Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before June 15, 2010

- **5110:** *Proposed by Kenneth Korbin, New York, NY.*

  Given triangle ABC with an interior point P and with coordinates $A(0,0), B(6,8)$, and $C(21,0)$. The distance from point P to side $\overline{AB}$ is a, to side $\overline{BC}$ is b, and to side $\overline{CA}$ is c where $a:b:c = \overline{AB} : \overline{BC} : \overline{CA}$.

  Find the coordinates of point P.

- **5111:** *Proposed by Michael Brozinsky, Central Islip, NY.*

  In Cartesianland where immortal ants live, it is mandated that any anthill must be surrounded by a triangular fence circumscribed in a circle of unit radius. Furthermore, if the vertices of any such triangle are denoted by A, B, and C, in counterclockwise order, the anthill's center must be located at the interior point P such that $\angle PAB = \angle PBC = \angle PCA$.

  Show $\overline{PA} \cdot \overline{PB} \cdot \overline{PC} \le 1$.

- **5112:** *Proposed by Juan-Bosco Romero Márquez, Madrid, Spain*

  Let $0 < a < b$ be real numbers with a fixed and b variable. Prove that

  $$\lim_{b\to a} \int_a^b \frac{dx}{\ln \dfrac{b+x}{a+x}} = \lim_{b\to a} \int_a^b \frac{dx}{\ln \dfrac{b(a+x)}{a(b+x)}}.$$

- **5113:** *Proposed by Paolo Perfetti, Mathematics Department, Tor Vergata University, Rome, Italy*

  Let x, y be positive real numbers. Prove that

  $$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} \le \sqrt{xy} + \frac{x+y}{2} + \frac{\left(\dfrac{x+y}{6} - \dfrac{\sqrt{xy}}{3}\right)^2}{\dfrac{2xy}{x+y}}.$$

- **5114:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let M be a point in the plane of triangle ABC. Prove that

$$\frac{\overline{MA}^2+\overline{MB}^2+\overline{MC}^2}{\overline{AB}^2+\overline{BC}^2+\overline{CA}^2}\geq\frac{1}{3}.$$

  When does equality hold?

- **5115:** *Proposed by Mohsen Soltanifar (student, University of Saskatchewan), Saskatoon, Canada*

  Let G be a finite cyclic group. Compute the number of distinct composition series of G.

Solutions

- **5092:** *Proposed by Kenneth Korbin, New York, NY*

  Given equilateral triangle ABC with altitude h and with cevian $\overline{CD}$. A circle with radius x is inscribed in $\triangle ACD$, and a circle with radius y is inscribed in $\triangle BCD$ with $x<y$. Find the length of the cevian $\overline{CD}$ if x,y and h are positive integers with $(x,y,h)=1$.

  Solution by David Stone and John Hawkins (jointly), Statesboro, GA;

  We let the length of cevian $=d$. Since the altitude of the equilateral triangle is h, the length of the side $\overline{AC}$ is $\dfrac{2h}{\sqrt{3}}$. Let F be the center of the circle inscribed in $\triangle ACD$. Let $\alpha=\angle ACF=\angle FCD$. Therefore $\angle ACD=2\alpha$.

  Let E be the point where the inscribed circle in $\triangle ACD$ is tangent to side $\overline{AC}$. Since $\overline{AF}$ bisects the base angle of 60°, we know that $\triangle AEF$ is a $30^\circ-60^\circ-90^\circ$ triangle, implying that $\overline{AE}=\sqrt{3}x$. Thus the length of $\overline{CE}$ is $\overline{AC}-\overline{AE}=\dfrac{2h}{\sqrt{3}}-\sqrt{3}x=\dfrac{2h-3x}{\sqrt{3}}$.

