434\}$ and hence $3r^2 + 2$ is not divisible by 13 (because each remainder of the division of 5, 14, 29, 50, 77, 110, 149, 194, 245, 302, 365, and 434 by 13 is not zero. The remainders are, respectively, 5, 1, 3, 11, 12, 6, 6, 12,11, 3, 1, and 5. Thus we have a contradiction showing that the expressions $3^{n-1} + (n-1)^2$, $3^n + n^2$, and $3^{n+1} + (n+1)^2$ cannot all be divisible by 13.

Solution 2 by Ed Gray, Highland Beach, FL

Suppose there were three consecutive integers, say, $n, n+1$ and $n+2$ for which $3^n + n^2$ is divisible by 13. Then we have the three congruences:

$$\begin{array}{ll} (1) & 3^n + n^2 \equiv 0 \pmod{13} \\ (2) & 3^{n+1} + n^2 + 2n + 1 \equiv 0 \pmod{13} \\ (3) & 3^{n+2} + n^2 + 4n + 4 \equiv 0 \pmod{13} \end{array}$$

Multiple (1) by 9, multiply (2) by 1 and multiply (3) by 3. Then

$$\begin{array}{ll} (4) & 9 \cdot 3^n + 9n^2 \equiv 0 \pmod{13} \\ (5) & 3 \cdot 3^n + n^2 + 2n + 1 \equiv 0 \pmod{13} \\ (6) & 27 \cdot 3^n + 3n^2 + 12n + 12 \equiv 0 \pmod{13} \end{array}$$

Adding the three congruences:

$$(7)\ \ 39 \cdot 3^n + 13n^2 + 14n + 13 \equiv 0 \pmod{13} \Longrightarrow 13 \Big| n,$$

which is equivalent to saying $n \equiv 0 \pmod{13}$. Therefore, if it were possible to have three consecutive integers such that $3^n + n^2$ were divisible by 13, then 13 would have to divide n and this implies (in eq. 1) that 13 divides 3^n , but this is impossible because the only divisors of 3^n are multiples of 3.

Solution 3 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

Suppose that there are positive integers n, x, y, z such that

$$3^n + n^2 = 13x, \quad 3^{n+1} + (n+1)^2 = 13y, \quad \text{and} \quad 3^{n+2} + (n+2)^2 = 13z.$$

Then,

$$\begin{aligned}13z &= 3\cdot 3^{n+1} + (n+1)^2 + (2n+3)\\ &= \left[3^{n+1} + (n+1)^2\right] + 2\cdot 3^{n+1} + 2n + 3\\ &= 13y + 6\cdot 3^n + 2n + 3\\ &= 13y + 6\left(13x - n^2\right) + 2n + 3\\ &= 13\left(y + 6x\right) - 6n^2 + 2n + 3.\end{aligned}$$

Hence,

$$13\left(6x + y - z\right) = 6n^2 - 2n - 3$$

which implies that

$$6n^2 - 2n - 3 \equiv 0 \pmod{13}.$$

However, as shown in the following table, this is impossible.

$n \pmod{13}$	$6n^2 - 2n - 3 \pmod{13}$
0	10
1	1
2	4
3	6
4	7
5	7
6	6
7	4
8	1
9	10
10	5
11	12
12	5

Therefore, no such n, x, y, z exist and $3^n + n^2$ is never divisible by 13 for three consecutive values of n.

Solution 4 by Kee-Wai Lau, Hong Kong, China

Suppose the contrary, that $3^m + m^2,\ 3^{m+1} + (m+1)^2, 3^{m+2} + (m+2)^2$ are divisible by 13 for some positive integer m. Hence their sum

$$13(3^m) + 3m^2 + 6m + 5$$

is also divisible by 13. However this contradicts the fact that $3m^2 + 6m + 5$ is congruent to $5, 1, 3, 11, 12, 6, 6, 12, 11, 3, 1, 5, 2$ modulo 13 according as m is congruent to $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ modulo 13. Hence the assertion of the problem.

Also solved by Arkady Alt, San Jose, CA; Brian D. Beasely, Presbyterian College, Clinton, SC; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY; Moti Levy, Rehovot, Israel; David E. Manes, SUNY College at Oneonta, Oneonta, NY; David Stone and John Hawkins of Georgia Southern University, Statesboro, Georgia; Titu Zvonaru, Comănesti, Romania (jointly with) Neculai Stanciu, "Geroge Emil Palade School," Buzău, Romania, and the proposer.

