Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before June 15, 2013

- **5254:** *Proposed by Kenneth Korbin, New York, NY*

Five different triangles, with integer length sides and with integer area, each have a side with length 169. The size of the angle opposite 169 is the same in all five triangles. Find the sides of the triangles.

- **5255:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

Let n be a natural number. Let $\phi(n), \sigma(n)$ and $\tau(n)$ be the Euler phi-function, the sum of the different divisors of n and the number of different divisors of n, respectively.
Prove:
(a) $\forall n \geq 2,\ \exists$ natural numbers a and b such that $\phi(a) + \tau(b) = n$.
(b) $\forall k \geq 1,\ \exists$ natural numbers a and b such that $\phi(a) + \sigma(b) = 2^k$.
(c) $\forall n \geq 2,\ \exists$ natural numbers a and b such that $\tau(a) + \tau(b) = n$.
(d) $\forall k \geq 1,\ \exists$ natural numbers a and b such that $\sigma(a) + \sigma(b) = 2^k$.
(e) $\forall n \geq 3,\ \exists$ natural numbers a, b and c such that $\phi(a) + \sigma(b) + \tau(c) = n$
(f) $\exists$ infinitely many natural numbers n such that $\phi(\tau(n)) = \tau(\phi(n))$.

- **5256:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

Let a be a positive integer. Compute:

$$\lim_{n\to\infty} n\left(a - e^{\dfrac{1}{n+1} + \dfrac{1}{n+2} + \ldots + \dfrac{1}{na}}\right).$$

- **5257:** *Proposed by Pedro H.O. Pantoja, UFRN, Brazil*

Prove that:

$$1 + \frac{1}{2}\cdot\sqrt{1+\frac{1}{2}} + \frac{1}{3}\cdot\sqrt[3]{1+\frac{1}{2}+\frac{1}{3}} + \cdots + \frac{1}{n}\cdot\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}} \sim \ln(n),$$

where $f(x) \sim g(x)$ means $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1$.

- **5258:** *Proposed by José Luis Díaz-Barrero and José Gibergans-Báguena, Polytechnical University of Catalonia, Barcelona, Spain*

  Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be real numbers such that $1 + \sum_{k=1}^{n} \cos^2 \alpha_k = n$. Prove that:

$$\sum_{1 \le i < j \le n} \tan \alpha_i \tan \alpha_j \le \frac{n}{2}.$$

Solutions

- **5236:** *Proposed by Kenneth Korbin, New York, NY*

  Given positive numbers (a, b, c, x, y, z) such that

$$\begin{aligned} x^2 + xy + y^2 &= a, \\ y^2 + yz + z^2 &= b, \\ z^2 + zx + x^2 &= c. \end{aligned}$$

  Express the value of the sum $x + y + z$ in terms of a, b, and c.

Solution 1 by David Diminnie, Texas Instruments, Inc., Dallas, TX and Charles R. Diminnie, Angelo State University, San Angelo, TX

From the first two equations, we get

$$\begin{aligned} a - b &= x^2 - z^2 + xy - yz \\ &= (x - z)(x + y + z). \end{aligned}$$

Similarly, combining other pairs of equations yields

$$b - c = (y - x)(x + y + z)$$

and

$$c - a = (z - y)(x + y + z).$$

Hence,

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = (x+y+z)^2 \left[(x-y)^2 + (y-z)^2 + (z-x)^2\right]. \qquad (1)$$

Also, by adding the three equations, we obtain

$$\begin{aligned} a + b + c &= 2\left(x^2 + y^2 + z^2\right) + (xy + yz + zx) \\ &= (x+y+z)^2 + \frac{1}{2}\left[(x-y)^2 + (y-z)^2 + (z-x)^2\right]. \qquad (2) \end{aligned}$$

Then, by (1) and (2),

$$(a+b+c)^2 = (x+y+z)^4 + (x+y+z)^2 \left[(x-y)^2 + (y-z)^2 + (z-x)^2\right]$$

$$
\begin{aligned}
&+\frac{1}{4}\left[(x-y)^2+(y-z)^2+(z-x)^2\right]^2 \\
= \;& (x+y+z)^4+\left[(a-b)^2+(b-c)^2+(c-a)^2\right] \\
&+\frac{1}{4}\left[(x-y)^2+(y-z)^2+(z-x)^2\right]^2.
\end{aligned}
$$

This in turn implies that

$$
\begin{aligned}
& (a+b+c)^2-2\left[(a-b)^2+(b-c)^2+(c-a)^2\right] \\
= \;& (x+y+z)^4-\left[(a-b)^2+(b-c)^2+(c-a)^2\right] \\
&+\frac{1}{4}\left[(x-y)^2+(y-z)^2+(z-x)^2\right]^2 \\
= \;& (x+y+z)^4-(x+y+z)^2\left[(x-y)^2+(y-z)^2+(z-x)^2\right] \\
&+\frac{1}{4}\left[(x-y)^2+(y-z)^2+(z-x)^2\right]^2 \\
= \;& \left[(x+y+z)^2-\frac{1}{2}\left((x-y)^2+(y-z)^2+(z-x)^2\right)\right]^2 \\
= \;& \left[3\left(xy+yz+zx\right)\right]^2.
\end{aligned}
$$

Since $x, y, z > 0$,

$$
3\left(xy+yz+zx\right)=\sqrt{(a+b+c)^2-2\left[(a-b)^2+(b-c)^2+(c-a)^2\right]}.
$$

As a result,

$$
\begin{aligned}
& a+b+c+\sqrt{(a+b+c)^2-2\left[(a-b)^2+(b-c)^2+(c-a)^2\right]} \\
= \;& \left[2\left(x^2+y^2+z^2\right)+\left(xy+yz+zx\right)\right]+3\left(xy+yz+zx\right) \\
= \;& 2\left[\left(x^2+y^2+z^2\right)+2\left(xy+yz+zx\right)\right] \\
= \;& 2\left(x+y+z\right)^2.
\end{aligned}
$$

Finally, since $x, y, z > 0$,

$$
x+y+z=\sqrt{\frac{a+b+c+\sqrt{(a+b+c)^2-2\left[(a-b)^2+(b-c)^2+(c-a)^2\right]}}{2}}.
$$

