Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before June 15, 2014

- **5301:** *Proposed by Kenneth Korbin, New York, NY*

  A convex cyclic quadrilateral with integer length sides is such that its area divided by its perimeter equals 2014.

  Find the maximum possible perimeter.

- **5302:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  If n is an even perfect number, $n > 6$, and $\phi(n)$ is the Euler phi-function, then show that $n - \phi(n)$ is a fourth power of an integer. Find infinitely many integers n such that $n - \phi(n)$ is a fourth power.

- **5303:** *Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

  Let a, b, c, d be positive real numbers. Prove that

  $$a^4 + b^4 + c^4 + d^4 + 4 \ \geq \ 4\left(\left(a^2b^2+1\right)\left(b^2c^2+1\right)\left(c^2d^2+1\right)\left(d^2a^2+1\right)\right)^{1/4}.$$

- **5304:** *Proposed by Michael Brozninsky, Central Islip, NY*

  Determine whether or not there exist nonzero constants a and b such that the conic whose polar equation is

  $$r = \sqrt{\frac{a}{\sin(2\theta) - b\cdot\cos(2\theta)}}$$

  has a rational eccentricity.

- **5305:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let x be a positive real number. Prove that

  $$\frac{[x]}{2x+\{x\}} + \frac{[x]\{x\}}{3x^2} + \frac{\{x\}}{2x+[x]} \ \leq \ \frac{1}{2},$$

  where $[x]$ is the greatest integer function and $\{x\}$ is the fractional part of the real number. I.e., $\{x\} = x - [x]$.

- **5306:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Calculate:
$$\int_0^1 \frac{\ln\left(1-x+x^2\right)}{x-x^2}dx.$$

Solutions

- **5283:** *Proposed by Kenneth Korbin, New York, NY*

  Find the sides of two different isosceles triangles that both have perimeter 162 and area 1008.

Solution 1 by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX

To begin, we will let the isosceles triangle be designated with sides $(a,\, a,\, x)$ and height h. With given perimeter 162,

$$\begin{aligned} x &= 162-2a & (1)\\ \frac{x}{2} &= 81-a, & (2) \end{aligned}$$

and, using the Pythagorean Theorem and (2),

$$\begin{aligned} h^2+\left(\frac{x}{2}\right)^2 &= a^2\\ h^2+(81-a)^2 &= a^2\\ h &= 9\sqrt{2a-81}. \end{aligned}$$

Thus, with given area 1008, (1), and (3),

$$\begin{aligned} \frac{1}{2}(162-2a)(9\sqrt{2a-81}) &= 1008\\ \frac{112}{81-a} &= \sqrt{2a-81}\\ 2\,a^3-405\,a^2+26,244\,a-543,985 &= 0. \end{aligned}$$

Using Mupad, the solutions are

$$a=\frac{275-7\sqrt{177}}{4}, \quad 65, \quad \frac{7\sqrt{177}+275}{4}.$$

Using (1), $a=\dfrac{7\sqrt{177}+275}{4}$ does not yield a triangle with perimeter 162. Hence, using (1), when $a=\dfrac{275-7\sqrt{177}}{4}$, $x=\dfrac{49+7\sqrt{177}}{2}$, and when $a=65$, $x=32$. Therefore, the isosceles triangles are $\left(\dfrac{275-7\sqrt{177}}{4}, \dfrac{275-7\sqrt{177}}{4}, \dfrac{49+7\sqrt{177}}{2}\right)$ and (65, 65, 32).

With some persistence, these solutions can be verified to yield an isosceles triangle with perimeter 162 and area 1008.

Solution 2 by Arkady Alt, San Jose, CA

Let b be length of the lateral sides and a be half of length of the base.

Then $\begin{cases} 2a+2b=162 \\ a\sqrt{b^2-a^2}=1008 \end{cases} \iff \begin{cases} a+b=81 \\ a\sqrt{b-a}=112 \end{cases} \iff \begin{cases} b=81-a \\ a\sqrt{81-2a}=112 \end{cases}$

We have $a\sqrt{81-2a}=112 \iff \begin{cases} 0<a\leq 81/2 \\ a^2(81-2a)=112^2 \end{cases}$ and the equation

$a^2(81-2a)=16^2\cdot 49 \iff 2a^3-81a^2+112^2=0.$

Since $2a^3-81a^2+112^2=(a-16)\left(2a^2-49a-784\right)$ and the quadratic equation

$2a^2-49a-784=0$ have only one positive root $a=\dfrac{49+7\sqrt{177}}{4}$ then we obtain two different isosceles triangles with side-lengths

$(b,2a,b)=(65,32,65)\,,\quad \left(\dfrac{275-7\sqrt{177}}{4},\dfrac{49+7\sqrt{177}}{2},\dfrac{275-7\sqrt{177}}{4}\right).$

Solution 3 by Kee-Wai Lau, Hong Kong, China

Let the sides of the isosceles triangles be $a,a,162-2a$. By Heron's formula for the area of a triangle we obtain

$$(81-a)\sqrt{2a-81}=112,$$

or

$$(81-a)^2(2a-81)-12544=0,$$

or

$$(a-65)((2a^2-275a+8369)=0.$$

Hence $a=65,\ \dfrac{275-7\sqrt{177}}{4}$. So the sides of the isosceles triangles are $65,\ 65,\ 32$ and $\dfrac{275-7\sqrt{177}}{4},\dfrac{275-7\sqrt{177}}{4},\dfrac{7(7+\sqrt{177})}{2}$.

Solution 4 by Brian D. Beasley, Presbyterian College, Clinton, SC

Any such triangle has sides with lengths x, x, and $162-2x$, where $81/2<x<81$. Heron's formula then implies

$$1008^2=81(81-x)^2(2x-81),$$

which in turn is equivalent to

$$2x^3-405x^2+26244x-543985=(x-65)(2x^2-275x+8369)=0.$$

We find three real solutions to this equation, namely $x=65$ and $x=(275\pm 7\sqrt{177})/4$; however, one of these yields $x\approx 92.032$, which is outside the necessary domain. Hence we obtain two triangles, corresponding to $x=65$ and $x\approx 45.468$:

$$(65,65,32);\qquad \left(\frac{275-7\sqrt{177}}{4},\frac{275-7\sqrt{177}}{4},\frac{49+7\sqrt{177}}{2}\right)\approx(45.468,45.468,71.064).$$

Question. In general, if we seek all isosceles triangles of the form $(x, x, P-2x)$ that have perimeter P and area A, then we obtain the equation

$$16Px^3 - 20P^2x^2 + 8P^3x - (P^4 + 16A^2) = 0.$$

The given values $P = 162$ and $A = 1008$ produce exactly two such triangles. For what values of P and A would we find no triangles, one triangle, two triangles, or three triangles?

