Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before
June 15, 2011

- **5158:** *Proposed by Kenneth Korbin, New York, NY*

  Given convex cyclic quadrilateral $ABCD$ with integer length sides $\overline{AB} = \overline{BC} = x$, and $\overline{CD} = \overline{DA} = x + 1$.

  Find the distance between the incenter and the circumcenter.

- **5159:** *Proposed by Kenneth Korbin, New York, NY*

  Given square $ABCD$ with point P on diagonal $\overline{AC}$ and with point Q at the midpoint of side $\overline{AB}$.

  Find the perimeter of cyclic quadrilateral $ADPQ$ if its area is one unit less than the area of square $ABCD$.

- **5160:** *Proposed by Michael Brozinsky, Central Islip, NY*

  In Cartesianland, where immortal ants live, there are n (where $n \geq 2$) roads $\{l_i\}$ whose equations are

  $$l_i:\ x\cos\left(\frac{2\pi i}{n}\right) + y\sin\left(\frac{2\pi i}{n}\right) = i, \text{where } i = 1, 2, 3, \ldots, n.$$

  Any anthill must be located so that the sum of the squares of its distances to these n lines is $\dfrac{n(n+1)(2n+1)}{6}$. Two queen ants are (im)mortal enemies and have their anthills as far apart as possible. If the distance between these queens' anthills is 4 units, find n.

- **5161:** *Proposed by Paolo Perfetti, Department of Mathematics, University "Tor Vergata, Rome, Italy*

  It is well known that for any function $f : \Re \rightarrow \Re$, continuous or not, the set of points on the y-axis where it attains a maximum or a minimum can be at most denumerable. Prove that any function can have at most a denumerable set of inflection points, or give a counterexample.

- **5162:** *Proposed by José Luis Díaz-Barrero and José Gibergans-Báguena, Barcelona, Spain*

Let a, b, c be the lengths of the sides of an acute triangle ABC. Prove that

$$\sqrt{\frac{b^2+c^2-a^2}{a^2+2bc}}+\sqrt{\frac{c^2+a^2-b^2}{b^2+2ca}}+\sqrt{\frac{a^2+b^2-c^2}{c^2+2ab}} \le \sqrt{3}.$$

- **5163:** *Proposed by Pedro H. O. Pantoja, Lisbon, Portugal*

Prove that for all $n \in N$

$$\int_0^\infty \frac{x^n}{2}\left(\coth\frac{x}{2}-1\right)dx = \sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\frac{1}{k_1\cdots k_n\,(k_1+\cdots+k_n)}.$$

Solutions

- **5140:** *Proposed by Kenneth Korbin, New York, NY*

Given equilateral triangle ABC with an interior point P such that

$$\begin{aligned}\overline{AP} &= 22+16\sqrt{2}\\ \overline{BP} &= 13+9\sqrt{2}\\ \overline{CP} &= 23+16\sqrt{2}.\end{aligned}$$

Find $\overline{AB}$.

Solution 1 Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX

Let α be the length of each side of triangle ABC with vertices $A\left(0, \frac{\sqrt{3}}{2}\alpha\right)$, $B\left(-\frac{\alpha}{2}, 0\right)$, and $C\left(\frac{\alpha}{2}, 0\right)$, and let $P(x, y)$ be an interior point of the triangle. Then,

$$\begin{aligned}\overline{AP}^2 &= (22+16\sqrt{2})^2 &= x^2+(y-\frac{\sqrt{3}}{2}\alpha)^2, \quad (1)\\ \overline{BP}^2 &= (13+9\sqrt{2})^2 &= (x+\frac{\alpha}{2})^2+y^2, \quad (2)\\ \overline{CP}^2 &= (23+16\sqrt{2})^2 &= (x-\frac{\alpha}{2})^2+y^2. \quad (3)\end{aligned}$$

Using (2)and (3), it follows that

$$(13+9\sqrt{2})^2-(23+16\sqrt{2})^2 = 2\alpha x,$$

and

$$x = \frac{(13+9\sqrt{2})^2-(23+16\sqrt{2})^2}{2\alpha}$$

$$
\begin{aligned}
&= \frac{-710-502\sqrt{2}}{2\alpha} \\
&= -\frac{355+251\sqrt{2}}{\alpha}. \qquad (4)
\end{aligned}
$$

Therefore, using (1), (2), and (3),

$$(13+9\sqrt{2})^2+(23+16\sqrt{2})^2-2(22+16\sqrt{2})^2=2\sqrt{3}\alpha y-\alpha^2, \text{ and}$$

$$
\begin{aligned}
y &= \frac{\alpha^2+(13+9\sqrt{2})^2+(23+16\sqrt{2})^2-2(22+16\sqrt{2})^2}{2\sqrt{3}\alpha} \\
&= \frac{\alpha^2-620-438\sqrt{2}}{2\sqrt{3}\alpha} \qquad (5)
\end{aligned}
$$