Solution 2 by David Diminnie, Texas Instruments, Incorporated, Dallas, TX

By summing the three equations in the problem statement we obtain

$$
2x^2+2y^2+2z^2+xy+yz+zx=a+b+c. \qquad (1)
$$

The cross terms may be eliminated from (1) via the change of variables

$$\begin{aligned} x &= \frac{1}{\sqrt{3}}x' - \frac{1}{\sqrt{2}}y' - \frac{1}{\sqrt{6}}z', \\ y &= \frac{1}{\sqrt{3}}x' + \frac{1}{\sqrt{2}}y' - \frac{1}{\sqrt{6}}z', \\ z &= \frac{1}{\sqrt{3}}x' + \sqrt{\frac{2}{3}}z', \end{aligned}$$

yielding

$$3x'^2 + \frac{3}{2}y'^2 + \frac{3}{2}z'^2 = a + b + c. \tag{2}$$

Note that the sum $x + y + z$ becomes

$$x + y + z = \sqrt{3}x' \tag{3}$$

in the new variables, and since x, y, and z are positive x' must also be positive. We may now rewrite the original problem statement in our new variables:

$$\begin{aligned} x'^2 - \sqrt{2}x'z' + \frac{1}{2}y'^2 + \frac{1}{2}z'^2 &= a, \\ x'^2 + \sqrt{\frac{3}{2}}x'y' + \frac{1}{\sqrt{2}}x'z' + \frac{1}{2}y'^2 + \frac{1}{2}z'^2 &= b, \\ x'^2 - \sqrt{\frac{3}{2}}x'y' + \frac{1}{\sqrt{2}}x'z' + \frac{1}{2}y'^2 + \frac{1}{2}z'^2 &= c. \end{aligned} \tag{4}$$

By subtracting the third equation from the second equation in (4) we obtain an expression for y' in terms of x':

$$\begin{aligned} \sqrt{6}x'y' &= b - c, \quad \text{or} \\ y' &= \frac{b - c}{\sqrt{6}x'}. \end{aligned} \tag{5}$$

Similarly, we may obtain an expression for z' in terms of x' by subtracting half the sum of the second and third equations from the first equation in (4): $-\frac{3}{\sqrt{2}}x'z' = a - \frac{1}{2}(b + c)$, or

$$z' = -\frac{\sqrt{2}}{3x'}\left(a - \frac{1}{2}(b + c)\right). \tag{6}$$

Substituting (5) and (6) into (2), we arrive at an equation for x' in terms of a, b, and c,

$$\frac{a^2 + b^2 + c^2 - (ab + ac + bc)}{3x'^2} + 3x'^2 = a + b + c,$$

or

$$9x'^4 - 3(a + b + c)x'^2 + a^2 + b^2 + c^2 - (ab + ac + bc) = 0. \tag{7}$$

The left side of (7) is quadratic in x'^2, so by applying the quadratic formula (or, if one prefers, by completing the square) we may solve for x'^2:

$$x'^2 = \frac{a + b + c\sqrt{-3(a^2 + b^2 + c^2) + 6(ab + ac + bc)}}{6}. \tag{8}$$

If we substitute the values of a, b, and c from the original problem statement into (8) and simplify the result, we see that the discriminant is positive (the discriminant simplifies to $9\left(xy+xz+yz\right)^2$ in the original variables) and that the solution involving the negative radical is spurious (since from (3)

$$x'^2 = \frac{1}{3}(x+y+z)^2 = \frac{1}{3}\left(x^2+y^2+z^2+2xy+2yz+2xz\right),$$

while the offending solution simplifies to

$$\frac{1}{3}\left(x^2+y^2+z^2-xy-xz-yz\right)$$

in the original variables).

We may now solve for x' in a straightforward manner (after rejecting the spurious solution) by taking square roots of both sides of (8):

$$x' = \sqrt{\frac{a+b+c+\sqrt{-3\left(a^2+b^2+c^2\right)+6(ab+ac+bc)}}{6}}, \qquad (9)$$

where this time we have rejected the negative branch because x' is positive. (Note that the quantity under the outermost radical is positive because each of its terms is positive.) By substituting (9) into (3) we finally obtain the desired sum,

$$x+y+z = \sqrt{\frac{a+b+c+\sqrt{-3\left(a^2+b^2+c^2\right)+6(ab+ac+bc)}}{2}}.$$

Solution 3 by Brian Beasley and Doug Daniel (jointly), Presbyterian College, Clinton, SC

Adding the three equations produces

$$2(x^2+y^2+z^2)+(xy+yz+zx) = a+b+c.$$

Since $(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx)$, we seek to express $xy+yz+zx$ in terms of a, b, and c. By the Law of Cosines, we note that x, y, and $\sqrt{a}$ may represent the lengths of the three sides of a triangle, with the angle between x and y having measure 120°. Similarly, we have two more triangles containing angles of measure 120°, one with sides of lengths y, z, and $\sqrt{b}$, and the other with sides of lengths z, x, and $\sqrt{c}$. Then we may combine these three triangles to create one triangle with sides of lengths $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$. By Heron's Formula, this new triangle has area

$$A = \sqrt{s(s-\sqrt{a})(s-\sqrt{b})(s-\sqrt{c})},$$

where $s = (\sqrt{a}+\sqrt{b}+\sqrt{c})/2$. By adding the areas of the three smaller triangles, we also obtain $A = (\sqrt{3}/4)(xy+yz+zx)$. Hence

$$(x+y+z)^2 = \frac{a+b+c-4A/\sqrt{3}}{2} + 2\left(\frac{4A}{\sqrt{3}}\right) = \frac{a+b+c}{2} + 2\sqrt{3}A,$$

so

$$x+y+z = \sqrt{\frac{a+b+c}{2} + 2\sqrt{3}A}$$

with A as given previously as a function of $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$.

Editor's comments: **David Stone and John Hawkins** approached the problem as in solution 3 above, and made the following comments about the problem and its solution.

The common vertex of our three interior triangles is often referred to the as the Steiner Point of the large triangle. The sides x, y, z form a minimal Spanning Tree (MST) of the large triangle, so the sum $x+y+z$ is the length of the MST. One would think that the length of this MST (in terms of the sides of the larger triangle is common knowledge, but we could not find it referenced.

We know that the larger triangle is actually the union of three interior triangles because we know the x, y, z and a, b, c are *all* given to satisfy the original equations. If we were simply given a, b, c then we might not have a triangle (or a solution x, y, z), or the Steiner point might be exterior to the triangle formed.

Also solved by Arkady Alt (two solutions), San Jose, CA; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Roberto de la Cruz Moreno, Centre de Recerca Matemàtica, Campus de Bellaterra, Barcelona, Spain; Adrian Naco, Polytechnic University, Tirana, Albania; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Georgia Southern University, Statesboro, GA, and the proposer.

- **5237:** *Proposed by Michael Brozinsky, Central Islip, NY*

Let $0 < R < 1$ and $0 < S < 1$, and define

$$\begin{aligned} a &= \sqrt{-2\sqrt{1-S^2}\sqrt{1-R^2}+2+2RS}, \\ b &= \sqrt{-R-S+1+RS}, \text{ and} \\ c &= \sqrt{R+S+1+RS}. \end{aligned}$$

Determine whether there is tuple (R, S) such that a, b, and c are sides of a triangle.