Also solved by Bruno Salgueiro Fanego, Viveriro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Angel Plaza, Universidad de Las Palmas, de Gran Canaria, Spain; Michael Thew, Student, St. George's School, Spokane, WA; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5284:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

Prove:

a) $3^{3^n} + 1 \equiv 0 \bmod 28,\ \forall \text{n} \geq 1,$

b) $3^{3^n} + 1 \equiv 0 \bmod 532,\ \forall \text{n} \geq 2,$

c) $3^{3^n} + 1 \equiv 0 \bmod 19684,\ \forall \text{n} \geq 3,$

d) $3^{3^n} + 1 \equiv 0 \bmod 3208492,\ \forall \text{n} \geq 4.$

Solution 1 by David E. Manes, SUNY College at Oneonta, Oneonta, NY

Note the following congruences:

$$\begin{aligned} 3 &\equiv -1 \bmod 4,\ \ 3^3 \equiv -1 \bmod 7,\ \ 3^9 \equiv -1 \bmod 19 \\ 3^{27} &\equiv -1 \bmod 37,\ \ 3^{81} \equiv -1 \bmod 63 \end{aligned}$$

.

Therefore,

(1) $3^{3^n} + 1 \equiv (-1)^{3^n} + 1 \equiv -1 + 1 \equiv 0 \pmod 4\ \ \forall n \geq 1,$

(2) $3^{3^n} + 1 \equiv \left(3^3\right)^{3^{n-1}} + 1 \equiv (-1)^{3^{n-1}} + 1 \equiv -1 + 1 \equiv 0 \pmod 7\ \ \forall n \geq 1,$

(3) $3^{3^n} + 1 \equiv \left(3^{3^2}\right)^{3^{n-2}} + 1 \equiv (-1)^{3^{n-2}} + 1 \equiv -1 + 1 \equiv 0 \pmod{19}\ \ \forall n \geq 2,$

(4) $3^{3^n} + 1 \equiv \left(3^{3^3}\right)^{3^{n-3}} + 1 \equiv (-1)^{3^{n-3}} + 1 \equiv -1 + 1 \equiv 0 \pmod{37}\ \ \forall n \geq 3,$

(5) $3^{3^n} + 1 \equiv \left(3^{3^4}\right)^{3^{n-4}} + 1 \equiv (-1)^{3^{n-4}} + 1 \equiv -1 + 1 \equiv 0 \pmod{163}\ \ \forall n \geq 4.$

Recall the elementary property of congruences : if $a \equiv b \bmod m$ and $a \equiv b \bmod n$ and $\gcd(m, n) = 1$, then $a \equiv b \bmod m \cdot n$

Therefore,

(a) since $\gcd(4, 7) = 1$, it follows from (1) and (2) that $3^{3^n} + 1 \equiv 0 \bmod 28\ \forall n \geq 1,$

(b) since $\gcd(19,28)=1$, it follows from (a) and (3) that $3^{3^n}+1\equiv 0 \bmod 532 = 19\cdot 28$ $\forall n\geq 2$,

(c) since $\gcd(37,532)=1$, it follows from (b) and (4) that $3^{3^n}+1\equiv 0 \bmod 19684$ $\forall n\geq 3$,

(d) since $\gcd(163,19684)=1$, it follows from (c) and (5) that $3^{3^n}+1\equiv 0 \bmod$ 3208492 $\forall n\geq 4$. This completes the solution.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We have
$28{=}2^2\times 7,\ 532=2^2\times 7\times 19,\ 19684=2^2\times 7\times 19\times 37,\ 3208492=2^2\times 7\times 19\times 37\times 163,$
$3^3\equiv -1(\text{mod } 28), 3^9\equiv -1(\text{mod } 19),\ 3^{27}\equiv -1(\text{mod } 37),\ 3^{81}\equiv -1(\text{mod } 163).$

Statement a) is true fore $n=1$, statement b) is true for $n=2$, statement c) is true for $n=3$, statement d) is true for $n=4$.

The general statment then follows by induction: If $3^{3^n}\equiv -1 \pmod{a}$ where $(a,3)=1$ then $3^{3^{n+1}}\equiv \left(3^{3^n}\right)^3\equiv (-1)^3\equiv -1(\text{mod} a)$.

Also solved by Arkady Alt, San Jose, CA; Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Brian D. Beasley, Presbyterian College, Clinton, SC; D.M. Bătinetu−Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "Geroge Emil Palade" General School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5285:** *Proposed by D.M. Bătinetu− Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "Geroge Emil Palade" General School, Buzu, Romania*

Let $\{a_n\}_{n\geq 1}$, and $\{b_n\}_n\geq 1$ be positive sequences of real numbers with
$\lim\limits_{n\to\infty}(a_{n+1}-a_n)=a\in\Re_+$ and $\lim\limits_{n\to\infty}\dfrac{b_{n+1}}{nb_n}=b\in\Re_+$.
For $x\in\Re$, calculate

$$\lim_{n\to\infty}\left(a_n^{\sin^2 x}\left(\left(\sqrt[n+1]{b_{n+1}}\right)^{\cos^2 x}-\left(\sqrt[n]{b_n}\right)^{\cos^2 x}\right)\right).$$

Solution 1 by Arkady Alt, San Jose, CA

Since the $\lim\limits_{n\to\infty}(a_{n+1}-a_n)=a$, then by the Stolz Theorem $\lim\limits_{n\to\infty}\dfrac{a_n}{n}=a$. Also note that

$$\lim_{n\to\infty}\frac{\dfrac{b_{n+1}}{(n+1)!}}{\dfrac{b_n}{n!}}=\lim_{n\to\infty}\frac{b_{n+1}}{(n+1)\,b_n}=\lim_{n\to\infty}\frac{b_{n+1}}{(n+1)\,b_n}\cdot\frac{n+1}{n}=\lim_{n\to\infty}\frac{b_{n+1}}{nb_n}=b.$$

By the Multiplicative Stolz Theorem $\lim\limits_{n\to\infty} \dfrac{\dfrac{b_n}{n!}}{\dfrac{b_{n+1}}{(n+1)!}} = b$ yields $\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{b_n}{n!}} = b.$

Let $c_n = \dfrac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} = \dfrac{\sqrt[n+1]{\dfrac{b_{n+1}}{(n+1)!}}}{\sqrt[n]{\dfrac{b_n}{n!}}} \cdot \dfrac{\dfrac{\sqrt[n+1]{(n+1)!}}{n+1}}{\dfrac{\sqrt[n]{n!}}{n}} \cdot \dfrac{n+1}{n}.$