- **5310:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

Let $a > 0$ and a sequence $\{E_n\}_{n\geq 0}$, be defined by $E_n = \sum_{k=0}^{n} \frac{1}{k!}$. Evaluate:

$$\lim_{n\to\infty} \sqrt[n]{n!}\left(a^{\sqrt[n]{E_n}-1} - 1\right).$$

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

We know that

$$e = \sum_{k=0}^{n} \frac{1}{k!} + \frac{c}{(n+1)!}, \quad c = e^{\xi}, \quad 0 \leq \xi < 1.$$

It follows that $\sqrt[n]{E_n} \to 1$ and then $\left(a^{\sqrt[n]{E_n}-1} - 1\right)/(\sqrt[n]{E_n} - 1) \to \ln a$, as well as

$$\lim_{n\to\infty} \sqrt[n]{n!}\left(a^{\sqrt[n]{E_n}-1} - 1\right) = \lim_{n\to\infty} \sqrt[n]{n!}(\sqrt[n]{E_n} - 1)\ln a.$$

Moreover,

$$\lim_{n\to\infty} n\left(E_n^{1/n} - 1\right) = 1,$$

and then

$$\lim_{n\to\infty} \sqrt[n]{n!}(\sqrt[n]{E_n} - 1)\ln a = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}\ln a.$$

Finally, the Cesaro–Stolz theorem yields

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}\ln a = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}}\ln a = \ln a \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!} = \ln a \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \frac{\ln a}{e}.$$

Solution 2 by Ed Gray, Highland Beach, FL

We first show that the limit to be evaluated is of the form $\infty \cdot 0$, and then we use L'Hospital's rule to evaluate it.

$$\lim_{n\to\infty} \sqrt[n]{E_n} = \lim_{n\to\infty} (E_n)^{1/n} = \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{1}{k!}\right)^{1/n} = \lim_{n\to\infty} \left\{\left(1 + \frac{1}{n}\right)^n\right\}^{1/n} = \lim_{n\to\infty} (1 + 1/n) = 1, \text{so,}$$

$$\lim_{n\to\infty} \left(a^{\sqrt[n]{E_n}-1} - 1\right) = 0.$$

Let $y = \lim_{n\to\infty} (n!)^{1/n}$. Then

$$\ln(y) \quad = \quad (1/n)\ln(n!) \to$$

$$
\begin{aligned}
\ln(y) &= (1/n)\ln(1\cdot 2\cdot 3\cdots n) \text{ and} \\
\ln(y) &= (1/n)\Big(\ln(1)+\ln(2)+\ln(3)+\cdots+\ln(n)\Big) \\
\ln(y) &\approx (1/n)\int_{x=1}^{x=n}\ln(x)dx \\
\ln(y) &= (1/n)\Big(x\ln(x)-x\Big)\Big|_{x=1}^{n} \\
\ln(y) &= (1/n)\Big(n\,(\ln(n))-n\Big) \\
\ln(y) &= \ln(n)-1 \\
\ln(y) &= \ln(n)-\ln e \\
\ln(y) &= \ln\left(\frac{n}{e}\right) \\
y &= \frac{n}{e}
\end{aligned}
$$

So we see that our problem, to evaluate $\lim_{n\to\infty} \sqrt[n]{n!}\left(a^{\sqrt[n]{E}_{n-1}}-1\right)$, is of the form $\infty\cdot 0$, and this allows us to use L'Hospital's rule, to differentiate the numerator and denominator separately with respect to n.

For the numerator, let $u = a^{1/n}-1$.

$$
\begin{aligned}
u &= a^{1/n}-1 \\
(u+1)^n &= a \\
n\ln(u+1) &= \ln(a) \\
\lim_{n\to\infty}\ln(u+1) &= \lim_{n\to\infty}(1/n)\ln(a) \\
\lim_{n\to\infty}\frac{1}{u+1}\frac{du}{dn} &= \lim_{n\to\infty}-\frac{\ln(a)}{n^2} \\
\frac{du}{dn} &= \frac{-(u+1)\ln(a)}{n^2} = -\left(\frac{a^{1/n}\ln(a)}{n^2}\right)
\end{aligned}
$$

For the denominator, $\frac{d}{dn}(e/n) = -\frac{e}{n^2}$.

So,

$$
\lim_{n\to\infty}\frac{\frac{-a^{1/n}\ln(a)}{n^2}}{-\frac{e}{n^2}} = \lim_{n\to\infty}\frac{a^{1/n}\ln(a)}{e}
$$

$$= \frac{\ln a}{e}.$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

It is known that for real x tending to zero, we have $e^x = 1 + x + O\left(x^2\right)$.