  Applying the Law of Sines in triangle $\triangle ADC$, we have

$$\frac{\sin 2\alpha}{\overline{AD}}=\frac{\sin 60^\circ}{d}=\frac{\sin(\angle ADC)}{\overline{AC}}. \qquad (1)$$

  Because $\angle ADC=180^\circ-60^\circ-2\alpha=120^\circ-2\alpha$, we have

$$\begin{aligned}\sin(\angle ADC)&=\sin(120^\circ-2\alpha)\\ &=\sin 120^\circ\cos 2\alpha-\cos 120^\circ\sin 2\alpha\\ &=\frac{\sqrt{3}}{2}\cos 2\alpha+\frac{1}{2}\sin 2\alpha\\ &=\frac{\sqrt{3}}{2}\left(\cos^2\alpha-\sin^2\alpha\right)+\frac{1}{2}(2\sin\alpha\sin\alpha).\end{aligned}$$

Thus from (1) we have

$$\frac{\sqrt{3}}{2d} = \frac{\left[\sqrt{3}\left(\cos^2\alpha - \sin^2\alpha\right) + \left(2\sin\alpha\sin\alpha\right)\right]\sqrt{3}}{4h}.$$

Therefore, we can solve for d in terms of h and α:

$$d = \frac{2h}{\left[\sqrt{3}\left(\cos^2\alpha - \sin^2\alpha\right) + \left(2\sin\alpha\sin\alpha\right)\right]}.$$

In the right triangle $\triangle EFC$, we have

$$\overline{FC} = \sqrt{x^2 + \left(\frac{2h-3x}{\sqrt{3}}\right)^2} = \sqrt{\frac{3x^2+4h^2-12hx+9x^2}{3}} = \frac{2}{\sqrt{3}}\sqrt{3x^2+h^2-3hx}.$$

Thus, $\sin\alpha = \dfrac{\sqrt{3}x}{2\sqrt{3x^2+h^2-3hx}}$ and $\cos\alpha = \dfrac{2h-3x}{2\sqrt{3x^2+h^2-3hx}}$. Therefore,

$$\begin{aligned} cos^2\alpha - \sin^2\alpha &= \frac{(2h-3x)^2}{4\left(3x^2+h^2-3hx\right)} - \frac{3x^2}{4\left(3x^2+h^2-3hx\right)} \\ &= \frac{4h^2-12hx+6x^2}{4\left(3x^2+h^2-3hx\right)} = \frac{2h^2-6hx+3x^2}{2\left(3x^2+h^2-3hx\right)}. \end{aligned}$$

and $2\sin\alpha\cos\alpha = \dfrac{\sqrt{3}x\left(2h-3x\right)}{2\left(3x^2+h^2-3hx\right)}$.

Therefore the denominator in the expression for d becomes

$$\frac{\sqrt{3}(2h^2-6hx+3x^2)}{2\left(3x^2+h^2-3hx\right)} + \frac{\sqrt{3}x\left(2h-3x\right)}{2\left(3x^2+h^2-3hx\right)} = \sqrt{3}\frac{2h^2-4hx}{2\left(3x^2+h^2-3hx\right)}.$$

Thus, $d = \dfrac{2h}{\dfrac{\sqrt{3}(2h^2-4hx)}{2(3x^2+h^2-3hx)}} = \dfrac{2(3x^2+h^2-3hx)}{\sqrt{3}(h-2x)}$.

Similarly, working in $\triangle BCD$, we can show that $d = \dfrac{2(3y^2+h^2-3hy)}{\sqrt{3}(h-2y)}$.

We note that x and y both satisfy the same equation when set equal to d. Thus for a given value of d, the equation should have two solutions. The smaller one can be used for x and the larger for y.

We also note that if x, h and y are integers, then d has the form $d = \dfrac{r}{\sqrt{3}}$, for r a rational number. We substitute this into the equation x:

$$d = \frac{2\left(3x^2+h^2-3hx\right)}{\sqrt{3}(h-2x)} = \frac{r}{\sqrt{3}}, \text{ so}$$

$$r = \frac{2(3x^2 + h^2 - 3hx)}{h - 2x}.$$

Now we solve this for x:

$$rh - 2xr = 6x^2 + 2h^2 - 6hx$$

$$6x^2 - (6h - 2r)\,x + 2h^2 - rh = 0$$

$$x = \frac{6h - 2r \pm \sqrt{36h^2 - 24hr + 4r^2 - 48h^2 + 24hr}}{12} = \frac{3h - r \pm \sqrt{r^2 - 3h^2}}{6}.$$

Of course we would have the exact same expression for y.

Thus we take $x = \dfrac{3h - r - \sqrt{r^2 - 3h^2}}{6}$ and $y = \dfrac{3h - r - \sqrt{r^2 - 3h^2}}{6}$ and find h and r so that x and y turn out to be positive integers.

Subtracting x from y gives $y - x = \dfrac{\sqrt{r^2 - 3h^2}}{3}$. Thus we need r and h such that $\dfrac{\sqrt{r^2 - 3h^2}}{3}$ is an integer.

