This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

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**5206:** Proposed by Kenneth Korbin, New York, NY

The distances from the vertices of an equilateral triangle to an interior point \( P \) are \( \sqrt{a}, \sqrt{b}, \text{ and } \sqrt{c} \) respectively, where \( a, b, \text{ and } c \) are positive integers.

Find the minimum and the maximum possible values of the sum \( a + b + c \) if the side of the triangle is 13.

**5207:** Proposed by Roger Izzard, Dallas, TX

Consider the following four algebraic terms:

\[
T_1 = a^2(b + c) + b^2(a + c) + c^2(a + b)
\]

\[
T_2 = (a + b)(a + c)(b + c)
\]

\[
T_3 = abc
\]

\[
T_4 = \frac{b + c - a}{a} + \frac{a + c - b}{b} + \frac{a + b - c}{c}
\]

Suppose that \( \frac{T_1 \cdot T_2}{(T_3)^2} = \frac{616}{9} \). What values would then be possible for \( T_4 \)?

**5208:** Proposed by D. M. Bătinețu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania

Let the sequence of positive real numbers \( \{a_n\}_{n \geq 1}, \ N \in Z^+ \) be such that \( \lim_{n \to \infty} \frac{a_{n+1}}{n^2 \cdot a_n} = b \). Calculate:

\[
\lim_{n \to \infty} \left( \frac{n+\sqrt[n]{a_{n+1}}}{n + 1} - \frac{\sqrt[n]{a_n}}{n} \right).
\]

**5209:** Proposed by Tom Moore, Bridgewater, MA

We noticed that 27 is a cube and 28 is an even perfect number. Find all pairs of consecutive integers such that one is cube and the other is an even perfect number.
• **5210**: Proposed by José Luis Díaz-Barrero, Barcelona, Spain

Let \(a, b, c, d\) be four positive real numbers. Prove that

\[
1 + \frac{1}{8} \left( \frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b} \right) > \frac{2\sqrt{3}}{3}.
\]

• **5211**: Proposed by Ovidiu Furdui, Cluj-Napoca, Romania

Let \(n \geq 1\) be a natural number and let

\[f_n(x) = x^{x^{\cdots^{x}}},\]

where the number of \(x\)'s in the definition of \(f_n\) is \(n\). For example

\[f_1(x) = x, \quad f_2(x) = x^x, \quad f_3(x) = x^{x^x}, \ldots.\]

Calculate the limit

\[
\lim_{x \to 1} \frac{f_n(x) - f_{n-1}(x)}{(1-x)^n}.
\]

**Solutions**

• **5188**: Proposed by Kenneth Korbin, New York, NY

Given \(\triangle ABC\) with coordinates \(A(-5, 0), B(0, 12)\) and \(C(9, 0)\). The triangle has an interior point \(P\) such that \(\angle APB = \angle APC = 120^\circ\). Find the coordinates of point \(P\).

**Solution 1 by Ercole Suppa, Teramo, Italy**

Let us construct equilateral triangles \(\triangle ABD, \triangle AEC\) externally on the sides \(AB, AC\) of triangle \(\triangle BAC\) and denote by \(\omega_1, \omega_2\) the circumcircles of \(\triangle ABD, \triangle AEC\). The point \(P\) is the intersection point of \(\omega_1, \omega_2\) different from \(O\). In order to find the coordinates of \(D, E\) we use complex numbers. If we denote respectively by \(a = -5, b = 12i, c = 9\) the affixes of \(A, B, C\) we get:

\[
d = a + (b-a)e^{\frac{\pi}{3}i} = -5 - 12\sqrt{3}i + \frac{12 + 5\sqrt{3}i}{2},
\]

\[
e = a + (c-a)e^{\frac{\pi}{3}i} = 2 - 7\sqrt{3}i
\]

so the coordinates of \(D, E\) are \(D\left(\frac{-5 - 12\sqrt{3}}{2}, \frac{12 + 5\sqrt{3}}{2}\right)\) and \(E\left(2, -7\sqrt{3}\right)\).

The equations of \(\omega_1, \omega_2\) are:

\[
\omega_1 : \quad 169\sqrt{3}x^2 + 169\sqrt{3}y^2 + \left(2028 + 845\sqrt{3}\right)x + \left(-845 - 2028\sqrt{3}\right)y + 10140 = 0
\]

\[
\omega_2 : \quad 196\sqrt{3}x^2 + 196\sqrt{3}y^2 - 784\sqrt{3}x + 2744y - 8820\sqrt{3} = 0
\]

2
and, after some calculations, we obtain

\[ P = \omega_1 \cap \omega_2 = \left( -2 \frac{(-981 + 112\sqrt{3})}{2353}, -21 \frac{(-896 + 263\sqrt{3})}{2353} \right). \]

**Solution 2 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain**

The isogonic center \( P \) is the point from which we see the sides of \( ABC \) under equal angles (that is 120° mod 180°). The isogonic center is the common intersection point of the three circumcircles to the equilateral triangles constructed on the sides of \( \triangle ABC \) (Napoleon’s theorem).