Since $\lim\limits_{n\to\infty} E_n = e$, so $\sqrt[n]{E_n} - 1 = e^{\frac{\ln E_n}{n}} - 1 = \dfrac{\ln E_n}{n} + O\left(\dfrac{1}{n^2}\right)$, and

$a^{\sqrt[n]{E_n}-1} - 1 = e^{\left(\sqrt[n]{E_n}-1\right)\ln a} - 1 = \dfrac{(\ln E_n)(\ln a)}{n} + O\left(\dfrac{1}{n^2}\right)$, where the last constant implied by O depends at most on a. Hence, by Stirling's formula

$n! = \sqrt{2\pi n}\left(\dfrac{n}{e}\right)^n\left(1 + O\left(\dfrac{1}{n}\right)\right)$ as n tends to infinity, we obtain

$$\lim_{n\to\infty} \sqrt[n]{n!}\left(a^{\sqrt[n]{E_n}-1} - 1\right) = \frac{\ln a}{e}.$$

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Moti Levy, Rehovot, Israel, and the proposers.

- **5311:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let x, y, z be positive real numbers. Prove that

$$\sum_{cyclic} \sqrt{\left(\frac{x^2}{3} + 3y^2\right)\left(\frac{2}{xy} + \frac{1}{z^2}\right)} \geq 3\sqrt{10}.$$

Solution 1 by Arkady Alt, San Jose, CA

Since by AM-GM Inequality $\dfrac{x^2}{3} + 3y^2 = \dfrac{x^2 + 9y^2}{3} \geq \dfrac{1}{3}\cdot 10\sqrt[10]{x^2\cdot (y^2)^9} = \dfrac{10}{3}\sqrt[5]{xy^9}$ and

$\dfrac{2}{xy} + \dfrac{1}{z^2} \geq 3\sqrt[3]{\left(\dfrac{1}{xy}\right)^2\cdot\dfrac{1}{z^2}} = \dfrac{3}{\sqrt[3]{x^2y^2z^2}}$ then

$\sqrt{\left(\dfrac{x^2}{3} + 3y^2\right)\left(\dfrac{2}{xy} + \dfrac{1}{z^2}\right)} \geq \sqrt{\dfrac{10}{3}\sqrt[5]{xy^9}\cdot\dfrac{3}{\sqrt[3]{x^2y^2z^2}}} \iff \dfrac{\sqrt{10}}{\sqrt[3]{xyz}}\cdot x^{\frac{1}{10}}y^{\frac{9}{10}}$ and, therefore,

using again AM-GM Inequality we obtain

$$\sum_{cyclic}\sqrt{\left(\tfrac{x^2}{3} + 3y^2\right)\left(\tfrac{2}{xy} + \tfrac{1}{z^2}\right)} \geq \frac{\sqrt{10}}{\sqrt[3]{xyz}}\cdot\sum_{cyclic} x^{\frac{1}{10}}y^{\frac{9}{10}} \geq$$

$$\frac{\sqrt{10}}{\sqrt[3]{xyz}}\cdot 3\sqrt[3]{x^{\frac{1}{10}}y^{\frac{9}{10}}\cdot y^{\frac{1}{10}}z^{\frac{9}{10}}\cdot z^{\frac{1}{10}}x^{\frac{9}{10}}} = \frac{\sqrt{10}}{\sqrt[3]{xyz}}\cdot 3\sqrt[3]{xyz} = 3\sqrt{10}.$$

Equality holds if $x = y = z$.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

By the AM-GM inequality

$\frac{x^3}{3}+3y^2=\frac{x^2}{3}+\overbrace{\frac{y^2}{3}+\ldots+\frac{y^2}{3}}^{9\text{ times}}\geq 10\sqrt[10]{\frac{x^2}{3}+\overbrace{\frac{y^2}{3}+\ldots+\frac{y^2}{3}}^{9\text{ times}}}=\frac{\sqrt[5]{10^5xy^9}}{3}$ with equality iff $\frac{x^2}{3}=\frac{y^2}{3}$, that is, iff $x=y$, and

$\frac{2}{xy}+\frac{1}{z^2}=\frac{1}{xy}+\frac{1}{xy}+\frac{1}{z^2}\geq 3\sqrt[3]{\frac{1}{xy}\cdot\frac{1}{xy}\cdot\frac{1}{z^2}}=\frac{3}{\sqrt[3]{x^2y^2z^2}}$ with equality iff $1xy=\frac{1}{z^2}$, that is, iff $xy=z^2$.