It must be the case that $r^2 - 3h^2 \geq 0$, which requires $0 < \sqrt{3}h \leq r$. In addition it must be true that

$$\begin{aligned}
3h - r - \sqrt{r^2 - 3h^2} &> 0 \\
9h^2 - 6hr + r^2 &> r^2 - 3h^2 \\
12h^2 - 6hr &> 0 \\
6h(2h - r) &> 0 \\
0 < r &< 2h. \quad \text{Thus,} \\
\sqrt{3}h &\leq r < 2h.
\end{aligned}$$

If we restrict our attention to integer values of r, then both h and r must be divisible by 3.

For $h = 3, 6$ and 9, no integer values of r divisible by 3 satisfy $\sqrt{3}h \leq r < 2h$. So the first allowable value of h is 12. Then the condition $12\sqrt{3} \leq r < 24$ forces $r = 21$. From this we find that $x = 2$ and $y = 3$ and $d = 7\sqrt{3}$. (Note that $(2, 3, 12) = 1$.)

This is only the first solution. We programmed these constraints and let MatLab check for integer values of h and appropriate integer values of r which make x and y integers satisfying $(x, y, h) = 1$. There are many solutions:

$$\begin{pmatrix} r & y & x & y & \text{cevian} \\ \\ 21 & 12 & 2 & 3 & 7\sqrt{3} \\ \\ 78 & 45 & 9 & 10 & 26\sqrt{3} \\ \\ 111 & 60 & 5 & 18 & 37\sqrt{3} \\ \\ 114 & 63 & 7 & 18 & 38\sqrt{3} \\ \\ 129 & 72 & 9 & 20 & 43\sqrt{3} \end{pmatrix}$$

Editor's note: David and John then listed another 47 solutions. They capped their search at $h = 1000$, but stated that solutions exist for values of $h > 1000$. They ended the write-up of their solution with a formula for expressing the cevian in terms of x, y and h.

$$\begin{aligned} y - x &= \frac{\sqrt{r^2 - 3h^2}}{3} \\ 9(y-x)^2 &= r^2 - 3h^2 \\ r^2 &= 3h^2 + 9(y-x)^2 \\ r &= \sqrt{3h^2 + 9(y-x)^2} \\ \text{Length of cevian } \frac{r}{\sqrt{3}} &= \sqrt{h^2 + 3(y-x)^2}. \end{aligned}$$

Ken Korbin, the proposer of this problem, gave some insights into how such a problem can be consructed. He wrote:

Begin with any prime number P congruent to 1(mod 6). Find positive integers $[a, b]$ such that $a^2 + ab + b^2 = P^2$. Construct an equilateral triangle with side $a + b$ and with Cevian P. The Cevian will divide the base of the triangle into segments with lengths a and b. Find the altitude of the triangle and the inradii of the 2 smaller triangles. Multiply the altitude, the inradii and the Cevian P by $\sqrt{3}$ and then by their LCD. This should do it. Examples: $P = 7, [a, b] = [3, 5]$. $P = 13, [a, b] = [7, 8]$. And so on.

- **5093:** *Proposed by Worapol Ratanapan (student), Montfort College, Chiang Mai, Thailand*

6 = 1 + 2 + 3 is one way to partition 6, and the product of 1, 2, 3, is 6. In this case, we call each of 1, 2, 3 a **part** of 6.

We denote the maximum of the product of all **parts** of natural number n as $N(n)$.

As a result, $N(6) = 3 \times 3 = 9, N(10) = 2 \times 2 \times 3 \times 3 = 36$, and $N(15) = 3^5 = 243$.

More generally, $\forall n \in N, N(3n) = 3^n, N(3n+1) = 4 \times 3^{n-1}$, and $N = (3n+2) = 2 \times 3^n$.

Now let's define $R(r)$ in the same way as $N(n)$, but each **part** of r is positive real. For instance $R(5) = 6.25$ and occurs when we write $5 = 2.5 + 2.5$

Evaluate the following:

$$i)\ R(2e)$$
$$ii)\ R(5\pi)$$

Solution by Michael N. Fried, Kibbutz Revivim, Israel

Let $R(r) = \prod_i x_i$, where $\sum_i x_i = r$ and $x_i > 0$ for all i. For any given r, find the maximum of $R(r)$.