Let \( \triangle AC'B, \triangle BA'C, \triangle ACB' \) be equilateral triangles constructed on the outside on the edges of \( \triangle ABC \), then \( AA', BB', CC' \) intersect in the isogonic center \( P \). Of course, it is enough to find two of these lines. In our case,

\[ B' = -5 + (14, 0) \cdot 1_{-\pi/3} = 2 - 7\sqrt{3}i \]
\[ C' = -5 + (5, 12) \cdot 1_{\pi/3} = -5/2 - 6\sqrt{3} + (6 + 5\sqrt{3}/2)i. \]

The intersection of lines \( BB' \) and \( CC' \) gives the solution

\[ P = \left( \frac{6(56 + 33\sqrt{3})}{504 + 295\sqrt{3}}, \frac{21(93 + 56\sqrt{3})}{504 + 295\sqrt{3}} \right). \]

**Solution 3 by Michael Brozinsky, Central Islip, NY**

Clearly \( AC = 14 \), and \( AB = 13 \). Consider the circumscribed circle of \( \triangle APC \). If we denote its center by \( O_1 \) and the midpoint of \( AC \) by \( M(2,0) \), then, since an inscribed angle is measured by one half of its intercepted arc, and a radius perpendicular to a chord bisects the chord and its arc, it readily follows that \( \triangle AO_1M \) is a 30, 60, 90 degree right triangle and so since \( MA = 7 \), the radius is \( 14 \sqrt{3}/3 \), \( O_1M = 7\sqrt{3}/3 \), and \( P \) lies on the circle

\[ (x - 2)^2 + \left( y + \frac{7}{3}\sqrt{3} \right)^2 = \frac{196}{3}. \]  \hspace{1cm} (1)

(Note that since the segment of this circle having minor arc \( AC \) contains \( P \), the center \( O_1 \) is below \( AC \).)

Similarly, consider the circumscribed circle of \( \triangle APB \). If we denote its center by \( O_2 \) and the midpoint of \( AB \) by \( N\left(\frac{-5}{2}, 6\right) \), then \( \triangle AO_2N \) is a 30, 60, 90 degree right triangle and so since \( NA = \frac{13}{2} \), the radius is \( 13\sqrt{3}/6 \) and \( O_2N = 13\sqrt{3}/6 \).

The perpendicular bisector of \( AB \) is

\[ y - 6 = -\frac{5}{12} \cdot \left( x + \frac{5}{2} \right) \implies y = -\frac{5}{12} x + \frac{119}{24}. \]  \hspace{1cm} (2)

If \((X, Y)\) are the coordinates of \( O_2 \) we have

\[ (X + \frac{5}{2})^2 + \left( -\frac{5}{12} X + \frac{119}{24} - 6 \right)^2 = \left( \frac{13\sqrt{3}}{6} \right)^2, \]
and thus, \( X = \frac{5}{2} \pm 2\sqrt{3} \). We choose \( X = -\frac{5}{2} - 2\sqrt{3} \) since \( O_2 \) is to the left of \( AB \), and from (2), \( Y = 6 + \frac{5}{6} \sqrt{3} \).

Thus \( P \) also lies on the circle

\[
\left( x + \frac{5}{2} + 2\sqrt{3} \right)^2 + \left( y - 6 - \frac{5}{6} \sqrt{3} \right)^2 = 169 \frac{3}{3}.
\] (3)

If we subtract equation (3) from equation (1) we obtain the line

\[-9x - 36 + \frac{19}{3} y\sqrt{3} - 4x\sqrt{3} - 20\sqrt{3} + 12y = 9 \] (4)

or equivalently,

\[ y = -\frac{3}{71} \left( -32 + 9\sqrt{3} \right) (x + 5). \] (5)

This line just found passes through the points of intersections of these two circles and thus \( P \) is that point that is interior to \( \triangle ABC \). Substituting (1) into (5), solving the resulting quadratic equation and rejecting \( x = -5 \), gives the \((x,y)\) coordinates of \( P \).

\[(x,y) = \left( \frac{1962}{2353}, \frac{224}{2353} \sqrt{3}, \frac{18816}{2353}, \frac{-5523}{2353} \sqrt{3} \right). \]

Also solved by Brian D. Beasley, Clinton, SC; Michael Brozinsky (two solutions), Central Islip, NY; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; John Nord, Spokane, WA; Titu Zvonaru, Comănești, Romania jointly with Neculai Stanciu, Buzău, Romania; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

5189: Proposed by Kenneth Korbin, New York, NY

Given triangle ABC with integer length sides and with \( \angle A = 60^\circ \) and with \((a,b,c) = 1\).

Find the lengths of \( b \) and \( c \) if

\[ i) \quad a = 13, \text{ and if} \]

\[ ii) \quad a = 13^2 = 169, \text{ and if} \]

\[ iii) \quad a = 13^4 = 28561. \]

Solution 1 by Kee-Wai Lau, Hong Kong, China

By the cosine formula, we have \( a^2 = b^2 + c^2 - bc \) so that \( c = \frac{b \pm \sqrt{4a^2 - 3b^2}}{2} \)

with \( 1 \leq b < \frac{2\sqrt{3}a}{3} \). A computer search yields the following solutions with \((a,b,c) = 1\):

\[ i) \quad (b,c) = (7,15), (8,15), (15,7), (15,8). \]
\( ii \) \((b,c) = (15, 176), (161, 176), (176, 15), (176, 161).\)

\( iii \) \((b,c) = (5055, 30751), (25696, 30751), (30751, 5055), (30751, 25696).\)

**Solution 2 by Albert Stadler, Herrliberg, Switzerland**

(Editor: Albert gave two solutions to this problem; in the first solution he used the above computer-aided approach. But in his second solution he used the complex roots of unity.)