Solution 1 by Ed Gray, Highland Beach, FL

Consider the squares of a, b, and c.

1) $c^2 = 1 + RS + R + S = (1+R)(1+S)$

2) $b^2 = 1 + RS - R - S = (1-R)(1-S)$, so

3) $b^2 + c^2 = 2 + 2RS$

4) $a^2 = 2 + 2RS - 2\sqrt{1-S^2)}\sqrt{1-R^2)}$

5) $b^2c^2 = (1-R)(1-S)(1+R)(1+S) = (1-R^2)(1-S^2)$

6) $bc = \sqrt{(1-R^2)(1-S^2)}$. So combining (3), (4), (6);

7) $a^2 = b^2 + c^2 - 2bc = (c-b)^2$, since $c > b$. Then

8) $a = c - b$ or

9) $c = a + b$

So there can be no triangle since the sum of two legs of a triangle is greater than the third.

Solution 2 by Kee-Wai Lau, Hong Kong, China

We show that no such tuples exist. Suppose, on the contrary, that there is a tuple (R, S) such that a, b, and c are the sides of a triangle. By the triangle inequality, we have $a > c - b > 0$. Hence,

$$a^2 > c^2 + b^2 - 2cb$$

$$\Longrightarrow -2\sqrt{1-S^2}\sqrt{1-R^2} + 2 + 2RS > -2\sqrt{R+S+1+RS}\sqrt{-R-S+1+RS} + 2 + 2RS$$

$$\Longrightarrow \sqrt{(1-S)(1+S)}\sqrt{(1-R)(1+R)} < \sqrt{(1+R)(1+S)}\sqrt{(1-R)(1-S)},$$

which is not true. Thus we obtain a contradiction and complete the solution.

Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Paul M. Harms, North Newton, KS; Roberto de la Cruz Moreno, Centre de Recerca Matemàtica, Campus de Bellaterra, Barcelona, Spain; Adrian Naco, Polytechnic University, Tirana, Albania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5238:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

It is fairly well-known that $(1111\ldots 1)_9$, a number written in base 9 with an arbitrary number of digits 1, always evaluates decimally to a triangular number. Find another base b and a single digit d in that base, such that $(ddd\ldots d)_b$, using k digits d, has the same property, $\forall k \geq 1$.

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

We begin by noting that triangular numbers are of the form $T(m) = \dfrac{m(m+1)}{2}$ for integers $m \geq 1$. Also, for decimally evaluating a base b number $(ddd\ldots d)_b$, with k digits, we use the formula for a geometric sum to get

$$(ddd\ldots d)_b = d + d\cdot b + d\cdot b^2 + \cdots + d\cdot b^{k-1} = d\cdot \frac{b^k - 1}{b-1}. \qquad (1)$$

Further, for $n \geq 1$,

$$(2n+1)^2 - T(n) \quad = \quad 4n(n+1) + 1 - \frac{1}{2}n(n+1)$$

$$= \frac{7}{2}n(n+1)+1$$

$$> 0.$$

Hence, for $n \geq 1$, we may consider $T(n)$ as a digit in base $(2n+1)^2$.

Then, there are an infinite number of choices for b and d which have the desired property for all $k \geq 1$. For $n \geq 1$, choose $d_n = T(n)$ and $b_n = (2n+1)^2$. Since $(2n+1)$ is odd, (1) implies that when k digits are used, with $k \geq 1$, we have

$$\begin{aligned} (T(n)\,T(n)\,T(n)\ldots T(n))_{(2n+1)^2} &= T(n)\cdot\frac{(2n+1)^{2k}-1}{(2n+1)^2-1} \\ &= \frac{n(n+1)}{2}\cdot\frac{\left[(2n+1)^k-1\right]\left[(2n+1)^k+1\right]}{4n(n+1)} \\ &= \frac{\left[(2n+1)^k-1\right]\left[(2n+1)^k-1+2\right]}{8} \\ &= \frac{1}{2}\left[\frac{(2n+1)^k-1}{2}\right]\left[\frac{(2n+1)^k-1}{2}+1\right] \\ &= T\left(\frac{(2n+1)^k-1}{2}\right). \end{aligned}$$

E. g., when $n = 1, 2, 3$, this yields

$$\begin{aligned} (111\ldots1)_9 &= T\left(\frac{3^k-1}{2}\right), \\ (333\ldots3)_{25} &= T\left(\frac{5^k-1}{2}\right), \\ (666\ldots6)_{49} &= T\left(\frac{7^k-1}{2}\right), \end{aligned}$$

when k digits are used in each situation.

Solution 2 by Roberto de la Cruz Moreno, Centre de Recerca Matemàtica, Campus de Bellaterra, Barcelona, Spain

We have:

$$(1111...1)_9 = 1 + 1\cdot 9 + ... + 1\cdot 9^{k-1} = 1\cdot\frac{9^k-1}{8} = \frac{1}{2}\cdot\frac{3^k-1}{2}\cdot\frac{3^k+1}{2} = \frac{m(m+1)}{2}$$

Thus, just search b, d such that $b = x^2$, $x \in Z^+$, and $(b-1) = 8d$, i.e., $x \in Z^+$ such that $(x^2-1) \equiv 0 \ (mod\ 8)$. But, $(x^2-1) \equiv 0 \ (mod\ 8) \Leftrightarrow x$ is odd.
Therefore, $\forall x = 2n+1, n \in Z^+$, $b = x^2$ and $d = \dfrac{b-1}{8}$ satisfy the property.

Examples: $(333...3)_{25}$, $(666...6)_{49}$.

Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC

Given a positive integer k, we seek a base b, a digit d in base b, and a positive integer n such that

$$(ddd\ldots d)_b = d\left(\frac{b^k-1}{b-1}\right) = \frac{n(n+1)}{2}.$$

Solving the resulting quadratic for n yields a discriminant of $(b-1)^2 + 8d(b-1)(b^k-1)$, and taking $d = (b-1)/8$ reduces this expression to $b^k(b-1)^2$. To make this a perfect square and to ensure that d is an integer, we let b be an odd square. Given any integer $m > 1$, we may take $b = (2m-1)^2$, so that $d = m(m-1)/2$. Then

$$(ddd\ldots d)_b = d\left(\frac{b^k-1}{b-1}\right) = \frac{b^k-1}{8} = \frac{n(n+1)}{2},$$

where $n = [(2m-1)^k - 1]/2$. In particular, letting $m = 3$ produces $b = 25$, $d = 3$, and

$$(333\ldots 3)_{25} = \frac{25^k-1}{8} = \frac{n(n+1)}{2}$$

for $n = (5^k-1)/2$; also, letting $m = 4$ produces $b = 49$, $d = 6$, and

$$(666\ldots 6)_{49} = \frac{49^k-1}{8} = \frac{n(n+1)}{2}$$

for $n = (7^k-1)/2$.