Hence, using (2), (4), and (5),

$$
\begin{aligned}
(13+9\sqrt{2})^2 &= \left(-\frac{355+251\sqrt{2}}{\alpha}+\frac{\alpha}{2}\right)^2+\left(\frac{\alpha^2-620-438\sqrt{2}}{2\sqrt{3}\alpha}\right)^2 \\
331+234\sqrt{2} &= \frac{(\alpha^2-710-502\sqrt{2})^2}{4\alpha^2}+\frac{(\alpha^2-620-438\sqrt{2})^2}{12\alpha^2} \\
12(331+234\sqrt{2})\alpha^2 &= 3(\alpha^2-710-502\sqrt{2})^2+(\alpha^2-620-438\sqrt{2})^2 \\
0 &= 4\alpha^4-(9472+6696\sqrt{2})\alpha^2+(3,792,412+2,681,640\sqrt{2}) \\
0 &= \alpha^4-(2368+1674\sqrt{2})\alpha^2+(948,103+670,410\sqrt{2}).
\end{aligned}
$$

Thus, using the quadratic formula and a computer algebra system, the solutions are

$$
\begin{aligned}
\alpha^2 &= 2147+1518\sqrt{2} \qquad \text{or} \qquad \alpha^2=221+156\sqrt{2} \\
\Rightarrow \alpha &= 33+23\sqrt{2} \qquad \text{or} \qquad \alpha=\sqrt{13}(3+2\sqrt{2}).
\end{aligned}
$$

The solution $\alpha=\sqrt{13}(3+2\sqrt{2})$ is extraneous since this will make $y<0$. Thus,

$$\overline{AB}=\alpha=33+23\sqrt{2}.$$

Remark: If we substitute $\alpha=33+23\sqrt{2}$ into (4) and (5), we obtain

$$x=-\frac{169+118\sqrt{2}}{31} \quad \text{and} \quad y=\frac{\sqrt{3}}{62}(237+173\sqrt{2}).$$

Solution 2 by Valmir Bucaj (student, Texas Lutheran University), Seguin, TX

This problem is of the same nature as Problem 5081. It can be solved using *Tripolar Coordinates* and a result from Euler.

Here, we us another method and instead solve the more general problem: Let ABC be an equilateral triangle and P an interior point such that $\overline{AB} = a, \overline{BO} = b, \overline{CP} = c$. We will give a general formula for the dimensions of $\triangle ABC$

Let $(x, y), (s, 0), (0, 0), \left(\frac{s}{2}, \frac{\sqrt{3}}{2}s\right)$, be the coordinates of the points P, A, B, C respectively. Then,

$$\begin{aligned} d(P,B) &= \sqrt{x^2+y^2} = b, && (1)\\ d(P,A) &= \sqrt{(x-s)^2+y^2} = a, && (2)\\ d(C,P) &= \sqrt{\left(\frac{s}{2}-x\right)^2 + \left(\frac{\sqrt{3}}{2}s - y\right)^2} = c. && (3) \end{aligned}$$

Solving (1) for y and substituting into (2) we obtain an expression for x:

$$x = \frac{s^2+b^2-a^2}{2s}.$$

Now, substituting in for $y = \sqrt{b^2-x^2}$ and x in (3) we eventually obtain the following biquadratic equation on s:

$$s^4 - (a^2+b^2+c^2)s^2 + (a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2) = 0. \qquad (4)$$

$$s = \pm\sqrt{\frac{a^2+b^2+c^2}{2} \pm \sqrt{b^2c^2 - \left(\frac{b^2+c^2-a^2}{2}\right)^2} \cdot \sqrt{3}}.$$

Finally, the length of the sides of the given equilateral triangle are calculated by

$$s = \sqrt{\frac{a^2+b^2+c^2}{2} + \sqrt{b^2c^2 - \left(\frac{b^2+c^2-a^2}{2}\right)^2} \cdot \sqrt{3}}.$$

For the given problem we have

$$s = \sqrt{1518\sqrt{2}+2147} = \sqrt{\left(33+23\sqrt{2}\right)^2} = 33+23\sqrt{2}.$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that $\overline{AB} = 33+23\sqrt{2}$.

We first note that the given conditions determine $\overline{AB}$ uniquely. Suppose that $\overline{AB} = x$ and $\overline{AB} = y$ are two distinct solutions, say $y > x$. Then all the angles $\angle APB, \angle BPC, \angle CPA$ for the solution $\overline{AB} = y$ will be greater than the corresponding angles $\angle APB, \angle BPC, \angle CPA$ for the solution $\overline{AB} = x$. Ths is impossible because $\angle APB + \angle BPC + \angle CPA = 2\pi$ for both solutions.