Since $\lim\limits_{n\to\infty} \dfrac{\sqrt[n]{n!}}{n} = \dfrac{1}{e}, \lim\limits_{n\to\infty} \sqrt[n]{\dfrac{b_n}{n!}} = b,\ \lim\limits_{n\to\infty} \dfrac{n+1}{n} = 1$ then $\lim\limits_{n\to\infty} c_n = 1,$ and, therefore, $\lim\limits_{n\to\infty} \dfrac{c_n^{\cos^2 x} - 1}{\ln\left(c_n^{\cos^2 x}\right)} = 1.$

Hence, $\lim\limits_{n\to\infty} n\left(c_n^{\cos^2 x} - 1\right) = \lim\limits_{n\to\infty} \left(n \ln\left(c_n^{\cos^2 x}\right) \cdot \dfrac{c_n^{\cos^2 x} - 1}{\ln\left(c_n^{\cos^2 x}\right)}\right) = \lim\limits_{n\to\infty} n \ln\left(c_n^{\cos^2 x}\right) =$

$\cos^2 x \lim\limits_{n\to\infty} n \ln c_n = \cos^2 x \ln\left(\lim\limits_{n\to\infty} c_n^n\right) = \cos^2 x \ln\left(\lim\limits_{n\to\infty} \dfrac{\sqrt[n+1]{b_{n+1}^n}}{b_n}\right).$

Since $\dfrac{\sqrt[n+1]{b_{n+1}^n}}{b_n} = \dfrac{b_{n+1}}{nb_n} \cdot \dfrac{1}{\sqrt[n+1]{\dfrac{b_{n+1}}{(n+1)!}}} \cdot \dfrac{n}{\sqrt[n+1]{(n+1)}}$, then $\lim\limits_{n\to\infty} \dfrac{\sqrt[n+1]{b_{n+1}^n}}{b_n} = b \cdot \dfrac{1}{b} \cdot e = e$

and, therefore, $\lim\limits_{n\to\infty} n\left(c_n^{\cos^2 x} - 1\right) = \cos^2 x.$

And since $a_n^{\sin^2 x}\left(\left(\sqrt[n+1]{b_{n+1}}\right)^{\cos^2 x} - \left(\sqrt[n]{b_n}\right)^{\cos^2 x}\right) =$

$\left(\dfrac{a_n}{n}\right)^{\sin^2 x} \cdot \left(\sqrt[n]{\dfrac{b_n}{n!}}\right)^{\cos^2 x} \cdot \left(\dfrac{\sqrt[n]{n!}}{n}\right)^{\cos^2 x} \cdot n\left(\left(\dfrac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^{\cos^2 x} - 1\right)$ then

$\lim\limits_{n\to\infty}\left(a_n^{\sin^2 x}\left(\left(\sqrt[n+1]{b_{n+1}}\right)^{\cos^2 x} - \left(\sqrt[n]{b_n}\right)^{\cos^2 x}\right)\right) =$

$a^{\sin^2 x} b^{\cos^2 x} e^{-\cos^2 x} \lim\limits_{n\to\infty} n\left(c_n^{\cos^2 x} - 1\right) = a^{\sin^2 x} b^{\cos^2 x} e^{-\cos^2 x} \cos^2 x.$

Solution 2 by Perfetti Paolo, Department of Mathematics, "Tor Vergata" University, Rome, Italy

$$\left(a_n^{\sin^2 x}\left(\left(\sqrt[n+1]{b_{n+1}}\right)^{\cos^2 x} - \left(\sqrt[n]{b_n}\right)^{\cos^2 x}\right)\right)$$

$$= \left(\frac{a_n}{n}\right)^{\sin^2 x} n^{\sin^2 x} b_n^{\frac{\cos^2 x}{n}} \left(\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^{\cos^2 x} - 1\right)$$

$$= \left(\frac{a_n}{n}\right)^{\sin^2 x} n \left(\frac{b_n^{\frac{1}{n}}}{n}\right)^{\cos^2 x} \left(\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^{\cos^2 x} - 1\right)$$

$$= \left(\frac{a_n}{n}\right)^{\sin^2 x} \left(\frac{b_n^{\frac{1}{n}}}{n}\right)^{\cos^2 x} \frac{\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^{\cos^2 x} - 1}{\ln\left(\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^{\cos^2 x}\right)} \ln\left(\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^{n\cos^2 x}\right).$$

By Cesaro-Stolz,

$$\lim_{n\to\infty} \frac{b_n^{1/n}}{n} = \lim_{n\to\infty} \left(\frac{b_n}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{b_{n+1}}{nb_n} \frac{n^{n+1}}{(n+1)^{n+1}} = \frac{b}{e}$$

and

$$\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} (a_{n+1} - a_n).$$

Now

$$\lim_{n\to\infty} \frac{\left(b_{n+1}\right)^{\frac{\cos^2 x}{n+1}}}{\left(b_n\right)^{\frac{\cos^2 x}{n}}} = \lim_{n\to\infty} \frac{\left(b_{n+1}\right)^{\frac{\cos^2 x}{n+1}}}{(n+1)^{\cos^2 x}} \frac{n^{\cos^2 x}}{(n+1)^{\cos^2 x}} \frac{n^{\cos^2 x}}{\left(b_n\right)^{\frac{\cos^2 x}{n}}} = \frac{b^{\cos^2 x}}{e^{\cos^2 x}} \cdot 1 \cdot \frac{e^{\cos^2 x}}{b^{\cos^2 x}} = 1.$$

Moreover,

$$\lim_{n\to\infty} \left(\frac{\left(b_{n+1}\right)^{\frac{\cos^2 x}{n+1}}}{\left(b_n\right)^{\frac{\cos^2 x}{n}}}\right)^n = \lim_{n\to\infty} \frac{\left(b_{n+1}\right)^{\cos^2 x}}{\left(b_{n+1}\right)^{\frac{\cos^2 x}{n+1}}} \frac{1}{\left(b_n\right)^{\cos^2 x}}$$

$$= \lim_{n\to\infty} \frac{\left(b_{n+1}\right)^{\cos^2 x}}{n^{\cos^2 x}\left(b_n\right)^{\cos^2 x}} \frac{n^{\cos^2 x}}{(n+1)^{\cos^2 x}} \frac{(n+1)^{\cos^2 x}}{\left(b_{n+1}\right)^{\frac{\cos^2 x}{n+1}}} = b^{\cos^2 x} \cdot 1 \cdot \frac{e^{\cos^2 x}}{b^{\cos^2 x}} = e^{\cos^2 x}$$

The limit is thus

$$a^{\sin^2 x} \cdot \frac{b^{\cos^2 x}}{e^{\cos^2 x}} \cdot 1 \cdot \cos^2 x.$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland

By assumption $a_{n+1} - a_n = a + o(1)$, $\dfrac{b_{n+1}}{nb_n} = be^{o(1)}$, as $n \to \infty$. So,

$$a_n = a_1 + \sum_{j=2}^{n} (a_j - a_{j-1}) = na + o(n),\ b_n = \frac{n!b_1}{n} \prod_{j=2}^{n} \frac{b_j}{(j-1)b_{j-1}} = n!b^n e^{o(n)} = n^n e^{-n} b^n e^{o(n)},\ \text{as } n \to \infty.$$

We have used a weak form of Stirling's formula, namely $n! = n^n e^{-n+o(n)}$ as $n \to \infty$.