Since for any given r and n the arithmetic mean of every set $\{x_i\}$ $i = 1, 2, 3 \ldots n$ is $\dfrac{r}{n}$ by assumption, the geometric-arithemetic mean inequality implies that

$$R(r) = \prod_{i=1}^{n} x_i \leq \left(\frac{r}{n}\right)^n.$$

Hence the maximum of $R(r)$ is a function of n. Let us then find the maximum of the function $R(x) = \left(\dfrac{r}{x}\right)^x$, which is the same as the maximum of

$$L(x) = \ln\left(R(x)\right) = x \ln r - x \ln x.$$

$L(x)$ indeed has a *single* maximum at $x = \dfrac{r}{e}$.

Let $m = \lfloor \dfrac{r}{e} \rfloor$ and $M = \lceil \dfrac{r}{e} \rceil$. Then the maximum value of $R(r)$ is

$$\max\left(\left(\frac{r}{m}\right)^m, \left(\frac{r}{M}\right)^M\right).$$

To make this concrete consider $r = 5,\ 2e$, and 5π.

For $r = 5, r/e = 1.8393\ldots$, so $\max R(5) = \max\left(5, (5/2)^2\right) = \max(5, 6.25) = 6.25$

For $r = 2e, r/e = 2$, so $\max R(2e) = e^2$.

For $r = 5\pi, r/e = 5.7786\ldots$, so $\max R(5\pi) = \max\left(\left(\dfrac{5\pi}{5}\right)^5, \left(\dfrac{5\pi}{6}\right)^6\right) = \left(\dfrac{5\pi}{6}\right)^6$.

Also solved by Brian D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins (jointly), Statesboro, GA; The Taylor University Problem Solving Group, Upland, IN, and the proposer.

- **5094:** *Proposed by Paolo Perfetti, Mathematics Department Tor Vergata University, Rome, Italy*

  Let a, b, c be real positive numbers such that $a + b + c + 2 = abc$. Prove that

$$2\left(a^2 + b^2 + c^2\right) + 2\left(a + b + c\right) \geq \left(a + b + c\right)^2.$$

Solution 1 by Ercole Suppa, Teramo, Italy

We will use the "magical" substitution given in "Problems from The Book" by Titu Andreescu and Gabriel Dospinescu, which is explained in the following lemma:

If a, b, c are positive real numbers such that $a+b+c+2 = abc$, then there exists three real numbers $x, y, z > 0$ such that

$$a = \frac{y+z}{x}, \; b = \frac{z+x}{y}, \text{ and } c = \frac{x+y}{z}. \qquad (*)$$

Proof: By means of a simple computation the condition $a+b+c+2 = abc$ can be written in the following equivalent form

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = 1.$$

Now if we take

$$x = \frac{1}{1+a}, \quad y = \frac{1}{1+b}, \text{ and } \mathrm{z} = \frac{1}{1+\mathrm{c}},$$

then $x+y+z = 1$ and $a = \dfrac{1-x}{x} = \dfrac{y+z}{x}$. Of course, in the same way we find $b = \dfrac{z+x}{y}$ and $c = \dfrac{x+y}{z}$.

By using the substitution $(*)$, after some calculations, the given inequality rewrites as

$$\frac{z^4(x-y)^2 + x^4(y-z)^2 + y^4(x-z)^2 + 2(x^3y^3 + x^3z^3 + y^3z^3 - 3x^2y^2z^2)}{x^2y^2z^2} \geq 0,$$

which is true since

$$x^3y^3 + x^3z^3 + y^3z^3 \geq 3x^2y^2z^2$$

by virtue of the AM-GM inequality.

Solution 2 by Shai Covo, Kiryat-Ono, Israel

First let $x = a+b$ and $y = ab$. Hence $x \geq 2\sqrt{y}$.

From $a+b+c+2 = abc$, we have $c = \dfrac{x+2}{y-1}$. Hence, $y > 1$.

Noting that $x^2 - 2y = a^2 + b^2$, it follows readily that the original inequality can be rewritten as

$$(y-2)^2 x^2 + 2\left(y^2 - 3y + 4\right) x - 4y^3 + 8y^2 \geq 0, \qquad (1)$$

where $y > 1$ and $x \geq 2\sqrt{y}$. For $y > 1$ arbitrary but fixed, we denote by $f_y(x)$, for $x \geq 2\sqrt{y}$, the function on the left-hand side of (1).