Hence,

$$\sqrt{\left(\frac{x^2}{3}+3y^2\right)\left(\frac{2}{xy}+\frac{1}{z^2}\right)}\geq\sqrt{\frac{\sqrt[5]{10^5xy^9}}{3}\frac{3}{\sqrt[3]{x^2y^2z^2}}}=\sqrt[30]{\frac{10^{15}x^3y^{27}}{x^{10}y^{10}z^{10}}}$$

with equality iff $x=y=z$, and cyclically. This and the AM-GM inequality prove the inequality, because

$$\begin{aligned}\sum_{cyclic}\sqrt{\left(\frac{x^2}{3}+3y^2\right)\left(\frac{2}{xy}+\frac{1}{z^2}\right)} &\geq \sum_{cyclic}\sqrt[30]{\frac{10^{15}x^3y^{27}}{x^{10}y^{10}z^{10}}}\\ &\geq 3\sqrt[3]{\prod_{cyclic}\sqrt[30]{\frac{10^{15}x^3y^{27}}{x^{10}y^{10}z^{10}}}}=3\sqrt[3]{\sqrt[30]{\frac{10^{45}x^{30}y^{30}z^{30}}{x^{30}y^{30}z^{30}}}}=3\sqrt{10},\end{aligned}$$

with equality iff $x=y=z$ and $\frac{10^{15}x^3y^{27}}{x^{10}y^{10}z^{10}}=\frac{10^{15}y^3z^{27}}{x^{10}y^{10}z^{10}}=\frac{10^{15}z^3x^{27}}{x^{10}y^{10}z^{10}}$, that is iff $x=y=z$.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, Titu Zvonaru, Comănesti, Romania (jointly with) Neculai Stanciu, "Geroge Emil Palade School," Buzău, Romania, and the proposer.

- **5312:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate:

$$\int_0^1 \ln|\sqrt{x}-\sqrt{1-x}|dx.$$

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

$$\int_0^1 \ln|\sqrt{x}-\sqrt{1-x}|dx = \int_{1/2}^1 \ln(\sqrt{x}-\sqrt{1-x})dx + \int_0^{1/2} \ln(\sqrt{1-x}-\sqrt{x})dx.$$

Moreover,

$$\int_0^{1/2} \ln(\sqrt{1-x}-\sqrt{x})dx \underbrace{=}_{1-x=y} \int_{1/2}^1 \ln(\sqrt{y}-\sqrt{1-y})dy$$

and then,

$$\begin{aligned}
\int_0^1 \ln|\sqrt{x}-\sqrt{1-x}|dx &= 2\int_0^{1/2} \ln(\sqrt{1-x}-\sqrt{x})dx \\
&= \int_0^{1/2} \ln(1-x)dx + 2\int_0^{1/2} \ln\left(1-\sqrt{\frac{x}{1-x}}\right)dx = \left[x = t^2/(1+t^2)\right] \\
&= \int_{1/2}^1 \ln x dx + 2\int_0^1 \ln(1-t)\frac{2t}{(1+t^2)^2}dt \\
&= (x\ln x - x)\Big|_{1/2}^1 + \lim_{a\to 1} 2\frac{t^2}{1+t^2}\ln(1-t)\Big|_0^a + \lim_{a\to 1}\int_0^a \frac{2t^2}{1+t^2}\frac{1}{1-t}dt. \qquad (*)
\end{aligned}$$

$$\begin{aligned}
2\int_0^a \frac{t^2}{1+t^2}\frac{1}{1-t}dt &= 2\int_0^a \left(\frac{1}{1-t} - \frac{1}{(1+t^2)(1-t)}\right)dt \\
&= \int_0^a \left(\frac{2}{1-t} - \frac{1}{1-t} - \frac{1+t}{1+t^2}\right)dt \\
&= \left(-\ln(1-t) - \arctan t - \frac{1}{2}\ln(1+t^2)\right)\Big|_0^a \\
&= -\ln(1-a) - \arctan a - \frac{1}{2}\ln(1+a^2).
\end{aligned}$$

The quantity (*) becomes

$$\frac{1}{2}\ln 2 - \frac{1}{2} + \lim_{a\to 1}\ln(1-a)\left(\frac{2a^2}{1+a^2}-1\right) - \frac{\pi}{4} - \frac{\ln 2}{2} = -\frac{1}{2} - \frac{\pi}{4}.$$