We now need only show that $\overline{AB} = 33+23\sqrt{2}$ is a solution. We let

$$\cos\angle APB = \frac{-5}{7},\ \cos\angle BPC = \frac{-\left(5+3\sqrt{2}\right)}{14},\ \cos\angle CPA = \frac{-\left(5-3\sqrt{2}\right)}{14},\ \text{and}$$

$$\sin\angle APB = \frac{\sqrt{24}}{7},\ \sin\angle BPC = \frac{\sqrt{150}-\sqrt{3}}{14},\ \sin\angle CPA = \frac{\sqrt{150}+\sqrt{3}}{14}.$$

By the standard compound angle formula, we readily check that

$$\begin{aligned}\cos\left(\angle APB + \angle BPC + \angle CPA\right) &= 1, \text{ so in fact}\\ \angle APB + \angle BPC + \angle CPA &= 2\pi.\end{aligned}$$

By the cosine formula, we obtain, $\overline{AB} = \overline{BC} = \overline{CA} = 33+23\sqrt{2}$ as well and this completes the solution.

Editor's comment: Several other solution paths were used in solving this problem. One used a theorem that states that in any equilateral triangle ABC with side length a and with P being any point in the plane whose distances to the vertices A, B, C are respectively p, q and t, then $3(p^4+q^4+t^4+a^4) = (p^2+q^2+t^2+a^2)^2$. **Bruno Salgueiro Fanego** stated that a reference for this theorem can be found in the article *Curious properties of the circumcircle and incircle of an equilateral triangle,* by Prithwijit De, *Mathematical Spectrum* 41(1), 2008-2009, 32-35. This solution path works but one ends up solving the following equation:

$$13108182 + 9268884\sqrt{2} + 3\overline{AB}^4 = \left(2368+1674\sqrt{2}\right)^2 + 2\left(2368+1674\sqrt{2}\right)\overline{AB}^2 + \overline{AB}^4.$$

Another solution path dealt with using Heron's formula for the area of a triangle and using the fact that the area of ABC equals the sum of the areas of the three interior triangles APB, BPC, and CPA.

Also solved by Brian D. Beasley, Clinton, SC; Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany; Paul M. Harms, North Newton, KS; Edwin Gray, Highland Beach, FL; Bruno Salgueiro, Fanego (two solutions), Viveiro, Spain; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA; Ercole Suppa, Teramo, Italy, and the proposer.

- **5141:** *Proposed by Kenneth Korbin, New York, NY*

  A quadrilateral with sides $259, 765, 285, 925$ is constructed so that its area is maximum. Find the size of the angles formed by the intersection of the diagonals.

  Solution by David E. Manes, Oneonta, NY

  Given the four sides of a quadrilateral, the one with maximum area is the convex, cyclic quadrilateral. However, the four sides can be permuted to give different quadrilateral shapes. Furthermore, there are only three different shapes that yield maximized convex quadrilaterals; all others are simply rotations and reflections of these three.

  For the given problem these three quadrilaterals can be denoted by

$$\begin{aligned} I:\quad a &= 259,\ b=765,\ c=285,\ d=925\\ II:\quad a &= 259,\ b=285,\ c=765,\ d=925\end{aligned}$$

$$III: a = 259,\ b = 765,\ c = 925,\ d = 285$$

All three quadrilaterals yield the same maximum area A given by Brahmagupta's formula; that is , if

$$s = \frac{1}{2}(a+b+c+d) = \frac{1}{2}(259+765+285+925) = 1117 \text{ is the semiperimeter, then}$$

$$\begin{aligned} A &= \sqrt{(s-a)(s-b)(s-c)(s-d)} \\ &= \sqrt{(1117-259)(1117-765)(1117-285)(1117-9250)} \\ &= 219648 \end{aligned}$$

Let θ denote the intersection angle of the diagonals. If $\theta \neq 90°$, then

$$A = \frac{|\tan\theta|}{4}\left|a^2+b^2-b^2-d^2\right|.$$

For the quadrilateral in case I,

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{4A}{|a^2+c^2-b^2-d^2|}\right) \\ &= \tan^{-1}\left(\frac{4\cdot 219648}{|259^2+285^2-765^2-925^2|}\right) = 34.21°. \end{aligned}$$

Similarly, for case II, $\theta = 72.05°$ and in case III, $\theta = 73.74°$.

Also solved by Scott H. Brown, Montgomery, AL; Bruno Salgueiro, Fanego, Viveiro, Spain; Edwin Gray, Highland Beach, FL; Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5142:** *Proposed by Michael Brozinsky, Central Islip, NY*

  Let CD be an arbitrary diameter of a circle with center O. Show that for each point A distinct from O, C, and D on the line containing CD, there is a point B such that the line from D to any point P on the circle distinct from C and D bisects angle APB.

  Solution by Paul M. Harms, North Newton, KS

  Consider the unit circle with its center at the origin C at $(-1,0)$ and D at $(1,0)$. Let A be at $(a,0)$ were $0 < a < 1$ and let B be at $(b,0)$, where $b > 1$.