A more inspired approach is based on Eisenstein integers (see e.g., http://en.wikipedia.org/wiki/Eisenstein_integer).

Let \( \omega = \frac{-1 + i\sqrt{3}}{2} \). The set of Eisenstein integers \( \mathbb{Z}[\omega] = \{a + b\omega | a, b \in \mathbb{Z}\} \) has the following properties:

- (i) \( \mathbb{Z}[\omega] \) forms a commutative ring of algebraic integers in the real number field \( \mathbb{Q}(\omega) \)
- (ii) \( \mathbb{Z}[\omega] \) is an Euclidean domain whose norm \( N \) is given by \( N(a + b\omega) = a^2 - ab + b^2 \). As a result of this \( \mathbb{Z}[\omega] \) is a factorial ring.
- (iii) The group of units in \( \mathbb{Z}[\omega] \) is the cyclic group formed by the sixth root of unity in the complex plane. Specifically, they are \( \{-1, \pm \omega, \pm \omega^2\} \). These are just the Eisenstein integers of norm one.
- (iv) An ordinary prime number (or rational prime) which is 3 or congruent to 1 mod 3 is of the form \( x^2 - xy + y^2 \) for some integers \( x, y \) and may therefore be factored into \((x + y\omega)(x + y\omega^2)\) and because of that it is not prime in the Eisenstein integers. Ordinary primes congruent to 2 mod 3 cannot be factored in this way and they are primes in the Eisenstein integers as well.

Based on the above we find the factorization \( 13 = (4 + \omega)(4 + \omega^2) \), where \( 4 + \omega \) and \( 4 + \omega^2 \) are two Eisenstein primes that are conjugate to each other. So \( 13^n = (4 + \omega)^n(4 + \omega^2)^n \), and this is (up to units) the only factorization into two factors of the form \( b + c\omega \) with \( b \) and \( c \) coprime. We find

\[
(4 + \omega)^2 = 16 + 8\omega + \omega^2 = 15 + 7\omega,
\]

\[
(4 + \omega)^2(\omega^2) = -15\omega^2 - 7 = 8 + 15\omega,
\]

\[
(4 + \omega)^4 = (15 + 7\omega)^2 = 225 + 210\omega + 49\omega^2 = 176 + 161\omega,
\]

\[
(4 + \omega)^4(\omega^2) = (176 + 161\omega)(-\omega^2) = -176\omega^2 - 161 = 15 + 176\omega,
\]

\[
(4 + \omega)^8 = (176 + 161\omega)^2 = 30976 + 56672\omega + 25921\omega^2 = 5055 + 30751\omega
\]

\[
(4 + \omega)^8(\omega) = (5055 + 30751\omega)(\omega) = -5055\omega - 30751\omega^2 = 30751 + 25696\omega.
\]
We note that
\[ N((4 + \omega)^2) = N((4 + \omega)^2(-\omega^2)) = 13^2 \]
\[ N((4 + \omega)^4) = N((4 + \omega)^4(-\omega^2)) = 13^4 \]
\[ N((4 + \omega)^8) = N((4 + \omega)^8(-\omega^2)) = 13^8 \]
\[ N(x + y\omega) = x^2 - xy + y^2, \]
and we get the same solutions as with the exhaustive computer search.

Also solved by Brian D. Beasley, Clinton, SC; Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie, San Angelo TX; Bruno Salgueiro Fanego, Viveiro Spain; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5190**: Proposed by Neculai Stanciu, Buzău, Romania

  Prove: If \( x, y \) and \( z \) are positive integers such that \( \frac{x(y + 1)}{x - 1} \in \mathbb{N}, \frac{y(z + 1)}{y - 1} \in \mathbb{N}, \) and \( \frac{z(x + 1)}{z - 1} \in \mathbb{N}, \) then \( xyz \leq 693. \)

  **Solution by Kee-Wai Lau, Hong Kong, China**

  Since two consecutive positive integers are relatively prime, so in fact
  \[
y + 1 = a(x - 1), \quad z + 1 = b(y - 1), \quad x + 1 = c(z - 1),
  \]
  where \( a, b, c \in \mathbb{N} \) and \( abc < 1. \) Solving (1) for \( x, y, z \) we obtain
  \[
x = \frac{1 + 2c + 2bc + abc}{abc - 1}, \quad y = \frac{1 + 2a + 2ac + abc}{abc - 1}, \quad \frac{1 + 2b + 2ab + abc}{abc - 1}. \quad (2)
  \]
  Also, we have from (1) that
  \[
  abc = \left(\frac{x + 1}{x - 1}\right) \left(\frac{y + 1}{y - 1}\right) \left(\frac{z + 1}{z - 1}\right) \leq (3)(3)(3) = 27. \quad (3)
  \]
  Using (3), we check with a computer that \( x, y, \) and \( z \) of (2) are positive integers if and only if

  \[
  (a, b, c) = (1, 1, 2), (1, 1, 3), (1, 1, 4), (1, 1, 7), (1, 2, 1), (1, 2, 3), (1, 3, 1), (1, 3, 5)(1, 4, 1),
  (1, 7, 1), (2, 1, 1), (2, 2, 2, ) (2, 3, 1), (3, 1, 1), (3, 1, 2), (3, 3, 3), (3, 5, 1), (4, 1, 1),
  (5, 1, 3), (7, 1, 1).
  \]

  Correspondingly,

  \[
  (x, y, z) = (11, 9, 7), (8, 6, 4), (7, 5, 3), (6, 4, 2), (9, 7, 11), (5, 3, 3), (6, 4, 8), (4, 2, 2), (5, 3, 7),
  (4, 2, 6), (7, 11, 9), (3, 3, 3), (3, 3, 5), (4, 8, 6), (3, 5, 3), (2, 2, 2), (2, 2, 4), (3, 7, 5),
  (2, 4, 2), (2, 6, 4).
  \]
Hence, $xyz \leq 693$ as desired.