Also solved by Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins (jointly), Georgia Southern University, Statesboro, GA, and the proposer.

- **5239:** *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany*

Determine all functions $f : \Re - \{-3, -1, 0, 1, 3\} \to \Re$, which satisfy the relation

$$f(x) + f\left(\frac{13+3x}{1-x}\right) = ax + b,$$

where a and b are given arbitrary real numbers.

Solution 1 by Adrian Naco, Polytechnic University, Tirana, Albania

If we let, $g(x) = \dfrac{13+3x}{1-x}$, then we have that,

$$(g\circ g)(x) \quad = \quad g(g(x)) = \frac{13+3g(x)}{1-g(x)} = \frac{x-13}{x+3} \qquad (1)$$

$$\textit{and} \qquad (g\circ g\circ g)(x) \quad = \quad g(g(g(x))) = \frac{g(x)-13}{g(x)+3} = x \qquad (2)$$

Considering the above and the given relation, it implies that,

$$f(x) + (f\circ g)(x) \quad = \quad ax + b, \qquad (3)$$

$$
\begin{aligned}
(f\circ g)(x)+(f\circ g\circ g)(x) &= ag(x)+b, \qquad (4)\\
(f\circ g\circ g)(x)+(f\circ g\circ g\circ g)(x) &= a(g\circ g)(x)+b,
\end{aligned}
$$

The last relation is simplified to

$$(f\circ g\circ g)(x)+f(x)=a(g\circ g)(x)+b, \qquad (5)$$

Adding equations (3) and (4) to (5) results that,

$$f(x)+(f\circ g)(x)+(f\circ g\circ g)(x)=\frac{a}{2}[x+g(x)+(g\circ g)(x)]+\frac{3b}{2}. \qquad (6)$$

Finally, if we subtract equation (4) from equation (6), then,

$$
\begin{aligned}
f(x) &= \frac{a}{2}[x-g(x)+(g\circ g)(x)]+\frac{b}{2} \quad \Rightarrow\\
f(x) &= \frac{a}{2}\left[x-\frac{13+3x}{1-x}+\frac{x-13}{x+3}\right]+\frac{b}{2} \quad \Rightarrow\\
f(x) &= \frac{a}{2}\cdot\frac{x^3+6x^2+5x+52}{(x-1)(x+3)}+\frac{b}{2}
\end{aligned}
$$

Solution 2 by Kee-Wai Lau, Hong Kong, China

Denote the given relationship by (1). Replacing x by $\dfrac{13+3x}{1-x}$ and $\dfrac{x-13}{x+3}$ in (1), we obtain respectively

$$f\left(\frac{13+3x}{1-x}\right)+f\left(\frac{x-13}{x+3}\right)=a\left(\frac{13+3x}{1-x}\right)+b \qquad (2)$$

and

$$f\left(\frac{x-13}{x+3}\right)+f(x)=a\left(\frac{x-13}{x+3}\right)+b. \qquad (3)$$

Now $(1)-(2)+(3)$ gives

$$2f(x)=(ax+b)-\left(a\left(\frac{13+3x}{1-x}\right)+b\right)+\left(a\left(\frac{x-13}{x+3}\right)+b\right).$$

Simplifying, we obtain

$$f(x)\frac{ax^3+(6a+b)x^2+(5a+2b)x+52a-3b}{2(x-1)(x+3)}.$$

Also solved by Arkady Alt, San Jose, CA; David Diminnie, Texas Instruments, Inc., Dallas TX and Charles Diminnie, Angelo State

University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Roberto de la Cruz Moreno, Centre de Recerca Matemàtica, Campus de Bellaterra, Barcelona, Spain; David Stone and John Hawkins (jointly), Georgia Southern University, Statesboro, GA, and the proposer.

- **5240:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

Let x be a positive real number. Prove that

$$\frac{x[x]}{(x+\{x\})^2}+\frac{x\{x\}}{(x+[x])^2}>\frac{1}{8},$$

where $[x]$ and $\{x\}$ represent the integral and fractional part of x, respectively.

Solution 1 by Armend Sh. Shabani, University of Prishtina, Republic of Kosova

Let $[x]=k$. Since $x=[x]+\{x\}$ we have that $\{x\}=x-k$, therefore we need to prove that $\dfrac{xk}{(2x-k)^2}+\dfrac{x(x-k)}{(x+k)^2}>\dfrac{1}{8}$, which is equivalent to

$$8kx(x+k)^2+8x(x-k)(2x-k)^2>(2x-k)^2(x+k)^2.$$

After calculations one obtains:

$$28x^4+59x^2k^2-60x^3k+2xk^3-k^4>0$$

which can be written as:

$$27x^4+59x^2k^2-60x^3k+2xk^3+x^4-k^4>0.$$

Clearly $x^4-k^4\geq 0$ and $2xk^3\geq 0$.

Consider the function

$$f(k)=59x^2k^2-60x^3k+27x^4.$$

Since $59x^2>0$ and $(60x^3)^2-4\cdot 59x^2\cdot 27x^4=-2772x^6<0$ we conclude that $f(k)>0$ for all k, which completes the proof.

Solution 2 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

We prove the stronger inequality

$$\frac{x[x]}{(x+\{x\})^2}+\frac{x\{x\}}{(x+[x])^2}>\frac{2}{5}$$

Rewrite it as

$$\frac{(x[x])^2}{x[x](x+\{x\})^2}+\frac{(x\{x\})^2}{x\{x\}(x+[x])^2}>\frac{2}{5}$$

Cauchy–Schwarz yields

$$\frac{(x[x])^2}{x[x](x+\{x\})^2}+\frac{(x\{x\})^2}{x\{x\}(x+[x])^2}\geq\frac{(x[x]+x\{x\})^2}{x[x](x+\{x\})^2+x\{x\}(x+[x])^2}>\frac{2}{5}$$

Clearing the denominators and taking into account that $[x] + \{x\} = x$ we come to

$$5[x]^2 + 5\{x\}^2 > 2x^2$$

and this follows by

$$5[x]^2 + 5\{x\}^2 \geq \frac{5}{2}([x] + \{x\})^2 = \frac{5}{2}x^2 > 2x^2$$

Solution 3 by Adrian Naco, Polytechnic University, Tirana, Albania

If $x \in (0;1)$ then we have that $[x] = 0$ and $x = \{x\}$. Thus the left side of the given inequality is valued by 1, and as a result, the inequality is true.