  To find the point B which satisfies the problem first consider P at $(a, \sqrt{1-a^2})$ just above point A. From the right triangle $APD, \tan\angle APD = \dfrac{1-a}{\sqrt{1-a^2}}$. The right triangle APB

has $\angle APB = 2\angle APD$. We have

$$\begin{aligned}
\tan\angle APB = \frac{b-a}{\sqrt{1-a^2}} &= \tan 2\angle APD \\
&= 2\left(\frac{\dfrac{1-a}{\sqrt{1-a^2}}}{1-\dfrac{(1-a)^2}{1-a^2}}\right) \\
&= \frac{\sqrt{1-a^2}}{a}.
\end{aligned}$$

Then, $b - a = \dfrac{1-a^2}{a}$, and $b = \dfrac{1}{a}$.

Now we check whether $B(1/a, 0)$ satisfies the problem for all points P on the upper half of the circle.

Let P be at $\left(t, \sqrt{1-t^2}\right)$ and let T be at $(t, 0)$ where $-1 < t < a$. Then

$$\begin{aligned}
\tan\angle APD = \tan\left(\angle TPD - \angle TPA\right) &= \frac{\dfrac{1-t}{\sqrt{1-t^2}} - \dfrac{a-t}{\sqrt{1-t^2}}}{1+\dfrac{(1-t)(a-t)}{1-t^2}} \\
&= \frac{(1-a)\sqrt{1-t^2}}{(1+a)(1-t)}.
\end{aligned}$$

Also

$$\begin{aligned}
\tan\angle DPB = \tan\left(\angle TPB - \angle TPD\right) &= \frac{\dfrac{\frac{1}{a}-t}{\sqrt{1-t^2}} - \dfrac{1-t}{\sqrt{1-t^2}}}{1+\dfrac{\left(\frac{1}{a}-t\right)(1-t)}{\sqrt{1-t^2}}} \\
&= \frac{\left(\frac{1}{a}-1\right)\sqrt{1-t^2}}{1+\frac{1}{a}-t-\frac{t}{a}} \\
&= \frac{(1-a)\sqrt{1-t^2}}{(1+a)(1-t)}
\end{aligned}$$

Since the tangents of the angles are equal we conclude that for the points P given above, $\angle APD$ bisects $\angle$APB.

Now consider points $P(t, \sqrt{1-t^2})$ where $a < t < 1$. The tangent of $\angle DPB$ is the same as above,

it is still $\dfrac{(1-a)\sqrt{1-t^2}}{(1+a)(1-t)}$. Now

$$\tan \angle APD = \tan\left(\angle TPD + \angle APT\right) = \frac{\dfrac{1-t}{\sqrt{1-t^2}} + \dfrac{t-a}{\sqrt{1-t^2}}}{1 - \dfrac{(1-t)(t-a)}{1-t^2}}$$

$$= \frac{(1-a)\sqrt{1-t^2}}{(1+a)(1-t)}.$$

From this we see that problem is also satisfied for these points. When P is on the bottom half of the circle at $\left(t, -\sqrt{1-t^2}\right)$, the triangles and angles used with P at $\left(t, \sqrt{1-t^2}\right)$, on the top half of the circle are congruent with those on the bottom half of the circle and should satisfy the problem.

If A is at $(a, 0)$, were $-1 < a < 0$, then using symmetry with respect the y-axis and point B at $\left(\dfrac{1}{a}, 0\right)$ satisfies the problem since all angles and triangles are congruent to those with A at $(|a|, 0)$.

If the circle has a radius of R, then we expect the same conclusion looking at similar triangles to those we used above. Consider C at $(-R, 0)$, D at $(R, 0)$, A at $(aR, 0)$, P at $(tR, R\sqrt{1-t^2})$, T at $(tR, 0)$, and B at $(R/a, 0)$ with the same restrictions on a and t as above. Using the new points, the tangents of the given lettered angles would have the same value as those given earlier since R would cancel for all the ratios involved with these tangents.

Also solved by Bruno Salgueiro, Fanego, Viveiro, Spain; Michael N. Fried, Kibbutz Revivim, Israel; Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5143:** *Proposed by Valmir Krasniqi (student), Republic of Kosova*

Show that

$$\sum_{n=1}^{\infty} Cos^{-1}\frac{1+\sqrt{n^2+2n}\cdot\sqrt{n^2-1}}{n\,(n+1)} = \frac{\pi}{2}.$$

$$\left(Cos^{-1} = Arccos\right)$$

Solution 1 by Bruno Sagueiro Fanego, Viveiro, Spain

For each $n = 1, 2, \ldots, \dfrac{1}{n+1} \in (0, 1]$ and $\dfrac{1}{n} \in (0, 1]$, there exists $\alpha_n, \beta_n \in \left[0, \dfrac{\pi}{2}\right)$ such that