Also solved by Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie, San Angelo TX; Paul M. Harms, North Newton, KS; Albert Stadler, Herrilberg, Switzerland; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5191**: Proposed by José Luis Díaz-Barrero, Barcelona, Spain
  
  Let $a, b, c$ be positive real numbers such that $ab + bc + ca = 3$. Prove that
  
  $$\frac{a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}}{a^4 + b^4 + c^4} \leq 1.$$ 

**Solution 1 by Paul M. Harms, North Newton, KS**

From inequalities of the type $\sqrt{bc} \leq \frac{b + c}{2}$ we see that

$$a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \leq a \frac{b + c}{2} + b \frac{c + a}{2} + c \frac{a + b}{2} = ab + bc + ca.$$ 

If we can show that the last expression is less than or equal to $(a^4 + b^4 + c^4)$, then the inequality of the problem is correct.

From the condition in the problem $ab + bc + ca = 3$, so we will prove that

$$\frac{3}{a^4 + b^4 + c^4} \leq 1.$$ 

Let $a \leq b \leq c$ with $b = ta$ and $c = sa$ where $1 \leq t \leq s$. Then

$$3 \leq a^4 + b^4 + c^4 = (a^2)^2 \left(1 + t^4 + s^4\right).$$

$$ab + bc + ca = 3 \Rightarrow a^2 = \frac{3}{t + ts + s}. \text{ We must prove that}$$

$$3 \leq \left[\frac{3}{t + ts + s}\right]^2 \left(1 + t^4 + s^4\right) \text{ or equivalently,}$$

$$(t + ts + s)^2 = t^2 + 2st^2 + s^2t^2 + 2st + 2s^2t + s^2 \leq 3 \left(1 + t^4 + s^4\right) \text{ for } 1 \leq t \leq s, \text{ i.e.,}$$

$$0 \leq 3t^4 + 3s^4 - t^2 - s^2 - 2t^2s - 2s^2t - t^2s - 2st + 3 \text{ for } 1 \leq t \leq s.$$ 

Let $f(t, s)$ be the right side of the last inequality. We use partial derivatives to help find the minimum of the function in the appropriate domain.

Subtracting the equations $f_t(t, s) = 0$ and $f_s(t, s) = 0$ we obtain:

$$12(t^3 - s^3) - 2(t-s) - 2(s^2 - t^2) - 2st(s-t) - 2(s-t) = 0 = 2(t-s) \left[6t^2 + 7st + 6s^2 + s + t\right].$$

The part in the brackets is clearly positive for $1 \leq t \leq s$ so we must check $t = s$ and other boundary points of the domain for a minimum of the function.
When \( t = s \),
\[
f(t, s) = f(s, s) = 5s^4 - 4s^3 - 4s^2 + 3 = (s - 1)^2 \left[ 5s^2 + 6s + 3 \right].
\]
The function has a minimum in this case for \( t = s = 1 \). For the boundary \( t = 1 \) with \( t = 1 \leq s \),
\[
f(1, s) = (s - 1)^2 \left[ 3s^2 + 6s + 5 \right]
\]
which again has a minimum for \( t = s = 1 \). Since \( f(s, t) \geq f(1, 1) = 0 \) for \( 1 \leq t \leq s \), the inequality of the problem has been proved.

**Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX**

Using the well-known result \( x^2 + y^2 \geq 2xy \), with equality if and only if \( x = y \), we obtain
\[
xy + yz + zx \leq \frac{1}{2} \left[ (x^2 + y^2) + (y^2 + z^2) + (z^2 + x^2) \right] = x^2 + y^2 + z^2, \tag{1}
\]
and consequently,
\[
(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) \leq 3 \left( x^2 + y^2 + z^2 \right). \tag{2}
\]
Further, equality is attained in (1) or (2) if and only if \( x = y = z \).

By (1),
\[
a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} = \sqrt{ab}\sqrt{ca} + \sqrt{bc}\sqrt{ab} + \sqrt{ca}\sqrt{bc} \leq ab + bc + ca = 3, \tag{3}
\]
with equality if and only if \( \sqrt{ab} = \sqrt{bc} = \sqrt{ca} \), i.e., if and only if \( a = b = c = 1 \).

Also, since \( a, b, c > 0 \), (1) and (2) imply that
\[
9 = (ab + bc + ca)^2 \leq (a^2 + b^2 + c^2)^2 \leq 3 \left( a^4 + b^4 + c^4 \right)
\]
and hence,
\[
a^4 + b^4 + c^4 \geq 3. \tag{4}
\]
Once again, equality is achieved in (4) if and only if \( a = b = c = 1 \).
Therefore, by (3) and (4),
\[
\frac{a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}}{a^4 + b^4 + c^4} \leq \frac{3}{3} = 1,
\]
with equality if and only if \(a = b = c = 1\).