Suppose that $x \geq 1$. Then $[x] \geq 1$ and $\{x\} \in [0;1)$. Let $\{x\} = q[x]$ where $q \in [0;1)$, then $x = (1+q)[x]$

Since, $q + 2 > 2q + 1$, then the left side of the inequality is transformed to

$$\begin{aligned} S &= \frac{x[x]}{(x+\{x\})^2} + \frac{x\{x\}}{(x+[x])^2} = \frac{q+1}{(2q+1)^2} + \frac{q(q+1)}{(q+2)^2} \\ &\geq \frac{q+1}{(q+2)^2} + \frac{q(q+1)}{(q+2)^2} = \left(\frac{q+1}{q+2}\right)^2 \geq \left(\frac{1}{2}\right)^2 > \frac{1}{8}. \end{aligned}$$

Editor's note: Most of the solvers noted that the right hand side of the inequality $\frac{1}{8}$ can be raised to $\frac{4}{9}$. Adrain Naco (see solution above) restated the problem as follows:

Let x be a positive number. Prove that

$$a) \qquad \inf_{x>0}\left\{\frac{1+[x]}{(1+2[x])^2} + \frac{[x]\,(1+[x])}{(2+[x])^2}\right\} = \inf_{x>0}\left\{\frac{x[x]}{(x+\{x\})^2} + \frac{x\{x\}}{(x+[x])^2}\right\} = \frac{4}{9},$$

$$b) \qquad \sup_{x>0}\left\{\frac{x[x]}{(x+\{x\})^2} + \frac{x\{x\}}{(x+[x])^2}\right\} = 1$$

where $[x]$ and $\{x\}$ represent the integral and fractional part of x, respectively.

Following are two additional proofs of the restated problem.

Solutions 4 and 5 by David Stone and John Hawkins of Georgia Southern University, Statesboro GA

For convenience we let $E(x) = \dfrac{x[x]}{(x+\{x\})^2} + \dfrac{x\{x\}}{(x+[x])^2}$.

Note that $E(n) = 1 + 0 = 1$, for any integer $n \geq 1$ and $E(x) = 0 + 1 = 1$, for any x with $0 < x < 1$.

We can describe precisely how the function E behaves: on the interval $[n, n+1), n \geq 1$, it *descends* strictly from a height of 1 towards the height $\dfrac{(n+1)n}{(n+2)^2} + \dfrac{n+1}{2n+1)^2}$. Thus the infimum on this interval is $\dfrac{(n+1)n}{n+2)^2} + \dfrac{n+1}{2n+1)^2}$. As n increases, these greatest lower bounds grow, so the smallest of these, $\dfrac{4}{9}$, occurs on the first interval, $[1,2)$.

Thus, $E(x) > \dfrac{4}{9}$ for all positive x, and the lower bound is sharp because $\lim_{x\to 2^-} E(x) = \dfrac{4}{9}$. Note that $E(x)$ barely dips below height 1 for large x.

To verify these claims, let $n < x < n+1$, with $x = n+f, n \geq 1, 0 < f < 1$.
Then $E(x) = \dfrac{(n+f)n}{(n+f+f)^2} + \dfrac{(n+f)f}{n+f+f)^2} = \dfrac{(n+f)n}{(n+2f)^2} + \dfrac{(n+f)f}{(2n+f)^2}$.
By letting $f \longrightarrow 1$ from the left, we see that $E(x) = \dfrac{(n+1)n}{(n+2)^2} + \dfrac{(n+1)}{2n+1)^2}$.
In particular, $\lim_{x\to 2^-} E(x) = \dfrac{(1+1)}{(1+2)^2} + \dfrac{(1+1)}{2+1)^2} = \dfrac{2}{9} + \dfrac{2}{9} = \dfrac{4}{9}$.

Also, $\lim_{x\to 3^-} E(x) = \dfrac{(2+1)2}{(2+2)^2} + \dfrac{(2+1)}{2\cdot 2+1)^2} = \dfrac{3}{8} + \dfrac{3}{25} = \dfrac{99}{200} = 0.495 > \dfrac{4}{9}$.

To verify that the function E decreases for $0 < f < 1$, we compute the derivative $\dfrac{dE}{df}$.

$$
\begin{aligned}
\frac{dE}{df} &= \frac{(n+2f)^2 - (n+f)n\cdot 2(n+2f)\cdot 2}{(n+2f)^4} + \frac{(2n+f)^2(n+2f) - (n+f)\cdot f\cdot 2(2n+f)}{(n+2f)^4} \\
&= \frac{n(2n+3f)}{(2n+f)^3} - \frac{n(3n+2f)}{(n+2f)^3} \\
&= n\frac{(2n+3f)(n+2f)^3 - (3n+2f)(2n+f)^3}{(2n+f)^3(n+2f)^3}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\frac{dE}{df} &< 0 \\
\iff \quad & (2n+3f)(n+2f)^3 - (3n+2f)(2n+f)^3 < 0 \\
\iff \quad & (2n+3f)(n+2f)^3 < (3n+2f)(2n+f)^3 \\
\iff \quad & \frac{2n+3f}{3n+2f)} < \frac{(2n+f)^3}{(n+2f)^3} \\
\iff \quad & 1 - \frac{n-f}{n+2f} < \left(1 + \frac{n-f}{n+2f}\right)^3.
\end{aligned}
$$

But $n \geq 1$ and $0 < f < 1$, so $n - f$ is positive. Hence the expression on the left of our inequality is less than 1 and the expression the right is larger than 1, so the final inequality is true.

Finally, we verify that the interval infima, $\dfrac{(n+1)n}{(n+2)^2} + \dfrac{(n+1)}{(2n+1)^2}$, form an increasing sequence (with limit 1):
regarded as a function of n, $\dfrac{(n+1)n}{(n+2)^2} + \dfrac{(n+1)}{(2n+1)^2}$, has derivative

$$\frac{3n+2}{(n+2)^3} - \frac{2n+3}{(2n+1)^3} = \frac{(n-1)(n+1)(22n^2+37n+22)}{(n+2)^3(2n+1)^3} \geq 0, \text{ for } n \geq 1.$$

Solution 5

This method verifies the proposed inequality, although it does not reveal as much information about the given expressions as does the preceding solution.
Recognizing that the expression of the left equals 1 when $0 < x < 1$ or when x is an integer, we consider $x > 1$ and write x in terms of its integral and fractional parts: let $n < x < n+1$ with $x = n + f,\ n \geq 1,\ 0 < f < 1$. Then we want to show

$$\begin{aligned} \frac{(n+f)n}{(n+2f)^2} + \frac{(n+f)f}{(n+2f)^2} &> \frac{4}{9} \\ &\Longleftrightarrow \left\{(n+f)\,n\,(2n+f)^2 + (n+f)\,f\,(n+2f)^2\right\} \\ &> 4(n+2f)^2(2+2f)^2. \end{aligned}$$

Upon division by n^4, this becomes an equivalent inequality in a single variable:

$$\Longleftrightarrow 9\left[\left(1+\frac{f}{n}\right)\left(2+\frac{f}{n}\right)^2 + \left[\frac{f}{n} + \left(\frac{f}{n}\right)^2\right]\left(1+2\frac{f}{n}\right)^2\right\} > 4\left(1+2\frac{f}{n}\right)^2\left(2+\frac{f}{n}\right)^2.$$

Letting $t = \dfrac{f}{n}$, so that $0 < t < \dfrac{f}{n} < 1$, we have more equivalent inequalities:

$$\begin{aligned} &\Longleftrightarrow \quad 9\left\{(1+t)\,(2+t)^2 + \left[t+t^2\right](1+2t)^2\right\} > 4(1+2t)^2(2+t)^2 \\ &\Longleftrightarrow \quad 9\left\{4t^4+9t^3+10t^2+9t+4\right\} > 4\left\{4t^4+20t^3+33t^2+20t+4\right\} \\ &\Longleftrightarrow \quad 20t^4+t^3-42t^2+t+20>0 \\ &\Longleftrightarrow \quad (t-1)^2(20t^2+41t+20)>0, \end{aligned}$$

which is certainly true.