$cos\ \alpha_n = \dfrac{1}{n+1}$, and $cos\ \beta_n = \dfrac{1}{n}$. That is, $\alpha_n = Cos^{-1}\dfrac{1}{n+1}$, and $\beta_n = Cos^{-1}\dfrac{1}{n}$. Hence,

$$sin\ \alpha_n = \sqrt{1-cos^2\ \alpha_n} = \sqrt{1-\frac{1}{(n+1)^2}} = \frac{\sqrt{n^2+2n}}{n+1} \text{ and}$$

$$sin\ \beta_n = \sqrt{1-cos^2\ \beta_n} = \sqrt{1-\frac{1}{(n)^2}} = \frac{\sqrt{n^2-1}}{n}.$$

Therefore,

$$\begin{aligned}\frac{1+\sqrt{n^2+2n}\cdot\sqrt{n^2-1}}{n\,(n+1)} &= \frac{1}{n\cdot(n+1)} + \frac{\sqrt{n^2+2n}\cdot\sqrt{n^2-1}}{n\cdot(n+1)}\\ &= \frac{1}{n+1}\cdot\frac{1}{n} + \frac{\sqrt{n^2-1}}{n}\cdot\frac{\sqrt{n^2+2n}}{n+1}\\ &= cos\ \alpha_n\cdot cos\ \beta_n + sin\ \alpha_n\cdot \sin\ \beta_n\\ &= cos\,(\alpha_n-\beta_n)\,.\end{aligned}$$

So,

$Cos^{-1}\dfrac{1+\sqrt{n^2+2n}\cdot\sqrt{n^2-1}}{n\,(n+1)} = Cos^{-1}cos(\alpha_n-\beta_n) = \alpha_n-\beta_n = Cos^{-1}\dfrac{1}{n+1} - Cos^{-1}\dfrac{1}{n}$, from where

$$\sum_{n=1}^{\infty} Cos^{-1}\frac{1+\sqrt{n^2+2n}\cdot\sqrt{n^2-1}}{n\,(n+1)} = \sum_{n=1}^{\infty} Cos^{-1}\frac{1}{n+1} - Cos^{-1}\frac{1}{n} = \lim_{m\to\infty}\sum_{n=1}^{m} Cos^{-1}\frac{1}{n+1} - Cos^{-1}\frac{1}{n}$$

$$= \lim_{m\to\infty} Cos^{-1}\frac{1}{2} - Cos^{-1}\frac{1}{1} + Cos^{-1}\frac{1}{3} - Cos^{-1}\frac{1}{2} + \cdots + Cos^{-1}\frac{1}{m} - Cos^{-1}\frac{1}{m-1} + Cos^{-1}\frac{1}{m+1} - Cos^{-1}\frac{1}{m}$$

$$= \lim_{m\to\infty} Cos^{-1}\frac{1}{m+1} - Cos^{-1}\frac{1}{1} = Cos^{-1}0 - Cos^{-1}1 = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

Solution 2 by Angel Plaza, University of Las Palmas de Gran Canaria, Spain
Let

$$\theta_n = \cos^{-1}\left(\frac{1}{n}\right) = \sin^{-1}\left(\frac{\sqrt{n^2-1}}{n}\right)$$

so that

$$\begin{aligned}\sum_{n=1}^{N}\cos^{-1}\left(\frac{1+\sqrt{n^2+2n}\cdot\sqrt{n^2-1}}{n(n+1)}\right) &= \sum_{n=1}^{N}\cos^{-1}\left(\cos\theta_{n+1}\cos\theta_n + \sin\theta_{n+1}\sin\theta_n\right)\\ &= \sum_{n=1}^{N}\cos^{-1}\left(\cos(\theta_{n+1}-\theta_n)\right) = \theta_{n+1}-\theta_1.\end{aligned}$$

which converges to $\pi/2 - 0 = \pi/2$ as $N\to\infty$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

Since $Cos^{-1}\dfrac{1+\sqrt{n^2+2n}\cdot\sqrt{n^2-1}}{n\,(n+1)} = Cos^{-1}\left(\dfrac{1}{n+1}\right) - Cos^{-1}\left(\dfrac{1}{n}\right)$, so for any positive integers N, $\sum\limits_{n=1}^{\infty} Cos^{-1}\dfrac{1+\sqrt{n^2+2n}\cdot\sqrt{n^2-1}}{n\,(n+1)} = Cos^{-1}\left(\dfrac{1}{N+1}\right)$.

The result of the problem follows as we allow N to tend to infinity.

Solution 4 by G. C. Greubel, Newport News, VA

Let $\cos x = \dfrac{1}{n+1}$ and $\cos y = \dfrac{1}{n}$. Using $\sin\theta = \sqrt{1-\cos^2\theta}$ with the corresponding cosine values leads to

$$\frac{1+\sqrt{n^2+2n}\sqrt{n^2-1}}{n(n+1)} = \cos x\cos y + \sin x\sin y = \cos(x-y).$$

From these reductions, now consider a finite version of the series in question.

$$\begin{aligned} S_m &= \sum_{n=1}^{m}\cos^{-1}\left(\frac{1+\sqrt{n^2+2n}\sqrt{n^2-1}}{n(n+1)}\right) \\ &= \sum_{n=1}^{m}\cos^{-1}(\cos(x-y)) \\ &= \sum_{n=1}^{m}(x-y). \end{aligned}$$