Trivially, $f_y(x) \geq 0$ for $y = 2$. For $y \neq 2$ (which we henceforth assume), $f_y(\cdot)$, when extended to $\Re$, is a quadratic function (parabola) attaining its minimum at $x_0 = \dfrac{-\left(y^2 - 3y + 4\right)}{(y-2)^2}$.

Noting that $x_0 < 0$, it follows that

$$\min_{\{x: x \geq 2\sqrt{y}\}} f_y(x) \quad = \quad f_y(2\sqrt{y})$$

$$= \quad 4\sqrt{y}\left(y^2 - 3y + 4 - 2y^{3/2} + 4\sqrt{y}\right).$$

Thus the inequality (1) will be proved if we show that

$$\varphi(y) := y^2 - 3y + 4 - 2y^{3/2} + 4\sqrt{y} \geq 0. \qquad (2)$$

This is trivial for $1 < y < 2$ since in this case both $y^2 - 3y + 4$ and $-2y^{3/2} + 4\sqrt{y}$ are greater than 0.

For $y > 2$, it is readily seen that $\varphi''(y) > 0$. Hence, $\varphi'(y)$ is increasing for $y > 2$. Noting that $\varphi'(4) = 0$, it thus follows that $\min_{\{y>2\}} \varphi(y) = \varphi(4)$. Since $\varphi(4) = 0$, inequality (2) is proved.

Solution 3 by Kee-Wai Lau, Hong Kong, China

Firstly, we have

$$2(a^2 + b^2 + c^2) + 2(a + b + c) - (a + b + c)^2 = (a + b + c)(a + b + c + 2) - 4(ab + bc + ca)$$

Let $p = a + b + c,\ q = ab + bc + ca,\ r = abc$, so that $r = p + 2$.

We need to show that $q \leq \dfrac{p(p+2)}{4}$ (1)

It is well known that $a,\ b$, and c are the positive real roots of the cubic equation

$$x^3 - px^2 + qx - r \quad = \quad 0 \ \text{ if, and only if,}$$

$$p^2q^2 - 4p^3r + 18pqr - 4q^3 - 27r^2 \quad \geq 0.$$

By substituting $r = p + 2$ and simplifying, we reduce the last inequality to $f(q) \leq 0$, where

$$\begin{aligned} f(q) &= 4q^3 - p^2q^2 - \left(36p + 18p^2\right) q + 4p^4 + 8p^3 + 27p^2 + 108p + 108 \\ &= (q + 2p + 3)\left(4q^2 - (p^2 + 8p + 12)q + 2p^3 + p^2 + 12p + 36\right). \text{ Thus} \end{aligned}$$

$$4q^2 - (p^2 + 8p + 12)q + 2p^3 + p^2 + 12p + 36 \leq 0. \qquad (2)$$

By the arithmetic mean-geometric inequality we have
$abc = a + b + c + 2 \geq 4(2abc)^{1/4}$ so that $abc \geq 8$ and $p = a + b + c \geq 6$.

From (2) we obtain $q \leq \dfrac{1}{8}\left(p^2 + 8p + 12 + \sqrt{(p+2)(p-6)^3}\right)$ and it remains to show that

$$\frac{1}{8}\left(p^2 + 8p + 12 + \sqrt{(p+2)(p-6)^3}\right) \leq \frac{p(p+2)}{4}. \qquad (3)$$

Now (3) is equivalent to $\sqrt{(p+2)(p-6^3)} \leq (p-6)(p+2)$ or, on squaring both sides and simplifying, $-8(p+2)(p-6)^2 \leq 0$.

Since the last inequality is clearly true, we see that (1) is true, and this completes the solution.

Also solved by Tom Leong, Scotrun, PA; Bruno Salgueiro Fanego, Viveiro, Spain, and the proposer.