Also solved by Arkady Alt, San Jose, CA; Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; David Diminnie, Texas Instruments, Incorporated, Dallas, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Roberto de la Cruz Moreno, Centre de Recerca Matemàtica,Campus de Bellaterra, Barcelona, Spain; Boris Rays, Brooklyn NY, and the proposer.

- **5241:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $\alpha \geq 0$ be a real number. Calculate

  $$\lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{x^n+\alpha}\ dx\right)^n.$$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria , Spain

For $x \in [0,1]$, $\alpha \leq x^n + \alpha$. Therefore,

$$\begin{aligned}\lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{\alpha}\ dx\right)^n &\leq \lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{x^n+\alpha}\ dx\right)^n\\ \alpha &\leq \lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{x^n+\alpha}\ dx\right)^n.\end{aligned}$$

On the other hand, since function $y = x^n$ is convex for $n \geq 1$, by Jensen's inequality

$$\begin{aligned}\left(\int_0^1 \sqrt[n]{x^n+\alpha}\ dx\right)^n &\leq \int_0^1 (x^n+\alpha)\ dx\\ \lim_{n\to\infty}\int_0^1 x^n+\alpha\ dx &\leq \lim_{n\to\infty}\frac{1}{n+1}+\alpha=\alpha.\end{aligned}$$

So, $\displaystyle\lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{x^n+\alpha}\ dx\right)^n = \alpha$.

Solution 2 by Arkady Alt, San Jose, CA

Let $a_n = \displaystyle\int_0^1 \sqrt[n]{x^n+\alpha}dx$. Note that $\displaystyle\lim_{n\to\infty} a_n = 1$.

Indeed, we have

$\sqrt[n]{\alpha} = \displaystyle\int_0^1 \sqrt[n]{\alpha}dx \leq a_n \leq \int_0^1 \sqrt[n]{1+\alpha}dx = \sqrt[n]{1+\alpha}$ and $\displaystyle\lim_{n\to\infty}\sqrt[n]{\alpha} = \lim_{n\to\infty}\sqrt[n]{1+\alpha} = 1$.

Since $\displaystyle\lim_{n\to\infty} a_n^n = \lim_{n\to\infty} e^{n\ln a_n}$ we will find $\displaystyle\lim_{n\to\infty} n\ln a_n$.

Since

$$\begin{aligned}\lim_{n\to\infty}(a_n-1) &= 0 \text{ we have}\\ \lim_{n\to\infty} n\ln a_n &= \lim_{n\to\infty} n\ln(1+(a_n-1))\\ &= \lim_{n\to\infty}\left(n(a_n-1)\cdot\frac{\ln(1+(a_n-1))}{(a_n-1)}\right)\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} n\left(a_n - 1\right) \text{ because} \\
&\quad \lim_{n\to\infty} \frac{\ln\left(1 + (a_n - 1)\right)}{(a_n - 1)} = 1.
\end{aligned}
$$

Thus, it suffices to find $\lim_{n\to\infty} n\left(a_n - 1\right)$.

Since

$$
\begin{aligned}
n\left(a_n - 1\right) &= n\left(\int_0^1 \sqrt[n]{x^n + \alpha}dx - 1\right) \\
&= n\int_0^1 \left(\left(\sqrt[n]{x^n + \alpha} - \sqrt[n]{\alpha}\right) + \left(\sqrt[n]{\alpha} - 1\right)\right) dx \\
&= n\left(\sqrt[n]{\alpha} - 1\right) + n\int_0^1 \left(\sqrt[n]{x^n + \alpha} - \sqrt[n]{\alpha}\right) dx \text{ and} \\
\lim_{n\to\infty} n\left(\sqrt[n]{\alpha} - 1\right) &= \lim_{n\to\infty} n\left(e^{\ln \alpha n} - 1\right) \\
&= \ln \alpha
\end{aligned}
$$

then it remains to find

$$
\lim_{n\to\infty} n\int_0^1 \left(\sqrt[n]{x^n + \alpha} - \sqrt[n]{\alpha}\right) dx.
$$

By the Mean Value Theorem

$$
\frac{\sqrt[n]{x^n + \alpha} - \sqrt[n]{\alpha}}{x^n} = \frac{1}{n\sqrt[n]{\theta^{n-1}}} \text{ where } \theta \in \left(\alpha, x^n + \alpha\right).
$$

Hence,

$$
\frac{\sqrt[n]{x^n + \alpha} - \sqrt[n]{\alpha}}{x^n} < \frac{1}{n\sqrt[n]{\alpha^{n-1}}} \iff \sqrt[n]{x^n + \alpha} - \sqrt[n]{\alpha} < \frac{x^n}{n\sqrt[n]{\alpha^{n-1}}}
$$

and, therefore,

$$
0 < n\int_0^1 \left(\sqrt[n]{x^n + \alpha} - \sqrt[n]{\alpha}\right) dx < n\int_0^1 \frac{x^n}{n\sqrt[n]{\alpha^{n-1}}}dx = \frac{1}{(n+1)\sqrt[n]{\alpha^{n-1}}}.
$$

Since

$$
\lim_{n\to\infty} \frac{1}{(n+1)\sqrt[n]{\alpha^{n-1}}} = 0,
$$

by the Squeeze Principle,

$$
\lim_{n\to\infty} n\int_0^1 \left(\sqrt[n]{x^n + \alpha} - \sqrt[n]{\alpha}\right) dx = 0.
$$

Thus,

$$\lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{x^n+\alpha}dx\right)^n = e^{\ln\alpha} = \alpha.$$

Solution 3 by Anastasios Kotronis, Athens, Greece

1. For $a = 0$ the limit is trivially $0 = a$.
2. For $a > 0$. We set $I_n^n = \left(\int_0^1 \sqrt[n]{x^n+a}\,dx\right)^n = \exp\left(n\ln\left(\int_0^1 \sqrt[n]{x^n+a}\,dx\right)\right) = e^{A_n}$.