Now using the values for x and y the finite series becomes

$$\begin{aligned} S_m &= \sum_{n=1}^{m}\left(\cos^{-1}\left(\frac{1}{n+1}\right) - \cos^{-1}\left(\frac{1}{n}\right)\right) \\ &= \cos^{-1}\left(\frac{1}{m+1}\right) - \cos^{-1}(1). \end{aligned}$$

The series stated in the problem can be obtained by taking the limit m goes to infinity. Considering this leads to

$$\begin{aligned} S_\infty &= \lim_{m\to\infty} S_m \\ &= \lim_{m\to\infty}\left[\cos^{-1}\left(\frac{1}{m+1}\right) - \cos^{-1}(1)\right] \\ &= \cos^{-1}(0) - \cos^{-1}(1) = \frac{\pi}{2} - 0 = \frac{\pi}{2}. \end{aligned}$$

That is, $\sum\limits_{n=1}^{\infty}\cos^{-1}\left(\dfrac{1+\sqrt{n^2+2n}\sqrt{n^2-1}}{n(n+1)}\right) = \dfrac{\pi}{2}$.

Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo State University, San Angelo, TX; Valmir Bucaj (student, Texas Luthern University), Seguin, TX; Edwin Gray, Highland

Beach, FL; David E. Manes, Oneonta, NY; Pedro H. O. Pantoja, Natal-RN, Brazil; Paolo Perfetti, Department of Mathematics, University "Tor Vergata, Rome, Italy; Boris Rays, Brooklyn, NY, and the proposer.

- **5144:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Compute

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left[1+\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right)^{1/n}\right].$$

Solution 1 by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany

Let $A=\int_0^1 \ln(x+\sqrt{1+x^2})$ and doing easy calculations we have

$$\begin{aligned} A &= \int_0^1 \ln(x+\sqrt{1+x^2}) \\ &= \int_0^1 \sinh^{-1}(x) \\ &= \left[-\sqrt{1+x^2}+x\sinh^{-1}(x)\right]_0^1 \\ &= 1-\sqrt{2}+\sinh^{-1}(1) \\ &\simeq 0.46716 \end{aligned}$$

Using the Riemann sums we have

$$\begin{aligned} \lim_{n\to\infty}\prod_{k=1}^{n}\left[1+\ln\left(\frac{k+\sqrt{k^2+n^2}}{n}\right)^{1/n}\right] &= \lim_{n\to\infty}\prod_{k=1}^{n}\left[1+\frac{1}{n}\ln\left(\frac{k+\sqrt{k^2+n^2}}{n}\right)\right] \\ &= \lim_{n\to\infty}\prod_{k=1}^{n}\left[1+\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\frac{k^2}{n^2}}\right)\right] \\ &= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{A^k}{k!} \\ &= \sum_{k=0}^{\infty}\frac{A^k}{k!} \\ &= e^A \\ &= e^{1-\sqrt{2}+\sinh^{-1}(1)} \\ &\simeq 1.5956 \end{aligned}$$

Solution 2 by Ovidiu Furdui, Cluj, Romania

More generally, we prove that if $f:[0,1]\to\Re$ is an integrable function then

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{1}{n}f\left(\frac{k}{n}\right)\right)=e^{\int_0^1 f(x)dx}.$$

Let

$$x_n=\prod_{k=1}^{n}\left(1+\frac{1}{n}f\left(\frac{k}{n}\right)\right).$$

Then,

$$\ln x_n=\sum_{k=1}^{n}\ln\left(1+\frac{1}{n}f\left(\frac{k}{n}\right)\right)=\sum_{k=1}^{n}\frac{\ln\left(1+\frac{1}{n}f\left(\frac{k}{n}\right)\right)}{\frac{1}{n}f\left(\frac{k}{n}\right)}\cdot\ln\left(1+\frac{1}{n}f\left(\frac{k}{n}\right)\right),$$

and it follows that

$$(1)\ \min_{1\le k\le n}\frac{\ln\left(1+\frac{1}{n}f\left(\frac{k}{n}\right)\right)}{\frac{1}{n}f\left(\frac{k}{n}\right)}\cdot\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\le\ln x_n\le\max_{1\le k\le n}\frac{\ln\left(1+\frac{1}{n}f\left(\frac{k}{n}\right)\right)}{\frac{1}{n}f\left(\frac{k}{n}\right)}\cdot\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right).$$

Since $\lim_{x\to 0}\dfrac{\ln(1+x)}{x}=0$, one has that

$$(2)\ \lim_{n\to\infty}\min_{1\le k\le n}\frac{\ln\left(1+\frac{1}{n}f\left(\frac{k}{n}\right)\right)}{\frac{1}{n}f\left(\frac{k}{n}\right)}=1\qquad\text{and}\qquad\lim_{n\to\infty}\max_{1\le k\le n}\frac{\ln\left(1+\frac{1}{n}f\left(\frac{\mathrm{k}}{\mathrm{n}}\right)\right)}{\frac{1}{n}f\left(\frac{\mathrm{k}}{\mathrm{n}}\right)}=1.$$

Letting $n\to\infty$ in (1) and using (2) one has that

$$\lim_{n\to\infty}\ln x_n=\int_0^1 f(x)dx,$$

and the problem is solved.