We conclude

$$\begin{aligned}
&\left(a_n^{\sin^2 x}\left(\left(\sqrt[n+1]{b_{n+1}}\right)^{\cos^2 x} - \left(\sqrt[n]{b_n}\right)^{\cos^2 x}\right)\right) = \\
&= n^{\sin^2 x}\left(a+o(1)\right)^{\sin^2 x}\left(\left((n+1)\,be^{-1+o(1)}\right)^{\cos^2 x} - \left(nbe^{-1+o(1)}\right)^{\cos^2 x}\right) = \\
&= n^{\sin^2 x+\cos^2 x}\left(a+o(1)\right)^{\sin^2 x}\left(be^{-1+o(1)}\right)^{\cos^2 x}\left(\left(1+\frac{1}{n}\right)^{\cos^2 x} - 1\right) = \\
&= n\left(a+o(1)\right)^{\sin^2 x}\left(be^{-1+o(1)}\right)^{\cos^2 x}\left(\frac{\cos^2 x}{n} + O\left(\frac{1}{n^2}\right)\right) \\
&= \left(a+o(1)\right)^{\sin^2 x}\left(be^{-1+o(1)}\right)^{\cos^2 x}\left(\cos^2 x + O\left(\frac{1}{n}\right)\right) \\
&\to a^{\sin^2 x} b^{\cos^2 x} e^{-\cos^2 x}\cos^2 x \text{ as } n \to \infty.
\end{aligned}$$

Comment by Bruno Salgueiro Fanego, Viveiro, Spain

A more general question can be seen in problem 75 from the journal *Mathproblems,* available at < http://mathproblems-ks.com/?wpfb_d1=11> (see page 2) and solved at < http://mathproblems-ks.com/?wpfb_d1=17>(see pages 6-8)>

Also solved by Kee-Wai Lau, Hong Kong, China, and the proposer.

- **5286:** *Proposed by Michael Brozinsky, Central Islip, NY*

In Cartesianland, where immortal ants live, an ant is assigned a specific equilateral triangle EFG and three distinct positive numbers $0 < a < b < c$. The ant's job is to find a unique point $P(x, y)$ such that the distances from P to the vertices E, F and G of his triangle are proportionate to $a : b : c$ respectively. Some ants are eternally doomed to an impossible search. Find a relationship between a, b and c that guarantees eventual success; i.e., that such a unique point P actually exists.

Solution 1 by David E. Manes, SUNY College at Oneonta, Oneonta, NY

Let s be the length of the side of $\triangle EFG$ and suppose we are given three distinct positive integers $0 < a < b < c$ such that $a + b > c, b + c > a$ and $c + a > b$.

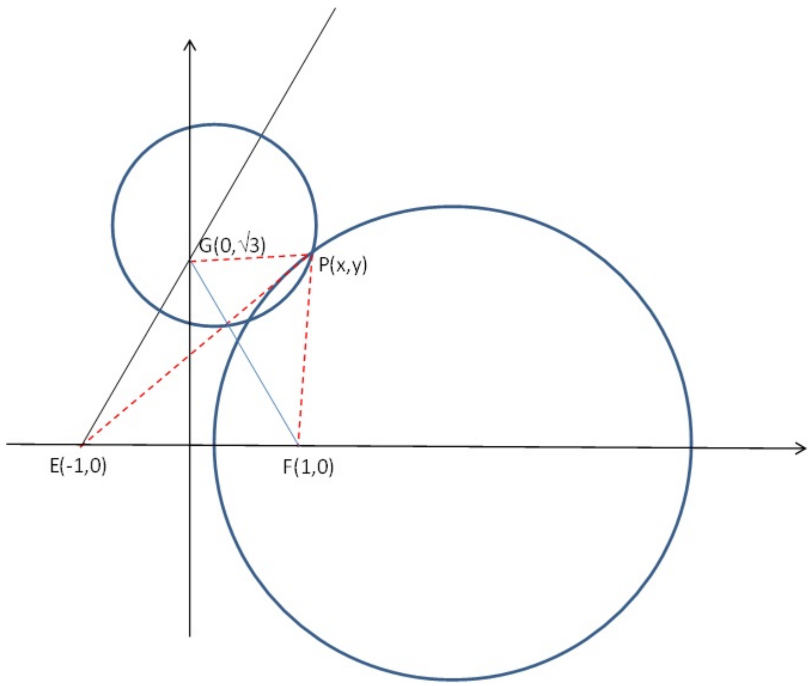
Recall the following: the symmetric equation

$$3\left(x^4 + y^4 + z^4 + w^4\right) = \left(x^2 + y^2 + z^2 + w^2\right)$$

relates the size of an equilateral triangle ABC to the distances of a point from its three vertices. Substituting a, b and c for x, y and z respectively and solving for w then gives the triangle's side (say $w = s'$) and the existence of a point P'. By Pompeiu's Theorem, if P' is an arbitrary point an equilateral triangle ABC, then there exists a triangle with sides of length $P'A, P'B, P'C$. Moreover, the theorem remains valid for any point P' in the plane of triangle ABC and that the triangle is degenerate if and only if P' lies on the circumcircle of $\triangle ABC$. Therefore, $a+b>c, b+c>a$ and $c+a>b$. Finally using a dilation transformation from $\triangle ABC$ to $\triangle EFG$ with a dilation factor of $\frac{s}{s'}$, it follows that there exists a point $P = P'\left(\frac{s}{s'}\right)$ whose distances from the three vertices are $PE = a\left(\frac{s}{s'}\right), PF = b\left(\frac{s}{s'}\right)$ and $PG = c\left(\frac{s}{s'}\right)$. Hence, $\frac{PE}{a} = \frac{PF}{b} = \frac{PG}{c} = \frac{s}{s'}$ so that the distancs from P to the vertices E, F and G are proportionate to $a:b:c$ respectively.