   Now, considering that $n \in [1, +\infty)$, since $0 < \sqrt[n]{x^n+a} \leq 1 + a$ and $\sqrt[n]{x^n+a} \stackrel{n\to+\infty}{\longrightarrow} 1$ for $x \in [0, 1]$, by dominated convergence theorem we get that $I_n \to 1$, thus $\ln I_n \to 0$.

   Furthermore, by Leibniz's rule we have that for $n \geq 1$

$$\frac{\partial I_n}{\partial n} = \int_0^1 \frac{\partial}{\partial n}\sqrt[n]{x^n+a}\,dx = \int_0^1 (x^n+a)^{\frac{1-n}{n}}\left(\frac{nx^n\ln x - (x^n+a)\ln(x^n+a)}{n^2}\right)dx.$$

   We also have that

$$\begin{aligned}\left|(x^n+a)^{\frac{1-n}{n}}\left((x^n+a)\ln(x^n+a) - nx^n\ln x\right)\right| &\leq \frac{1+a}{a}\left(\left|(x^n+a)\ln(x^n+a)\right| + \left|nx^n\ln x\right|\right. \\ &\leq \frac{1+a}{a}\left(\max\{e^{-1},(1+a)\ln(1+a)\} + e^{-1}\right)\end{aligned}$$

   and since

$$(x^n+a)^{\frac{1-n}{n}}\left((x^n+a)\ln(x^n+a) - nx^n\ln x\right) \to \begin{cases}\ln(1+a), & \text{if } x = 1\\ \ln a, & \text{if } x \in [0,1)\end{cases}$$

   by the dominated convergence theorem it is $-n^2\dfrac{\partial I_n}{\partial n} \to \ln a$.

   Now applying De l' Hospital's rule we get

$$\lim_{n\to+\infty} A_n = \lim_{n\to+\infty}\frac{\ln I_n}{n^{-1}} = \lim_{R\ni n\to+\infty}\frac{\ln I_n}{n^{-1}} \stackrel{0/0}{=} \lim_{n\to+\infty} I_n^{-1}\cdot\left(-n^2\frac{\partial I_n}{\partial n}\right) \to \ln a,$$

so the required limit in each case is a.

Solution 4 by Adrian Narco, Polytechnic University, Tirana, Albania

The function, $f(x) = \sqrt[n]{x^n+\alpha} = (x^n+\alpha)^{\frac{1}{n}}$, is strictly increasing and everywhere continuous on $[0; 1]$, thus we can apply the mean value theorem for integral, that is,

$$\exists c \in (0;1) : \int_0^1 \sqrt[n]{x^n+\alpha}\mathrm{dx} \quad = \quad f(c)(1-0) = (c^n+\alpha)^{\frac{1}{n}}$$

$$\Rightarrow \quad \lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{x^n+\alpha}\,\mathrm{dx}\right)^{\mathrm{n}}$$

$$= \lim_{n\to\infty}\left((c^n+\alpha)^{\frac{1}{n}}\right)^n = \lim_{n\to\infty}(c^n+\alpha) = \alpha$$

since $c \in (0,1)$ and $c^n \overset{n\to+\infty}{\longrightarrow} 0$.

Also solved by Kee-Wai Lau, Hong Kong, China; Carl Libis (two solutions; one alone and one with Tom Dunion), Ivy Bridge College of Tiffin University, Toledo, OH and Bentley University, Waltham, MA (respectively); Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy, and the proposer.

Mea Culpa

The names of Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany were inadvertently not listed as having solved problem 5232.

The featured solutions to Problem 5229 have turned out to be in error, or perhaps more correctly stated, incomplete. Following is a note received from **Arkady Alt of San Jose, CA.**

I'm writing you about problem 5229. I think that there are some issues with the proposed solutions and I wanted to give a few arguments to prove this point.
Also, below, I'm attaching my solution that I have not posted after realizing that it is not complete, although I did obtain the desired limit.

There are two main approaches to finding limits. Both are in two steps.

The first way is to prove that limit exists and then find it;

The second way is to find the value of the limit assuming that it exists, and then prove that the obtained value is indeed a limit.

The second way isn't complete without such a proof, because there are counterexamples of sequences which have no limit, but when assuming that it exists we can obtain a value.

For example: let $a_1 = 1$ and $a_{n+1} = a_n^2 + 3a_n + 1, n \geq 1$ then obviously $\lim\limits_{n\to\infty} a_n = \infty$.

But assuming that $(a_n)_{n\geq 1}$ is convergent and denoting $a = \lim\limits_{n\to\infty} a_n$ we immediately obtain

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(a_n^2+3a_n+1\right) = \lim_{n\to\infty} a_n^2 + 3\lim_{n\to\infty} a_n + 1 = a^2+3a+1a^2+2a+1 = 0 \iff a = -1.$$

Also, the Stolz Theorem cannot be inverted.

Example:

Let $a_n = \sum_{k=1}^{n} \sin k$, then $\frac{a_{n+1} - a_n}{n+1-n} = \sin(n+1)$ and the sequence $(\sin n)_{n \in N}$ isn't convergent, but

since $\sum_{k=1}^{n} \sin k = \frac{\sin\left(\frac{n+1}{2}\right) \sin \frac{n}{2}}{\sin \frac{1}{2}}$ ($2a_n \sin \frac{1}{2} = \sum_{k=1}^{n} \left(\cos\left(k - \frac{1}{2}\right) - \cos\left(k + \frac{1}{2}\right)\right) =$ $\cos \frac{1}{2} - \cos\left(n + \frac{1}{2}\right) = 2 \sin\left(\frac{n+1}{2}\right) \sin \frac{n}{2}$) then $\lim_{n\to\infty} \frac{a_n}{n} = 0$ because $\left|\sin\left(\frac{n+1}{2}\right) \sin \frac{n}{2}\right| \leq 1$.

Here is my solution, which I decided not to send because it is missing the crucial "proof" points that are mentioned above and it is only based on an assumption. (Note that the published solutions 2 and 3 for problem 5229 are incomplete for the same reason).

Solution 1 is also incomplete (for another reason) because it is based on an unproved assumption about the asymptotic behavior of $(x_n)_{n \geq 1}$, namely that $x_n \sim kn^{\alpha}$, for some k and α.

This assumption is basically equivalent to the problem statement.

I have a slight suspicion that a "simple" solution from the proposer was originally the rationale for the publication of this problem.

So, in my opinion this problem has not been solved as of yet.