In particular, when $f(x)=\ln(x+\sqrt{1+x^2})$ one has that

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\ln\left(\frac{k+\sqrt{k^2+n^2}}{n}\right)^{1/n}\right)=e^{\int_0^1\ln(x+\sqrt{1+x^2})}=e^{\ln(1+\sqrt{2})+1-\sqrt{2}}=(1+\sqrt{2})e^{1-\sqrt{2}}.$$

Solution 3 by Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy

Answer: $(1+\sqrt{2})e^{1-\sqrt{2}}$

Proof: By taking the logarithm we obtain

$$\ln\left\{\prod_{k=1}^{n}\left[1+\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right)^{1/n}\right]\right\} = \sum_{k=1}^{n}\ln\left[1+\frac{1}{n}\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right)\right]$$

We observe that

$$\frac{k+\sqrt{n^2+k^2}}{n} \le \frac{n+\sqrt{2}n}{n} = 1+\sqrt{2}$$

thus the increasing monotonicity implies

$$\frac{1}{n}\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right) \le \frac{\ln(1+\sqrt{2})}{n}$$

By employing

$$\ln(1+x) = x + O(x^2), \quad x \to 0$$

we may write

$$\ln\left[1+\frac{1}{n}\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right)\right] = \frac{1}{n}\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right) + O(n^{-2})$$

and this in turn implies

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left[1+\frac{1}{n}\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right)\right] &= \lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{n}\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right)+O(\frac{1}{n^2})\right] \\
\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right) &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\frac{k^2}{n^2}}\right)
\end{aligned}$$

Of course we have

$$\sum_{k=1}^{n}\frac{1}{n}O(\frac{1}{n^2}) = O(\frac{1}{n^2})$$

Now it is easy to recognize that the last limit is actually the Riemann sum of the following integral

$$\int_0^1 \ln(x+\sqrt{1+x^2})dx$$

and the integral is easily calculated by the standard methods. Integrating by parts

$$\begin{aligned}
\int_0^1 \ln(x+\sqrt{1+x^2})dx &= x\ln(x+\sqrt{1+x^2})|_0^1 - \int_0^1 \frac{x}{\sqrt{1+x^2}}dx \\
&= \ln(1+\sqrt{2}) - (1+x^2)^{1/2}|_0^1 = \ln(1+\sqrt{2}) - \sqrt{2} + 1 = \ln\left(1+\sqrt{2}\right)e^{1-\sqrt{2}}
\end{aligned}$$

By exponentiating we obtain the desired result.

Solution 4 by Bruno Salgueiro Fanego, Viveiro, Spain

Let us denote $P = \lim\limits_{n\to} P_n$ and $P_n = \prod\limits_{k=1}^{n}\left[1+\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right)^{1/n}\right]$.

Then

$$\begin{aligned}\ln P = \lim_{n\to\infty}\ln P_n &= \lim_{n\to\infty}\sum_{k=1}^{n}\ln\left[1+\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right)^{1/n}\right]\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\ln\left[1+\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right)\right].\end{aligned}$$

Taking, for each $k = 1, 2, \ldots, n,\ x = \frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right)$ in the

inequalities $\frac{x}{1+x} \le \ln(1+x) \le x$ (for any $x > 0$), and being, for any $k = 1, 2, \ldots, n$,

$\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2} \le 1+\sqrt{1+1^2} = 1+\sqrt{2}$, we obtain:

$$\begin{aligned}\frac{\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right)}{1+\frac{1}{n}\ln\left(1+\sqrt{2}\right)} &\le \frac{\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right)}{1+\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right)}\\ &\le \ln\left[1+\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right)\right]\\ &\le \frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right). \qquad (\mathbf{1})\end{aligned}$$

From **(1)** we obtain:

$$\begin{aligned}\sum_{k=1}^{n}\frac{\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right)}{1+\frac{1}{n}\ln\left(1+\sqrt{2}\right)} &\le \sum_{k=1}^{n}\ln\left[1+\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right)\right]\\ &\le \sum_{k=1}^{n}\frac{1}{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right) \qquad (\mathbf{2})\end{aligned}$$

And from (**2**) we obtain:

$$\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}\ln\left(1+\sqrt{2}\right)}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right)\le\ln P\le\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\sqrt{1+\left(\frac{k}{n}\right)^2}\right).$$

That is:

$$\frac{1}{1+0}\int_0^1\ln\left(x+\sqrt{1+x^2}\right)dx\le\ln P\le\int_0^1\ln\left(x+\sqrt{1+x^2}\right)dx$$

So,

$$\begin{aligned}\ln P=\int_0^1\ln\left(x+\sqrt{1+x^2}\right)dx &= \left[x\ln\left(x+\sqrt{1+x^2}\right)-\sqrt{1+x^2}\right]_0^1\\ &= \ln(1+\sqrt{2})+1-\sqrt{2},\end{aligned}$$

and therefore,

$$P=e^{\ln(1+\sqrt{2})+1-\sqrt{2}}=\left(1+\sqrt{2}\right)e^{1-\sqrt{2}.}$$

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China, and the proposer.