Solution 2 by Michael Fried, Ben Gurion University, Beer-Sheva, Israel

Since this is Cartesianland, we might as well place the equilateral triangle in the Cartesian plane and give the vertices convenient coordinates, say, $E = (-1, 0)$, $F = (1, 0)$, and $G = (0, \sqrt{3})$ (see figure below.)



Let us set $\alpha = b/c = PE/PF$, $\beta = a/c = PG/PF$, and $\gamma = a/b = PG/PE$.

Then the locus of points P with $PE/PF = \alpha$ is the Apollonius circle:

$$\alpha^2\left((x-1)^2+y^2\right) - \left((x+1)^2+y^2\right) = 0$$

Similarly, the locus of points P with $PG/PE = \gamma$ is the Apollonius circle:

$$\gamma^2\left((x+1)^2+y^2\right) - \left(x^2+(y-\sqrt{3})^2\right) = 0$$

The condition that the system of equations,

$$\alpha^2\left((x-1)^2+y^2\right) - \left((x+1)^2+y^2\right) = 0$$

$$\gamma^2\left((x+1)^2+y^2\right) - \left(x^2+(y-\sqrt{3})^2\right) = 0$$

has a solution, that is, that the two Apollonius circles have an intersection is (after some messy but routine algebra) is:

$$\Delta = 16\left[\left(\gamma^2\gamma^2 + (\alpha^2+1)\right)^2 - \left((2\alpha^2\gamma^2 - (\alpha^2+1))^2 - 3(\alpha^2-1)^2\right)\right] \geq 0$$

After some further manipulation, this come down to the inequality:

$$\left(\alpha^2\gamma^2 - (\alpha+1)^2\right)\left(\alpha^2\gamma^2 - (\alpha-1)^2\right) \leq 0$$

From which we have the condition:

$$\left(1 - \frac{1}{\alpha}\right)^2 \leq \gamma^2 \leq \left(1 + \frac{1}{\alpha}\right)^2$$

Or going back to the definition $\alpha = b/c, \gamma = a/b$, we have:

$$\left(1 - \frac{c}{b}\right)^2 \leq \frac{a^2}{b^2} \leq \left(1 + \frac{c}{b}\right)^2$$

So that,

$$(b-c)^2 \leq a^2 \leq (b+c)^2$$

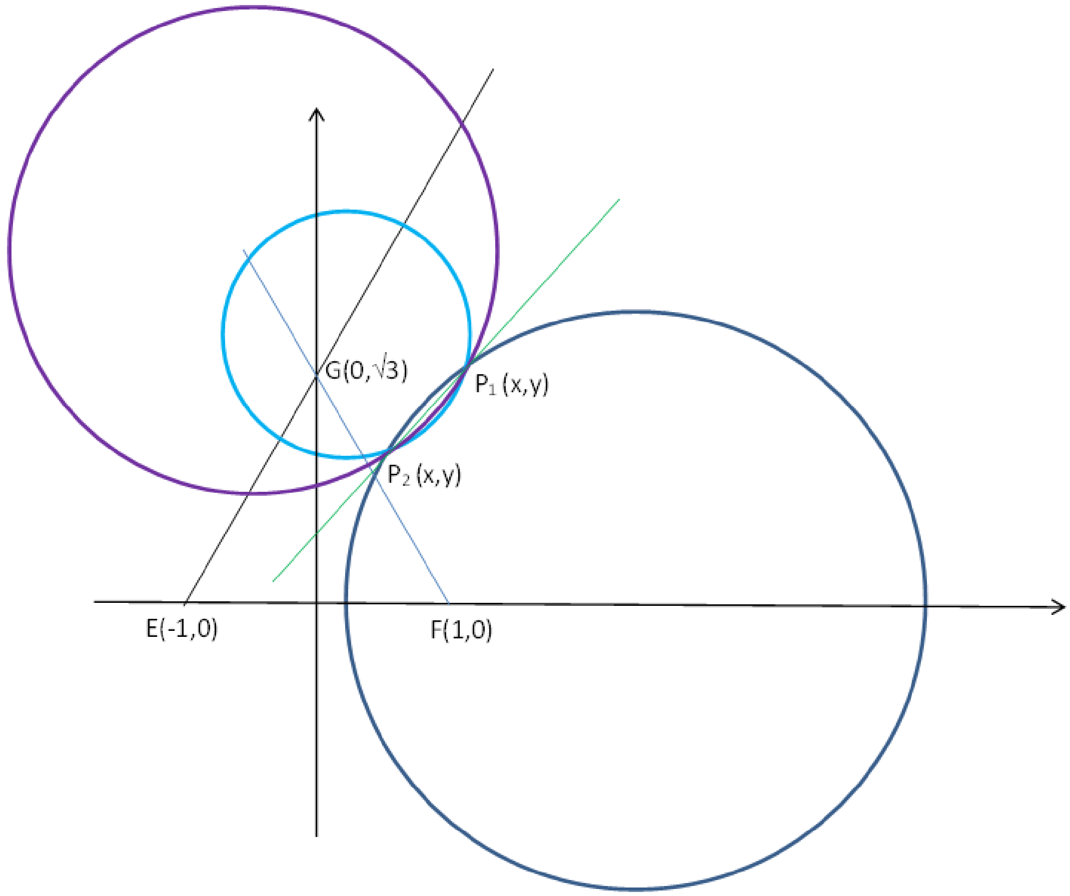
Since a, b, c are positive numbers, and since this must be true no matter which Apollonius circle ratio we begin with, we have the triangle-like inequalities:

$$a \leq b + c$$

$$b \leq a + c$$

$$c \leq a + b$$

One should note that if the circles $\alpha^2\left((x-1)^2 + y^2\right) - \left((x+1)^2 + y^2\right) = 0$ and $\gamma^2\left((x+1)^2 + y^2\right) - \left(x^2 + (y - \sqrt{3})^2\right) = 0$ intersect, they will generally intersect in *two* points P_1 and P_2, where both P_1G/PF and $P_2G/PF = a/c$, and a single Apollonius circle with respect to G and F will pass through these points. Observe too, the three circles then have the same radical axis, namely, P_1P_2 (see figure below).



Comments

1. Ken Korbin, New York, NY

Given $0 < a < b < c$. If it is possible to construct a triangle with sides (a, b, c) in which each of the angles is less than $120°$, them there is a unique point P.