5229. Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

Let $\beta, a > 0$ be a real numbers and let $\{x_n\}_{n \in N}$ be the sequence defined by the recurrence
relation

$$x_1 = a, \text{ and } x_{n+1} = x_n + \tfrac{n^{2\beta}}{x_1 + x_2 + \ldots + x_n} \quad \text{for } n \geq 1.$$

1. Prove that $\lim_{n\to\infty} x_n = \infty$;
2. Calculate $\lim_{n\to\infty} \frac{x_n}{n^{\beta}}$.

Solution by Arkady Alt, San Jose ,CA

1. Let $S_n := x_1 + x_2 + \ldots + x_n, n \in N$. It is easy to see (by Math. Induction) that $x_n > 0$ for all $n \in N$.

Also, note that sequence $\{x_n\}_{n \in N}$ is increasing, since

$$x_{n+1} - x_n = \frac{n^{2\beta}}{S_n} > 0 \iff x_{n+1} > x_n \ , n \in N.$$

Then $x_{n+1}^2 - x_n^2 = \frac{n^{2\beta}(x_n + x_{n+1})}{S_n} > \frac{2n^{2\beta} x_n}{n x_n} = 2n^{2\beta - 1}, n \in N$ and, therefore,

$$x_{n+1}^2 - x_1^2 = \sum_{k=1}^{n} \left(x_{k+1}^2 - x_k^2\right) > 2 \sum_{k=1}^{n} k^{2\beta-1} > \frac{n^{2\beta}}{\beta} x_{n+1}^2 > a + \frac{n^{2\beta}}{\beta} > \frac{n^{2\beta}}{\beta} x_n > \frac{(n-1)^{\beta}}{\sqrt{\beta}}.$$

Thus, $\lim_{n\to\infty} x_n = \infty$.

We can prove that sequence $\dfrac{x_n}{n^\beta}$ has an upper bound.

Indeed, since $x_n > \dfrac{(n-1)^\beta}{\sqrt{\beta}}$ then $S_n > \sum_{k=1}^{n} \dfrac{(k-1)^\beta}{\sqrt{\beta}} > \dfrac{1}{\sqrt{\beta}} \sum_{k=1}^{n-1} k^\beta > \dfrac{(n-1)^{\beta+1}}{(\beta+1)\sqrt{\beta}}$

and, therefore ,

$x_{n+1} - x_n = \dfrac{n^{2\beta}}{S_n} < \dfrac{n^{2\beta}(\beta+1)\sqrt{\beta}}{(n-1)^{\beta+1}} = n^{\beta-1} \cdot \left(1 + \dfrac{1}{n-1}\right)^{\beta+1} (\beta+1)\sqrt{\beta} < Kn^{\beta-1}$,

where

$K = e(\beta+1)\sqrt{\beta}$, because $\left(1+\dfrac{1}{n-1}\right)^{\beta+1} < \left(1+\dfrac{1}{n-1}\right)^{n-1} < e$ for any n bigger then some $n_0 > 0$.

Then $x_{n+1} - x_{n_0} < \dfrac{K(n+1)^\beta}{\beta} \dfrac{x_n}{n^\beta} < \dfrac{x_{n_0}}{n^\beta} + \dfrac{K}{\beta}, n \geq n_0$.

If I can prove that $\dfrac{x_n}{n^\beta}$ is increasing, then we can conclude that $\lim\limits_{n\to\infty} \dfrac{x_n}{n^\beta}$ exists. Attempts to do so failed.

Or, assuming that $\left(\dfrac{x_n}{n^\beta}\right)_{n\in N}$ is convergent we can try to find $L = \lim\limits_{n\to\infty} \dfrac{x_n}{n^\beta}$, but later we must prove that the obtained value is really the desired limit. Value of L can be obtained repeatedly using Stolz Theorem:

Indeed, using* $\lim\limits_{n\to\infty} \dfrac{(n+1)^\alpha - n^\alpha}{\alpha n^{\alpha-1}} = 1, \alpha > 0$ we obtain

$$L = \lim_{n\to\infty} \frac{x_n}{n^\beta} = \lim_{n\to\infty} \frac{x_{n+1}-x_n}{(n+1)^\beta - n^\beta} = \lim_{n\to\infty} \frac{x_{n+1}-x_n}{\beta n^{\beta-1}} = \lim_{n\to\infty} \frac{n^{2\beta}}{\beta n^{\beta-1} S_n} = \frac{1}{\beta} \lim_{n\to\infty} \frac{n^{\beta+1}}{S_n} =$$

$$\frac{1}{\beta} \lim_{n\to\infty} \frac{(n+1)^{\beta+1} - n^{\beta+1}}{S_{n+1} - S_n} = \frac{1}{\beta} \lim_{n\to\infty} \frac{(n+1)^{\beta+1} - n^{\beta+1}}{S_{n+1} - S_n} = \frac{\beta+1}{\beta} \lim_{n\to\infty} \frac{n^\beta}{x_{n+1}} =$$
$$\frac{\beta+1}{\beta} \lim_{n\to\infty} \left(\frac{x_n}{x_{n+1}} \cdot \frac{n^\beta}{x_n}\right) = \frac{\beta+1}{\beta} \lim_{n\to\infty} \frac{n^\beta}{x_n} = \frac{\beta+1}{\beta} . \frac{1}{L} L = \sqrt{\frac{\beta+1}{\beta}}.$$

(here, the chain of equalities according to Stolz Theorem works from the right to the left).

But attempts to prove that $\lim\limits_{n\to\infty} \dfrac{x_n}{n^\beta} = \sqrt{\dfrac{\beta+1}{\beta}}$ failed as well.

(*) By Mean Value Theorem $(n+1)^\alpha - n^\alpha = \alpha c_n^{\alpha-1}$, where $c_n \in (n, n+1)$ and, therefore, $\alpha \min\left\{n^{\alpha-1}, (n+1)^{\alpha-1}\right\} < (n+1)^\alpha - n^\alpha < \alpha \max\left\{n^{\alpha-1}, (n+1)^{\alpha-1}\right\}$.

Hence, $\alpha \min\left\{1, \dfrac{(n+1)^{\alpha-1}}{n^{\alpha-1}}\right\} < \dfrac{(n+1)^\alpha - n^\alpha}{n^{\alpha-1}} < \alpha \max\left\{1, \dfrac{(n+1)^{\alpha-1}}{n^{\alpha-1}}\right\}$.

Editor again: I sent Arkady's comments to Ovidiu (proposer of the problem), and he answered as follows:

"I have read Prof. Alt's comments on problem 5229 and he is right, namely the applicability of the Stolz-Cesaro lemma is valid provided that $\lim \dfrac{x_{\{n+1\}} - x_n}{(n+1)^\beta - n^\beta}$ exists, which I failed to prove. It seems hard to establish the existence of this limit. It appears that the solution of this problem is incomplete, as Prof. Alt has observed."

Ovidiu went on to say that he had communicated the above to some of his colleagues, but to date, they had not been able to solve, or circumvent the glitch.