**Solution 3 by Albert Stadler, Herrliberg, Switzerland**

The homogeneous form of this inequality reads as
\[
(a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}) (ab + bc + ca) \leq 3 \left( a^4 + b^4 + c^4 \right)
\]
or equivalently as
\[
a^2b^2c^2 + a^2b^2c^2 + a^2b^2c^2 + a^2b^2c^2 + a^2b^2c^2 + a^2b^2c^2 + a^2b^2c^2 + a^2b^2c^2 \leq 3 \left( a^4 + b^4 + c^4 \right).
\]

By the weighted AM-GM inequality
\[
a^4b^4c^4 \leq ra^4 + sb^4 + tc^4 \tag{1}
\]
for all tuples \((r,s,t)\) of positive real numbers \(r, s, \) and \(t\) such that \(r + s + t = 1\). We write down the nine inequalities that result from (1) by choosing:
\[
(r, s, t) = \left( \frac{1}{2}, \frac{3}{8}, \frac{1}{8} \right), \left( \frac{3}{8}, \frac{1}{2}, \frac{1}{8} \right), \left( \frac{3}{8}, \frac{1}{8}, \frac{3}{2} \right), \left( \frac{3}{8}, \frac{1}{8}, \frac{3}{4} \right), \\
\left( \frac{3}{8}, \frac{1}{8}, \frac{3}{4} \right), \left( \frac{1}{2}, \frac{3}{8}, \frac{1}{8} \right), \left( \frac{3}{8}, \frac{1}{2}, \frac{1}{8} \right), \left( \frac{3}{8}, \frac{1}{8}, \frac{3}{4} \right), \left( \frac{3}{8}, \frac{1}{8}, \frac{3}{2} \right).
\]

and add them up. The result follows.

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Ercole Suppa, Teramo, Italy; Paolo Perfetti, Department of Mathematics, University “Tor Vergata,” Rome, Italy; Boris Rays, Brooklyn, NY; Titu Zvonaru, Comănești, Romania jointly with Neculai Stanciu, Buzău, Romania, and the proposer.

**5192: Proposed by G. C. Greubel, Newport News, VA**

Let \([n] = [n]_q = \frac{1 - q^n}{1 - q}\) be a \(q\) number and \(\ln_q(x) = \sum_{n=1}^{\infty} \frac{x^n}{[n]}\) be a \(q\)-logarithm. Evaluate the following series:

\[
i) \sum_{k=0}^{\infty} \frac{q^{mk}}{[mk + 1][mk + m + 1]} \quad \text{and}
\]

\[
ii) \sum_{k=1}^{\infty} \frac{x^k}{[k][k + m]}.
\]

**Solution 1 by Kee-Wai Lau, Hong Kong, China**

For \(0 < |q| < 1\) and for \(0 < |x| < 1\), we have
i) 

\[
\sum_{k=0}^{\infty} \frac{q^{mk}}{mk+1[mk+m+1]} = \frac{1}{q[m]} \sum_{k=0}^{\infty} \left( \frac{1}{mk+1} - \frac{1}{mk+m+1} \right) \\
= \frac{1}{q[m]} (1 - (1 - q)) \\
= \frac{1}{m}.
\]

and ii) 

\[
\sum_{k=1}^{\infty} \frac{x^k}{k[k+m]} = \frac{1}{m} \sum_{k=1}^{\infty} \left( \frac{x^k}{[k]} - \frac{q^m x^k}{[k+m]} \right) \\
= \frac{1}{m} \left( \sum_{k=1}^{\infty} \frac{x^k}{[k]} - \left( \frac{q}{x} \right)^m \sum_{k=1}^{\infty} \frac{x^{k+m}}{[k+m]} \right) \\
= \frac{1}{m} \left( \ln_q(x) - \left( \frac{q}{x} \right)^m \ln_q(x) + \left( \frac{q}{x} \right)^m \sum_{k=1}^{m} \frac{x^k}{[k]} \right).
\]

Solution 2 by Albert Stadler, Herrliberg, Switzerland

i) 

\[
\sum_{k=0}^{\infty} \frac{q^{mk}}{mk+1[mk+m+1]} = (1 - q)^2 \sum_{k=0}^{\infty} \frac{q^{mk}}{(1 - q^{mk+1})(1 - q^{mk+m+1})} \\
= (1 - q)^2 \sum_{k=0}^{\infty} q^{mk} \left( \frac{1}{1 - q^{mk+1}} - \frac{1}{1 - q^{mk+m+1}} \right) \frac{1}{q^{mk+1} - q^{mk+m+1}} \\
= \frac{(1 - q)^2}{q - q^{m+1}} \sum_{k=0}^{\infty} \left( \frac{1}{1 - q^{mk+1}} - \frac{1}{1 - q^{mk+m+1}} \right) \\
= \frac{(1 - q)^2}{q - q^{m+1}} \sum_{k=0}^{\infty} \left( \frac{1}{1 - q^{mk+1}} - \frac{1}{1 - q^{m(k+1)+1}} \right)
\]
a) Prove that 

\[
\frac{(1 - q)^2}{q - q^{m+1}} \cdot \left( \frac{1}{1 - q} - 1 \right)
\]

\[
= \frac{(1 - q)}{1 - q^m}
\]

ii)