2. Bruno Salgueiro Fanego, Viveiro, Spain

In the article by Oene Bottema *On the distances of a point to the vertices of a triangle.* journal *Crux Mathematicorum,* $1984, 10(8), 242-246$, it is proved (among other things) the following relationship between the lengths of the sides
$\alpha_1 = \angle A_2A_1A_3, \alpha_2 = \angle A_3A_2A_1, \alpha_3 = \angle A_1A_3A_2$ and any point P in the plane of $\triangle A_1A_2A_3$ with distances to the vertices $d_1 = PA_1, d_2 = PA_2, d_3 = PA_3$, then:

$$a_1^2d_1^4 + a_2^2d_2^4 + a_3^2d_3^4 - 2a_2a_3\cos\alpha_1 d_2^2d_3^2 - 2a_3a_1\cos\alpha_2 d_3^2d_1^2 - 2a_1a_2\cos\alpha_3 d_1^2d_2^2 -$$

$$2a_1^2a_2a_3\cos\alpha_1 d_1^2 - 2a_1a_2^2a_3\cos\alpha_2 d_2^2 - 2a_1a_2a_3^2\cos\alpha_3 d_3^2 + a_1^2a_2^2a_3^2 = 0$$

called identity (6) and reciprocally. That is, that if d_1, d_2, d_3 are positive numbers satisfying identity (6) then there is a unique point P such that
$PA_1 = d_1, PA_2 = d_2, PA_3 = d_3$.

This implies that identity(6) is the relationship which solves a problem more generally than the one proposed.

Note: In particular, if we suppose that $A_1A_2A_3$ is the equilateral triangle EFG of the statement of the problem, with sides $e = a_1 = a_2 = a_3$ and k is the constant of proportionality such that $d_1 = ka, d_2 = kb, d_3 = kc$ then identity (6), when divided by e^2 becomes

$$k^4\left(a^4+b^4+c^4\right)+e^4-k^4\left(a^2b^2+a^2c^2+b^2c^2\right)-k^2\left(a^2+b^2+c^2\right)e^2+e^4=0,$$

which is the required relationship in the original statement of the problem.

On the other hand, if we suppose that a point P exists and k is the constant of proportionality, such aht $PE = ka, PF = kb$, and $PG = kc$, using the identity which appears in the editor's comment of SSM problem 5140, or its equivalent,

$$PE^4+PF^4+PG^4+EF^4 = PE^2PF^2+PE^2PG^2+PF^2PG^2+PE^2EF^2+PF^2EF^2+PG^2EF^2,$$

we obtain directly the relationship which is required in the problem, that is,

$$k^4\left(a^4+b^4+c^4\right)+e^4=k^4\left(a^2b^2+a^2c^2+b^2c^2\right)+k^2\left(a^2+b^2+c^2\right)e^2,$$

which is also equivalent to equality (4) in the published solution #2 to 5140.

Also solved by Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS, and the proposer.

- **5287:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let u, v, w, x, y, z be complex numbers. Prove that

$$2\text{Re}(ux+vy+zw) \le 3\left(|u|^2+|v|^2+|w|^2\right)+\frac{1}{3}\left(|x|^2+|y|^2+|z|^2\right).$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We note that

$$0 \le \left|\sqrt{3}u - \frac{1}{\sqrt{3}}\overline{x}\right|^2 = \left(\sqrt{3}u - \frac{1}{\sqrt{3}}\overline{x}\right)\left(\sqrt{3}\overline{u} - \frac{1}{\sqrt{3}}x\right) = 3|u|^2+\frac{1}{3}|x|^2-2\text{Re}(ux).$$

So, $2\text{Re}(ux) \le 3|u|^2+\frac{1}{3}|x|^2$.

Similarly, $2\text{Re}(vy) \le 3|v|^2+\frac{1}{3}|y|^2$, and $2\text{Re}(zw) \le 3|w|^2+\frac{1}{3}|z|^2$.

The statement follows by adding these inequalities.

Solution 2 by David Diminnie and Tatyana Savchuk, Texas Instruments, Inc., Dallas, TX

We will prove the equivalent statement

$$0 \le 3\left(|u|^2+|v|^2+|w|^2\right)-2\text{Re}\left(ux+vy+zw\right)+\frac{1}{3}\left(|x|^2+|y|^2+|z|^2\right). \quad (1)$$

Let u_1, u_2 denote the real and imaginary parts of u, respectively, and similarly for v, w, x, y, z. Then the right side of (1) becomes

$$3\left(u_1^2+u_2^2+v_1^2+v_2^2+w_1^2+w_2^2\right)-2\left(u_1x_1-u_2x_2+v_1y_1-v_2y_2+w_1z_1-w_2z_2\right)$$
$$+\frac{1}{3}\left(x_1^2+x_2^2+y_1^2+y_2^2+z_1^2+z_2^2\right),$$

which we rewrite as

$$\left(3u_1^2-2u_1x_1+\frac{1}{3}x_1^2\right)+\left(3u_2^2+2u_2x_2+\frac{1}{3}x_2^2\right)+\left(3v_1^2-2v_1y_1+\frac{1}{3}y_1^2\right)+\left(3v_2^2+2v_2y_2+\frac{1}{3}y_2^2\right)$$
$$+\left(3w_1^2-2w_1z_1+\frac{1}{3}z_1^2\right)+\left(3w_2^2+2w_2z_2+\frac{1}{3}z_2^2\right). \qquad (2)$$

Noting that $3a^2+2ab+\frac{1}{3}b^2$ and $3a^2-2ab+\frac{1}{3}b^2$ may be rewritten as $\left(\sqrt{3}a+\dfrac{1}{\sqrt{3}}b\right)^2$ and $\left(\sqrt{3}a-\dfrac{1}{\sqrt{3}}b\right)^2$, respectively, (2) becomes

$$\left(\sqrt{3}u_1-\frac{1}{\sqrt{3}}x_1\right)^2+\left(\sqrt{3}u_2+\frac{1}{\sqrt{3}}x_2\right)^2+\left(\sqrt{3}v_1-\frac{1}{\sqrt{3}}y_1\right)^2+\left(\sqrt{3}v_2+\frac{1}{\sqrt{3}}y_2\right)^2$$
$$+\left(\sqrt{3}w_1-\frac{1}{\sqrt{3}}z_1\right)^2+\left(\sqrt{3}w_2+\frac{1}{\sqrt{3}}z_2\right)^2. \qquad (3)$$

Since (3) is a sum of squares of real numbers the expression must be nonnegative, and therefore (1) holds.

Solution 3 by Paul M. Harms, North Newton, KS

We know that the real part of a finite sum of complex numbers is less than or equal to the modulus of the sum which is less than or equal to the sum of the moduli. Also the modulus of a finite product of complex numbers equals the product of the moduli.