Solution 2 by Kee-Wai Lau, Hong Kong, China

By the substitution $x = \sin^2(\theta/2)$ we have

$$\int_0^1 \ln\left|\sqrt{x}-\sqrt{1-x}\right|dx$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^1 \ln\left(\left(\sqrt{x}-\sqrt{1-x}\right)^2\right) dx \\
&= \frac{1}{2}\int_0^1 \ln\left(1-2\sqrt{x(1-x)}\right) dx \\
&= \frac{1}{4}\int_0^\pi \ln(1-\sin\theta)\sin\theta d\theta \\
&= \frac{-1}{4}\left[\ln(1-\sin\theta)\cos\theta\right]_0^\pi - \frac{1}{4}\int_0^\pi \frac{\cos^2\theta}{1-\sin\theta}d\theta \\
&= \frac{-1}{4}\int_0^\pi (1+\sin\theta)d\theta \\
&= \frac{-1}{4}\left[\theta-\cos\theta\right]\Big|_0^\pi \\
&= -\frac{\pi+2}{4}.
\end{aligned}
$$

Solution 3 by Moti Levy, Rehovot, Israel

Using the symmetry of the integrand and substituting $u = 2x$,

$$
\begin{aligned}
\int_0^1 \ln\left|\sqrt{x}-\sqrt{1-x}\right| dx &= 2\int_0^{\frac{1}{2}} \ln\left(\sqrt{\frac{1}{2}+x}-\sqrt{\frac{1}{2}-x}\right) dx \\
&= -1+\ln\sqrt{2}-\int_0^1 \ln\left(\sqrt{1+u}+\sqrt{1-u}\right) du. \qquad (1)
\end{aligned}
$$

To evaluate the integral in (1), we substitute $u = \cos 2x$, integrate by parts and use the trigonometric equality,

$$
\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)\cos 2x = 1-\sin 2x.
$$

$$
\begin{aligned}
&\int_0^1 \ln\left(\sqrt{1+u}+\sqrt{1-u}\right) du \\
&= 2\int_0^{\frac{\pi}{4}} \ln\left(\sqrt{1+\cos 2x}+\sqrt{1-\cos 2x}\right) \sin 2x dx \\
&= 2\int_0^{\frac{\pi}{4}} \ln\left(\sqrt{2}\left(\cos x+\sin x\right)\right) \sin 2x dx \\
&= -\ln\left(\sqrt{2}\left(\cos x+\sin x\right)\right)\cos 2x\Big]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)\cos 2x dx \\
&= \ln\sqrt{2} + \int_0^{\frac{\pi}{4}} \left(1-\sin 2x\right) dx \\
&= \ln\sqrt{2} + \frac{\pi}{4} - \frac{1}{2}. \qquad (2)
\end{aligned}
$$

By (1) and (2), we obtain,

$$\int_0^1 \ln\left|\sqrt{x}-\sqrt{1-x}\right| dx = -\frac{\pi}{4} - \frac{1}{2}.$$

Solution 4 by Brian D. Beasely, Presbyterian College, Clinton, SC

We denote the given integral by I and let $A = \int_0^{1/2} \ln(\sqrt{1-x}+\sqrt{x})dx$ and $B = \int_0^{1/2} \ln(\sqrt{1-x}-\sqrt{x})dx$. We then show that $A+B = -1/2$ and $A-B = \pi/4$, so we conclude that

$$I = 2B = -1/2 - \pi/4.$$

Using L'Hopital's Rule, we have

$$A+B = \int_0^{1/2} \ln(1-2x)dx = \frac{(1-2x)\ln(1-2x)-(1-2x)}{-2} = -\frac{1}{2}.$$

Next, we integrate by parts to calculate $A-B$:

$$
\begin{aligned}
\int \ln\left(\frac{\sqrt{1-x}+\sqrt{x}}{\sqrt{1-x}-\sqrt{x}}\right) dx &= \left(x-\frac{1}{2}\right)\ln\left(\frac{\sqrt{1-x}+\sqrt{x}}{\sqrt{1-x}-\sqrt{x}}\right) + \int \frac{1}{2\sqrt{x(1-x)}}dx \\
&= \left(x-\frac{1}{2}\right)\ln\left(\frac{\sqrt{1-x}+\sqrt{x}}{\sqrt{1-x}-\sqrt{x}}\right) + \sin^{-1}(\sqrt{x}) + C.
\end{aligned}
$$

Using L'Hopital once again, we conclude $A-B = \pi/4$ as needed.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego,Viveiro, Spain; Ed Gray, Highland Beach, FL, and the proposer.