\[
\sum_{k=1}^{\infty} \frac{x^k}{[k][k + m]} = (1 - q)^2 \sum_{k=1}^{\infty} \frac{x^k}{(1 - q^k)(1 - q^{k+m})}
\]

\[
= (1 - q)^2 \sum_{k=1}^{\infty} x^k \left( \frac{1}{1 - q^k} - \frac{1}{1 - q^{k+m}} \right) \frac{1}{q^k - q^{k+m}}
\]

\[
= \frac{(1 - q)^2}{1 - q^m} \sum_{k=1}^{\infty} \left( \frac{x}{q} \right)^k \left( \frac{1}{1 - q^k} - \frac{1}{1 - q^{k+m}} \right)
\]

\[
= \frac{(1 - q)^2}{1 - q^m} \sum_{k=1}^{\infty} \left[ x^k \left( \frac{1}{q^k} + \frac{1}{1 - q^k} \right) - x^k q^m \left( \frac{1}{q^{k+m}} + \frac{1}{1 - q^{k+m}} \right) \right]
\]

\[
= \frac{(1 - q)^2}{1 - q^m} \sum_{k=1}^{\infty} \left[ x^k \left( \frac{1}{1 - q^k} \right) - x^k q^m \left( \frac{1}{1 - q^{k+m}} \right) \right]
\]

\[
= \frac{(1 - q)}{1 - q^m} \sum_{k=1}^{\infty} \frac{x^k}{[k]} - \frac{(1 - q)^2 q^m}{(1 - q^m)x^m} \sum_{k=1}^{\infty} x^{k+m} \left( \frac{1}{1 - q^{k+m}} \right)
\]

\[
= \frac{(1 - q)}{1 - q^m} \sum_{k=1}^{\infty} \frac{x^k}{[k]} - \frac{(1 - q)^2 q^m}{(1 - q^m)x^m} \left[ \frac{x}{1 - q} - \frac{x^2}{1 - q^2} - \cdots - \frac{x^m}{1 - q^m} + \sum_{k=1}^{\infty} x^k \left( \frac{1}{1 - q^k} \right) \right]
\]

\[
= \frac{1 - q}{1 - q^m} \sum_{k=1}^{\infty} \frac{x^k}{[k]} + \frac{(1 - q)^2 q^m}{(1 - q^m)x^m} \left[ \frac{x}{1 - q} + \frac{x^2}{1 - q^2} + \cdots + \frac{x^m}{1 - q^m} \right] - \frac{(1 - q)q^m}{(1 - q^m)x^m} \sum_{k=1}^{\infty} \frac{x^k}{[k]}
\]

\[
= \frac{1 - q}{1 - q^m} \left( 1 - \frac{q}{x} \right) \ln_q(x) + \frac{(1 - q)q^m}{(1 - q^m)} \left[ \frac{x^{1-m}}{1} + \frac{x^{2-m}}{1+q} + \cdots + \frac{1}{1+q+q^2+\cdots+q^{m-1}} \right].
\]

Also solved by Arkady Alt, San Jose, CA, and the proposer.

• 5193: Proposed by Ovidiu Furdui, Cluj-Napoca, Romania

Let \( f \) be a function which has a power series expansion at 0 with radius of convergence \( R \).

a) Prove that \( \sum_{n=1}^{\infty} n f^{(n)}(0) \left( e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} \right) = \int_0^x e^{x-t} f'(t) dt, \quad |x| < R. \)
b) Let $\alpha$ be a non-zero real number. Calculate $\sum_{n=1}^{\infty} n\alpha^n \left( e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} \cdots - \frac{x^n}{n!} \right)$.

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

a) Let $S(x)$ be the sum of the series. Then, by differentiation, and for $|x| < R$,

$$S'(x) = \sum_{n=1}^{\infty} nf^{(n)}(0) \left( e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} \cdots - \frac{x^n}{n!} \right) = S(x) + \sum_{n=1}^{\infty} n^2 f^{(n)}(0) \cdot \frac{x^n}{n!}.$$ 

It follows that $S'(x) = S(x) + xf'(x)$, and hence

$$S(x) = \int_0^x e^{x-t} f'(t) dt + Ce^x,$$

where $C$ is a constant of integration. Because $S(0) = 0$, we have $C = 0$ and

$$S(x) = \int_0^x e^{x-t} f'(t) dt.$$ 

b) Note that if $f(x) = e^{\alpha x}$ then $f^{(n)}(0) = \alpha^n$, for $n \geq 1$. Hence, by part a), the sum of the given series is

$$\int_0^x e^{x-t} \alpha e^{\alpha t} dt = \frac{x^2 e^x}{2}$$ if $\alpha = 1$. If $\alpha \neq 1$, the sum of the series is

$$\int_0^x e^{x-t} \alpha e^{\alpha t} dt = \frac{\alpha x e^{\alpha x}}{\alpha - 1} + \frac{\alpha^2 (e^x - e^{\alpha x})}{(\alpha - 1)^2}.$$ 

Solution 2 by Anastasios Kotronis, Athens, Greece

a) From the problem’s assumptions we have that

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n$$ and $$f'(x) = \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!} x^{n-1}$$ for $|x| < R$, so, for $|x| < R$ we obtain

$$\int_0^x e^{x-t} f'(t) dt = \int_0^x e^{x-t} \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!} t^n dt$$

$$= e^x \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!} \int_0^x t^n e^{-t} dt$$

$$= e^x \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!} I_n. \quad (1)$$