We have $0\le\left(3|u|-|x|\right)^2+\left(3|v|-|y|\right)^2+\left(3|w|-|z|\right)^2$. After squaring the three parts, moving terms and dividing by 3, we can obtain,

$$2\left(|u|\,|x|+|v|\,|y|+|z|\,|w|\right)\le 3\left(|u|^2+|v|^2+|w|^2\right)+\frac{1}{3}\left(|x|^2+|y|^2+|z|^2\right).$$

From what was said and shown above,

$$2\mathrm{Re}(ux+vy+zw)\le 2\left(|u|\,|x|+|v|\,|y|+|z|\,|w|\right)\le 3\left(|u|^2+|v|^2+|w|^2\right)+\frac{1}{3}\left(|x|^2+|y|^2+|z|^2\right).$$

Solution 4 by Kee-Wai Lau, Hong Kong, China

We have

$$3|u|^2+\frac{1}{3}|x|^2-2\mathrm{Re}(ux)\ge 3|u|^2+\frac{1}{3}|x|^2-2|u||x|=\frac{1}{3}\left(3|u|-|x|\right)^2\ge 0,$$

and similarly,

$$3|v|^2+\frac{1}{3}|y|^2-2\mathrm{Re}(vy)\ge 0,\quad 3|\mathrm{z}|^2+\frac{1}{3}|\mathrm{w}|^2-2\mathrm{Re}(zw)\ge 0.$$

The inequality of the problem follows by adding up the three inequalities above.

Solution 5 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

$$2\text{Re}(ux+vy+zw) \leq 2(|ux|+|vy|+|zw|) = 2(|u|\cdot|x|+|v|\cdot|y|+|z|\cdot|w|)$$

and

$$|z|\cdot|w| \leq 3|z|^2+\frac{1}{3}|w|^2$$

is simply the AGM.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL and the proposer.

- **5288:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a, b, c \geq 0$ be real numbers. Find the value of

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\sqrt{i^2+j^2+ai+bj+c}}.$$

Solution 1 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

Answer: $2\ln(\sqrt{2}+1)$

Proof: We show that the limit is independent on a, b, c allowing us to set $a=b=c=0$ for evaluating it. If $Q=[0,1]\times[0,1]$, the limit becomes

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i,j=1}^{n} \frac{1}{\sqrt{i^2+j^2}} = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j=1}^{n} \frac{1}{\sqrt{\frac{i^2}{n^2}+\frac{j^2}{n^2}}} = \int\int_Q \frac{1}{\sqrt{x^2+y^2}} dxdy.$$

By writing the integral as $2\int_0^1 \left(\int_0^x \frac{1}{\sqrt{x^2+y^2}}dy\right) dx$ and passing to polar coordinates we have

$$2\int_{\pi/4}^{\pi/2} \left(\int_0^{1/\sin\theta} \frac{\rho}{\rho} d\rho\right) d\theta = 2\int_{\pi/4}^{\pi/2} \frac{1}{\sin\theta} d\theta = 2\ln\tan\frac{\theta}{2}\Big|_{\pi/4}^{\pi/2} = 2\ln(\sqrt{2}+1).$$

To show that the limit is independent by a, b, c, we prove

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i,j=1}^{n} \frac{1}{\sqrt{i^2+j^2+ai+bj+c}} = \lim_{n\to\infty} \frac{1}{n} \sum_{i,j=1}^{n} \frac{1}{\sqrt{i^2+j^2+a'i+b'j+c'}}$$

for any a, b, c, a', b', c'. We introduce a number of positive constants C_k, $k=0,1,\ldots$.

Since $i|a'-a|+j|b'-b|+|c'-c| \leq C_0(i+j)$ and $i^2+j^2+ai+bj+c \leq C_1(i^2+j^2)$ we have the bound

$$\left|\frac{1}{\sqrt{i^2+j^2+ai+bj+c}}-\frac{1}{\sqrt{i^2+j^2+a'i+b'j+c'}}\right|=$$

$$\left|\frac{(a'-a)+j(b'-b)+c'-c}{(i^2+j^2+ai+bj+c)(i^2+j^2+a'i+b'j+c')}\right|\times$$

$$\times\left(\frac{1}{\sqrt{i^2+j^2+ai+bj+c}}+\frac{1}{\sqrt{i^2+j^2+a'i+b'j+c'}}\right)^{-1}\leq$$

$$\leq\frac{C_0(i+j)}{(i^2+j^2)^2}\frac{\sqrt{i^2+j^2}}{C_1}=C_2\frac{i+j}{(i^2+j^2)^{3/2}}$$

Thus

$$\frac{1}{n}\sum_{i,j=1}^{n}\frac{i+j}{(i^2+j^2)^{3/2}}\leq\frac{1}{n}\sum_{j=1}^{\infty}\sum_{i=1}^{n}\frac{i}{(2ij)^{3/2}}+\frac{1}{n}\sum_{i=1}^{\infty}\sum_{j=1}^{n}\frac{j}{(2ij)^{3/2}}\leq C_3/\sqrt{n}$$

and it follows that for any a,b,c,a',b',c'

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i,j=1}^{n}\left(\frac{1}{\sqrt{i^2+j^2+ai+bj+c}}-\frac{1}{\sqrt{i^2+j^2+a'i+b'j+c'}}\right)=0.$$

In particular we can take $a'=b'=c'=0$ and write

$$\frac{1}{\sqrt{i^2+j^2+ai+bj+c}}=\left(\frac{1}{\sqrt{i^2+j^2+ai+bj+c}}-\frac{1}{\sqrt{i^2+j^2}}\right)+\frac{1}{\sqrt{i^2+j^2}}$$

The conclusion is that for any a,b,c the limit assumes the same value $2\ln(\sqrt{2}+1)$.