- **5095:** *Proposed by Zdravko F. Starc, Vršac, Serbia*

Let F_n be the Fibonacci numbers defined by

$$F_1 = F_2 = 1,\ F_{n+2} = F_{n+1} + F_n\ (n = 1, 2, \cdots).$$

Prove that

$$\sqrt{F_{n-2}F_{n-1}} + 1 \leq F_n \leq \sqrt{(n-2)F_{n-2}F_{n-1}} + 1 \quad (n = 3, 4, \cdots).$$

Solution 1 by Valmir Bucaj (student, Texas Lutheran University), Seguin, TX

First, using mathematical induction, we show that

$$F_n^2 = F_{n-1}F_{n+1} + (-1)^{n+1}, \text{for } n = \mathit{2}, \mathit{3}, \ldots \qquad (\mathit{2}).$$

For $n = 2$ we have:

$$F_2^2 = 1 = 1 \cdot 2 - 1 = F_1F_3 + (-1)^3.$$

Assume that (2) holds for n. We show that it is true also for $n + 1$.

$$\begin{aligned} F_nF_{n+2} + (-1)^{n+2} &= F_n\left(F_n + F_{n+1}\right) + (-1)^{n+2} \\ &= F_n^2 + F_nF_{n+1} + (-1)^{n+2} \\ &= F_{n-1}F_{n+1} + F_nF_{n+1} + (-1)^{n+1} + (-1)^{n+2} \\ &= F_{n+1}\left(F_{n-1} + F_n\right) = F_{n+1}^2. \end{aligned}$$

So (2) hold for any $n \geq 2$.

Next we show that,

$$\sqrt{F_{n-2}F_{n-1}} + 1 \leq F_n, \text{ holds.}$$

By applying (2) several times we obtain:

$$\begin{aligned} F_n^2 &= F_{n-1}F_{n+1} + (-1)^{n+1} \\ &= F_{n-1}\left(F_n + F_{n-1}\right) + (-1)^{n+1} \\ &= F_{n-1}F_n + F_{n-1}^2 + (-1)^{n+1} \\ &= F_{n-1}F_n + F_{n-2}F_n + (-1)^n + (-1)^{n+1} \\ &= F_{n-1}F_n + F_{n-2}F_{n-1} + F_{n-2}^2 \\ &= 2F_{n-1}F_{n-2} + F_{n-2}F_n + F_{n-2}^2 + (-1)^{n+1} \\ &= 3F_{n-1}F_{n-2} + 2F_{n-2}^2 + (-1)^{n+1} \\ &= F_{n-1}F_{n-2} + 2F_{n-1}F_{n-2} + 2F_{n-2}^2 + (-1)^{n+1} \end{aligned}$$

$$
\begin{aligned}
&\geq F_{n-1}F_{n-2}+2\sqrt{F_{n-1}F_{n-2}}+1\\
&= \left(\sqrt{F_{n-1}F_{n-2}}+1\right)^2
\end{aligned}
$$

Taking the square root of both sides we obtain:

$$F_n \geq \sqrt{F_{n-1}F_{n-2}}+1,$$

which is the first part of (1).

To prove the second part of (1), we proceed similarly. That is:

$$
\begin{aligned}
F_n^2 &= F_{n-1}F_{n+1}+(-1)^{n+1}\\
&= F_{n-1}\left(F_n+F_{n-1}\right)+(-1)^{n+1}\\
&= F_{n-1}F_n+F_{n-1}^2+(-1)^{n+1}\\
&= F_{n-1}F_n+F_{n-2}F_n+(-1)^n+(-1)^{n+1}\\
&= F_{n-1}F_n+F_{n-2}F_{n-1}+F_{n-2}^2\\
&= 2F_{n-1}F_{n-2}+F_{n-2}F_n+F_{n-2}^2+(-1)^{n+1}\\
&= 3F_{n-1}F_{n-2}+2F_{n-2}^2+(-1)^{n+1}\\
&\leq 3F_{n-1}F_{n-2}+2F_{n-1}F_{n-2}+1\\
&= 5F_{n-1}F_{n-2}+1\\
&\leq (n-2)\,F_{n-1}F_{n-2}+1 \text{ for } n\geq 7.
\end{aligned}
$$

Taking the square root of both sides we obtain:

$$F_n \leq \sqrt{(n-2)F_{n-1}F_{n-2}+1} \leq \sqrt{(n-2)F_{n-1}F_{n-2}}+1, \qquad (4)$$

which proves the second part of (1) for $n \geq 7$.