Now $I_n = -\int_0^x t^n (e^{-t})' dt = -x^n e^{-x} + nI_{n-1}$, so it is easily verified by induction that

$$I_n = -e^{-x} \left( x^n + nx^{n-1} + \cdots + n! x^0 \right) + n!$$

With the above, (1) will give
\[
\int_0^x e^{x-t} f'(t) \, dt = e^x \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!} \left( -e^{-x} \left( x^n + nx^{n-1} + \cdots + n!x^0 \right) + n! \right) \\
= \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!} \left( n!e^x - x^n - nx^{n-1} - \cdots - n!x^0 \right) \\
= \sum_{n=1}^{+\infty} n f^{(n)}(0) \left( e^x - 1 - \frac{x}{1!} - \cdots - \frac{x^n}{n!} \right).
\]

2) From (1) with \( f(x) = e^{\alpha x} \) we obtained that
\[
\sum_{n=1}^{+\infty} n \alpha^n \left( e^x - 1 - \frac{x}{1!} - \cdots - \frac{x^n}{n!} \right) = \int_0^x e^{x-t} \alpha te^{\alpha t} \, dt \\
= I_\alpha.
\]

So,
\[
\begin{cases}
\int_0^x e^{x-t} t e^t \, dt = \frac{x^2 e^x}{2}, & \text{for } \alpha = 1 \\
I_\alpha = \alpha e^x \left( \int_0^x t \left( \frac{e^{(\alpha-1)t}}{\alpha-1} \right) \, dt \right), & \text{for } \alpha \neq 1 \\
= \frac{\alpha e^{\alpha x}}{\alpha-1} \left( x - \frac{1}{\alpha-1} \right) + \frac{\alpha e^x}{(\alpha-1)^2}.
\end{cases}
\]

Solution 3 by Arkady Alt, San Jose, CA

a) Let
\[ a_n(x) = e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}, \ n \in N \cup \{0\} \] and \( F(x) = \sum_{n=1}^{\infty} n f^{(n)}(0) a_n(x). \)

Noting that
\[ a'_n(x) = e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^{n-1}}{(n-1)!} \]
\[ = a_{n-1}(x), \ n \in N \]
we obtain
\[
F'(x) = \left( \sum_{n=1}^{\infty} n f^{(n)}(0) a_n(x) \right)' \\
= \sum_{n=1}^{\infty} n f^{(n)}(0) a'_n(x) \\
= \sum_{n=1}^{\infty} n f^{(n)}(0) a_{n-1}(x).
\]
\[ F(x) - F'(x) = \sum_{n=1}^{\infty} n f^{(n)}(0) (a_n(x) - a_{n-1}(x)) \]
\[ = \sum_{n=1}^{\infty} n f^{(n)}(0) \left( -\frac{x^n}{n!} \right) \]
\[ = -\sum_{n=1}^{\infty} f^{(n)}(0) \frac{x^n}{(n-1)!} \]
\[ = -x \sum_{n=0}^{\infty} f^{(n+1)}(0) \frac{x^n}{n!} \]
\[ = -xf'(x). \]

Multiplying equation \( F'(x) - F(x) = xf'(x) \) by \( e^{-x} \) we obtain
\[ F'(x) e^{-x} - F(x) e^{-x} = e^{-x}xf'(x) \iff (F(x) e^{-x})' = e^{-x}xf'(x). \]

Hence,
\[ F(x) e^{-x} = \int_0^x e^{-t}t f'(t) \, dt \]
\[ \iff F(x) = \int_0^x e^{x-t}t f'(t) \, dt. \]

b) Let \( f(x) = e^{\alpha x} \) then \( f^{(n)}(0) = \alpha^n \) and, using the result we obtained in part (a) we get,
\[ \sum_{n=1}^{\infty} n \alpha^n \left( e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} \right) = \int_0^x e^{x-t}t \alpha e^{\alpha t} \, dt \]
\[ = \alpha e^x \int_0^x t e^{t(\alpha-1)} \, dt. \]

If \( \alpha = 1 \) then
\[ \int_0^x t e^{t(\alpha-1)} \, dt = \frac{x^2}{2} \]
and, therefore,
\[ \sum_{n=1}^{\infty} n \alpha^n \left( e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} \right) = \sum_{n=1}^{\infty} n \left( e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} \right) \]
\[ = \alpha e^x \frac{x^2}{2}. \]

If \( \alpha \neq 1 \) then
\[ \int_0^x t e^{t(\alpha-1)} \, dt = \frac{x \alpha e^{x(\alpha-1)} x}{\alpha - 1} - \frac{e^{x(\alpha-1)} x}{(\alpha - 1)^2}. \]
\[ e^{(a-1)x} \frac{x (\alpha - 1) - 1}{(\alpha - 1)^2}. \]

Hence,
\[
\sum_{n=1}^{\infty} n\alpha^n \left( e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \ldots - \frac{x^n}{n!} \right) = \frac{\alpha e^{\alpha x} (x (\alpha - 1) - 1)}{(\alpha - 1)^2}.
\]

Solution 4 by Paolo Perfetti, Department of Mathematics, University “Tor Vergata,” Rome, Italy

a) We need the two lemmas:

Lemma 1 \( m!n! \leq (n + m)! \)

Proof by Induction. Let \( m \) be fixed. If \( n = 0 \) evidently holds true. Let’s suppose that the statement is true for any \( 1 \leq n \leq r \). For \( n = r + 1 \) we have

\[ m!(r + 1)! = m!r!(r + 1) \leq (m + r)!(r + 1) \leq (m + r)!(m + r + 1) = (m + r + 1)! \]

which clearly holds for any \( m \geq 0 \). Since the inequality is symmetric, the induction on \( m \) proceeds along the same lines. q.e.d.