- **5145:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $k\ge 1$ be a natural number. Find the sum of

$$\sum_{n=1}^{\infty}\left(\frac{1}{1-x}-1-x-x^2-\cdots-x^n\right)^k,\quad\text{for } |x|<1.$$

Solution by Michael C. Faleski, University Center, MI

The summation can be rewritten as

$$\sum_{n=1}^{\infty}\left(\frac{1}{1-x}-\left(1+x+x^2+\cdots+x^n\right)\right)^k$$

where we can write the second term in the parentheses using the geometric series expression. That is, $(1+x+x^2+\cdots+x^n)=\dfrac{1-x^{n+1}}{1-x}$. Substitution of this result yields the original sum to

$$\sum_{n=1}^{\infty}\left(\frac{1}{1-x}-\left(1+x+x^2+\cdots+x^n\right)\right)^k=\sum_{n=1}^{\infty}\left(\frac{1}{1-x}-\frac{1-x^{n+1}}{1-x}\right)^k=\sum_{n=1}^{\infty}\left(\frac{x^{n+1}}{1-x}\right)^k.$$

Since the denominator has no "n" dependence, we now have

$$\sum_{n=1}^{\infty}\left(\frac{x^{n+1}}{1-x}\right)^k=\left(\frac{1}{(1-x)^k}\right)\sum_{n=1}^{\infty}x^{(n+1)k}.$$

Once again, we use the geometric series relation with multiplicative term x^k yielding a result of

$$\sum_{n=1}^{p} x^{(n+1)k} = \frac{x^{2k} - x^{(p+2)k}}{1 - x^k}.$$

Now, as the upper limit of the sum is $p \to \infty$, then $x^{(p+2)k} \to 0$ since $|x| < 1$ and $k \geq 1$. Hence

$$\sum_{n=1}^{\infty} x^{(n+1)k} = \frac{x^{2k}}{1 - x^k}.$$

So, finally, we have our result of the original sum as

$$\sum_{n=1}^{\infty} \left(\frac{1}{1-x} - 1 - x - x^2 - \cdots - x^n \right)^k = \frac{x^{2k}}{(1-x)^k (1-x^k)}.$$

Also solved by Dionne Bailey, Elsie Campbell, Charles Diminnie and Andrew Siefker (jointly), San Angelo, TX; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Bruno Salgueiro, Fanego, Viveiro,Spain; Edwin Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; Angel Plaza, Gran Canaria, Spain; Paolo Perfetti, Department of Mathematics, University "Tor Vergata, Rome, Italy; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

From the Editor; Mea Culpa

In the March 2011 issue of the column I inadvertently forgot to mention that **Enkel Hysnelaj, of the University of Technology, in Sydney, Australia and Elton Bojaxhiu of Kriftel, Germany** had also solved problems 5136 and 5139. Sorry.

Mistakes happen to all of us; they are embarrassing, but they are part of life. **Albert Stadler of Herrliberg, Switzerland** pointed out an error in the first solution to 5138 that appeared in last month's column. The problem challenged us to prove for all natural numbers $n \geq 2$ that

$$\frac{n}{F_n F_{n+1}} \leq \frac{1}{(n-1)F_1^2 + F_2^2} + \cdots + \frac{1}{(n-1)F_n^2 + F_1^2} \leq \frac{1}{n} \sum_{k=1}^{n} \frac{1}{F_k^2},$$

where F_n is the n^{th} Fibonacci number defined in the usual way with $F_0 = 0, F_1 = 1$. At a crucial step in the proof Muirhead's inequality was employed and this is what triggered Albert's suspicions; the conditions to use the inequality were not met. (Muirhead's inequality generalizes the arithmetic-geometric means inequality. See: http://en.wikipedia.org/wiki/Muirhead) In further correspondence with Albert he pointed out a paper by Yufei Zhao (yufeiz@mit.edu) on Inequalities that contains two practical notes with respect to the Muirhead inequality. Zhao wrote: "Don't try to apply Muirhead when there are more than 3 variables, since mostly likely you won't succeed (and never, ever try to use Muirhead when the inequality is only cyclic but not symmetric, since it is incorrect to use Muirhead there) (2) when writing up your solution, it is probably safer to just deduce the inequality using weighted AM-GM by finding the appropriate weights, as this can always be done. The reason is that it is not always clear that Muirhead will be accepted as a quoted theorem." The second solution listed for 5138 is correct.