Solution 2 by Ed Gray, Highland Beach, FL

Consider the integral

$$\lim_{n\to\infty}\frac{1}{n}\int_{x=1}^{x=n}\int_{y=1}^{y=n}\frac{dxdy}{\sqrt{x^2+y^2}}.$$

(*Editor's comment: Ed used intuition in moving from the double summation to the double integral by reasoning that the linear terms in the summation wouldn't contribute much to the summation for very large values of n. His intuition was right on target, as seen in Paolo's solution above. Ed evaluated the double integral in the usual manner, by first integrating the inside integral with respect to x treating y as a constant, and then integrating that answer with respect to y, treating x as a constant.*

$$\int_{x=1}^{n}\frac{dx}{\sqrt{x^2+y^2}} \quad = \quad \ln\left(\sqrt{x^2+y^2}+x\right)-\ln y\Big|_{x=1}^{n}$$

$$= \ln\left(\sqrt{n^2+y^2}+n\right) - \ln y - \ln\left(\sqrt{1^2+y^2}+1\right) + \ln y$$

$$= \ln\left(\sqrt{n^2+y^2}+n\right) - \ln\left(\sqrt{1^2+y^2}+1\right)$$

And now we compute:

$$\int_{y=1}^{n} \ln\left(\sqrt{n^2+y^2}+n\right) dy - \int_{y=1}^{n} \ln\left(\sqrt{1^2+y^2}+1\right) dy.$$

$$\int_{y=1}^{n} \ln\left(\sqrt{n^2+y^2}+n\right) dy = y\ln\left(\sqrt{n^2+y^2}+n\right) + n\ln\left(\sqrt{n^2+y^2}+y\right) - y\Bigg|_{y=1}^{n}$$

$$\left[n\ln\left(\sqrt{n^2+n^2}+n\right) + n\ln\left(\sqrt{n^2+n^2}+n\right) - n\right] - \left[(1)\ln\left(\sqrt{n^2+1}+n\right) + n\ln\left(\sqrt{n^2+1}+1\right) - 1\right]$$

Let's called this **A.** And evaluating

$$\int_{y=1}^{n} \ln\left(\sqrt{y^2+1}+1\right) dy = y\ln\left(\sqrt{y^2+1}+1\right) - y + \ln\left(y+\sqrt{1+y^2}\right)\Bigg|_{y=1}^{n}$$

we obtain

$$n\left[\ln(\sqrt{n^2+1}+1)\right] - n + \ln\left(n+\sqrt{n^2+1}\right) - \left[(1)\left(\ln\left(\sqrt{2}+1\right)\right) - 1 + \ln(1+\sqrt{2})\right].$$

And let's call this **B.**

We now evaluate $\frac{1}{n}\lim_{n\to\infty}\mathbf{A} - \frac{1}{n}\lim_{n\to\infty}\mathbf{B}$. Doing this gives us $2\ln(\sqrt{2}+1)$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that the limit equal $2\ln(1+\sqrt{2})$, independent of a, b, c.

We first note that

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\sqrt{i^2+j^2+ai+bj+c}} - \sum_{i=1}^{n}\sum_{j=1^n}\frac{1}{\sqrt{i^2+j^2}}\right|$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{ai+bj+c}{\left(\sqrt{i^2+j^2+ai+bj+c}\right)\left(\sqrt{i^2+j^2}\right)\left(\sqrt{i^2+j^2+ai+bj+c}+\sqrt{i^2+j^2}\right)}$$

$$\le \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{ai+bj+c}{(i^2+j^2)^{3/2}}$$

$$= \quad O\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{ai+bj+c}{(ij)^{3/2}}\right)$$

$$= \quad O\left(\sum_{i=1}^{n}\frac{1}{i^{1/2}}\sum_{j=1}^{n}\frac{1}{j^{3/2}}\right)+O\left(\sum_{i=1}^{n}\frac{1}{i^{3/2}}\sum_{j=1}^{n}\frac{1}{j^{1/2}}\right)+O\left(\sum_{i=1}^{n}\frac{1}{i^{3/2}}\sum_{j=1}^{n}\frac{1}{j^{3/2}}\right)$$

$$= \quad O(\sqrt{n}).$$

The constants implied by O depend at most on a, b, and c. It follows that the limit of the problem in fact equals $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\sqrt{i^2+j^2}}$. Now the last limit equals

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\sqrt{\left(\frac{i}{n}\right)^2+\left(\frac{j}{n}\right)^2}}=\int_0^1\int_0^1\frac{dydx}{\sqrt{x^2+y^2}},$$

which we are going to evaluate. It is easy to check that

$$\frac{d}{dy}\left(\ln\left(y+\sqrt{x^2+y^2}\right)=\frac{1}{\sqrt{x^2+y^2}}\right.$$

and

$$\frac{d}{dx}\left(\ln\left(x+\sqrt{x_{+}^{2}1}\right)+x\ln\left(1+\sqrt{x^2+1}\right)-\ln x\right)=\ln\left(1+\sqrt{x^2+y^2}\right)-\ln x.$$

Hence

$$\int_0^1\int_0^1\frac{dydx}{\sqrt{x^2+y^2}}=\int_0^1\left(\ln\left(1+\sqrt{x^2+1}\right)-\ln x\right)dx=2\ln\left(1+\sqrt{2}\right),$$

where we have used the fact that $\lim_{x\to0^+}(x\ln x)=0$.
This completes the solution.

Solution 4 by Anastasios Kotronis, Athens, Greece

Let

$$a_n=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\sqrt{i^2+j^2+ai+bj+c}}.$$

We have

$$a_{n+1}-a_n \quad = \quad \sum_{i=1}^{n+1}\frac{1}{\sqrt{i^2+(n+1)^2+ai+b(n+1)+c}}+\sum_{j=1}^{n+1}\frac{1}{\sqrt{(n+1)^2+j^2+a(n+1)+bj+c}}$$

$$-\frac{1}{\sqrt{2(n+1)^2+(a+b)(n+1)+c}}$$

$$= \quad b_{n+1}+c_{n+1}-d_{n+1}$$

But

$$\begin{aligned} b_n &= \sum_{i=1}^{n}\frac{1}{\sqrt{i^2+n^2+ai+bn+c}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{(i/n)^2+1+ai/n^2+b/n+c/n^2}} \\ &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{(i/n)^2+1}}1+\mathcal{O}(n^{-1}) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{(i/n)^2+1}}+\mathcal{O}(n^{-1}) \\ &\to \int_0^1 \frac{1}{\sqrt{x^2+1}}\,dx = \ln(1+\sqrt{2}) \end{aligned}$$

and by symmetry, the same holds for c_n. Since clearly $d_n \to 0$, by Cezàro Stolz

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\sqrt{i^2+j^2+ai+bj+c}} \to 2\ln(1+\sqrt{2}).$$

Comment by Bruno Salgueiro Fanego, Viveiro, Spain

This problem and its solution appeared as challenge exercise $U114$ in the journal *Mathematical Reflections.* See:
< https://www.awesomemath.org/wp-ontent/uploads/reflections/2009_2/MR_2_2009_Solutions.pdf >. Pages 36-38.

The required value is $2\ln\left(\sqrt{2}+1\right)$.

Also solved by Arkady Alt, San Jose, CA; Albert Stadler, Herrliberg, Switzerland, and the proposer.

Mea Culpa

The name of Michael Thew, a student at St. George's School in Spokane, WA was inadvertently omitted from the list of those who had solved 5277 and 5279.