Lemma 2 The power series
\[
\sum_{n=1}^{\infty} n f^{(n)}(0) \left( e^x - 1 - x \frac{1}{1!} - \frac{x^2}{2!} - \ldots - \frac{x^n}{n!} \right) = \sum_{n=1}^{\infty} n f^{(n)}(0) \sum_{k=n+1}^{\infty} \frac{x^k}{k!}
\]
converges for \( |x| < R \) and is differentiable.

Proof:
\[
\sum_{k=n+1}^{\infty} \frac{x^k}{k!} = \frac{x^{n+1}}{(n + 1)!} \sum_{k=n+1}^{\infty} \frac{x^{k-n-1}(n + 1)!}{k!}.
\]

By using the Lemma 1 we can bound
\[
\sum_{k=n+1}^{\infty} \frac{|x|^{k-n-1}(n + 1)!}{k!} \leq \sum_{k=n+1}^{\infty} \frac{|x|^{k-n-1}}{(k-n-1)!} = \sum_{k=0}^{\infty} \frac{|x|^k}{k!} = e^{|x|} \leq e^R.
\]

Thus we can write
\[
\sum_{n=0}^{\infty} n |f^{(n)}(0)| \sum_{k=n+1}^{\infty} \frac{|x|^k}{k!} \leq eR|x| \sum_{n=0}^{\infty} n |f^{(n)}(0)| \frac{|x|^n}{n!} \frac{n!}{(n + 1)!}
\]

Since
\[
\lim_{n \to \infty} \left| \frac{f^{(n)}(0)}{n!} \right|^{1/n} = R^{-1} \quad \Longrightarrow \quad \lim_{n \to \infty} \left| \frac{f^{(n)}(0)}{n!} \right|^{1/n} = R^{-1}
\]
the series
\[
\sum_{n=1}^{\infty} n f^{(n)}(0) \sum_{k=n+1}^{\infty} \frac{x^k}{k!}
\]
converges for any $|x| < R$. Its differentiability is a consequence of the standard theory on power–series so we don’t write it here. q.e.d.

The function $\int_0^x e^{x-t} t f'(t) dt$ is also differentiable by the fundamental theorem of calculus and the derivative yields

$$\left( \int_0^x e^{x-t} t f'(t) dt \right)' = x f'(x) + \int_0^x e^{x-t} t f'(t) dt$$

namely it satisfies the ordinary differential equation $Q'(x) = Q(x) + x f'(x)$, $Q(0) = 0$.

The derivative of the series in question a) is

$$\sum_{n=1}^{\infty} n f^{(n)}(0) \left( e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^{n-1}}{(n-1)!} \right)$$

that is

$$\sum_{n=1}^{\infty} n f^{(n)}(0) \sum_{k=n+1}^{\infty} \frac{x^k}{k!} + \sum_{n=1}^{\infty} n f^{(n)}(0) \frac{x^n}{n!}$$

which is in turn equals

$$= \sum_{n=1}^{\infty} n f^{(n)}(0) \sum_{k=n+1}^{\infty} \frac{x^k}{k!} + x f'(x)$$

Moreover $\left( \sum_{n=1}^{\infty} n f^{(n)}(0) \sum_{k=n+1}^{\infty} \frac{x^k}{k!} \right) \bigg|_{x=0} = 0$ thus the functions $\int_0^x e^{x-t} t f'(t) dt$ and

$$\sum_{n=1}^{\infty} n f^{(n)}(0) \left( e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} \right)$$

satisfy the same differential equation with the same initial condition. By the uniqueness theorem for ODE, they are the same function. This concludes the proof.

b) $\alpha^n = (e^{\alpha x})^{(n)} \bigg|_{x=0}$ thus

$$\sum_{n=1}^{\infty} n \alpha^n \left( e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} \right) = \int_0^x e^{x-t} t \alpha e^{\alpha t} dt$$

If $\alpha = 1$ we obtain $\int_0^x t e^x dt = \frac{x^2 e^x}{2}$.

If $\alpha \neq 1$ we obtain integrating by parts

$$\alpha e^x \int_0^x t e^{t(a-1)} dt = \alpha e^x \left( \frac{1}{\alpha - 1} t e^{t(a-1)} \bigg|_0^x - \frac{1}{\alpha - 1} \int_0^x e^{t(a-1)} dt \right)$$

$$= \frac{\alpha x e^{\alpha x}}{\alpha - 1} - \frac{\alpha e^{\alpha x}}{(\alpha - 1)^2} + \frac{\alpha e^x}{(\alpha - 1)^2}.$$
Also solved by Dionne T. Bailey, Elsie M. Campbell, Charles Diminnie, and Andrew Siefker, San Angelo, TX; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland, and the proposer.

Mea Culpa

The name of Achilleas Sinefakopoulos of Larissa, Greece was inadvertently omitted in the March issue of the column as having solved problem 5184. I am terrible sorry for this oversight—Ted.