On can easily show that (4) also holds for $n=3,4,5$, and 6 by checking each of these cases separately. So combining (3) and (4) we have proved that:

$$\sqrt{F_{n-2}F_{n-1}}+1 \leq F_n \leq \sqrt{(n-2)F_{n-2}F_{n-1}}+1 \quad (n=3,4,\cdots).$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Given $n=3,4,\cdots$, we can use (because all the F_n are positive) the Geometric Mean-Arithmetic Mean Inequality applied to $F_i, i=n-1,n-2$, the facts that $F_n=F_{n-1}+F_{n-2}$ and $F_n \geq 2$ with equality if, and only if, $n=3$, to obtain:

$$\sqrt{F_{n-2}F_{n-1}}+1 \leq \frac{F_{n-2}+F_{n-1}}{2}+1 = \frac{F_n}{2}+1 \leq F_n,$$

which is the first inequality to prove, with equality if, and only if, $n = 3$.

The second inequality, if $n = 3, 4, \cdots$ can be proved using that $F_n = \sum_{i=1}^{n-2} F_i + 1$, the Quadratic Mean-Arithmetic Mean inequality applied to the positive numbers F_i, $i = 1, 2, \cdots, n-2$, and that $F_{n-2}F_{n-1} = \sum_{i=1}^{n-2} F_i^2$, because

$$F_n = \sum_{i=1}^{n-2} F_i + 1 \le \sqrt{(n-2)\sum_{i=1}^{n-2} F_i^2} + 1 = \sqrt{(n-2)F_{n-2}F_{n-1}} + 1,$$

with equality if, and only if, $n = 3$ or $n = 4$.

Solution 3 by Shai Covo, Kiryat-Ono, Israel

The left inequality is trivial. Indeed, for any $n \ge 3$,

$$\sqrt{F_{n-2}F_{n-1}} + 1 \le \sqrt{F_{n-1}F_{n-1}} + F_{n-2} = F_n.$$

As for the right inequality, the result is readily seen to hold for $n = 3, 4, 5, 6$. Hence, it suffices to show that for any $n \ge 7$ the following inequality holds:

$$F_n = F_{n-2} + F_{n-1} < \sqrt{5F_{n-2}F_{n-1}}.$$

With x and y playing the role of F_{n-2} and F_{n-1} ($n \ge 7$), respectively, it thus suffices to show that $x + y < \sqrt{5xy}$, subject to $x < y < 2x$ ($x \ge F_5 = 5$).
It is readily checked that, for any fixed $x > 0$ (real), the function $\phi_x(y) = \sqrt{5xy} - (x+y)$, defined for $y \in [x, 2x]$, has a global minimum at $y = 2x$, where $\phi_x(y) = (\sqrt{10} - 3)x > 0$. The result is now established.

Solution 4 by Brian D. Beasley, Clinton, SC

Let $L_n = \alpha\sqrt{\alpha F_{n-2}F_{n-1}} - 1$ and $U_n = \alpha\sqrt{\alpha F_{n-2}F_{n-1}} + 1$, where $\alpha = (1+\sqrt{5})/2$. We prove the stronger inequalities $L_n \le F_n \le U_n$ for $n \ge 3$, with improved lower bound for $n \ge 5$ and improved upper bound for $n \ge 7$.

First, we note that the inequalities given in the original problem hold for $3 \le n \le 6$. Next, we apply induction on n, verifying that $L_3 \le F_3 \le U_3$ and assuming that $L_n \le F_n \le U_n$ for some $n \ge 3$. Then $(F_n - 1)^2 \le \alpha^3 F_{n-2}F_{n-1} \le (F_n + 1)^2$, which implies

$$(F_{n+1} - 1)^2 = (F_n - 1)^2 + 2F_{n-1}(F_n - 1) + F_{n-1}^2 \le \alpha^3 F_{n-2}F_{n-1} + 2F_{n-1}(F_n - 1) + F_{n-1}^2$$

and

$$(F_{n+1} + 1)^2 = (F_n + 1)^2 + 2F_{n-1}(F_n + 1) + F_{n-1}^2 \ge \alpha^3 F_{n-2}F_{n-1} + 2F_{n-1}(F_n + 1) + F_{n-1}^2.$$

Since $\alpha^3 F_{n-1}F_n = \alpha^3 F_{n-2}F_{n-1} + \alpha^3 F_{n-1}^2$, it suffices to show that

$$2F_{n-1}(F_n - 1) + F_{n-1}^2 \le \alpha^3 F_{n-1}^2 \le 2F_{n-1}(F_n + 1) + F_{n-1}^2,$$

that is, $2(F_n - 1) + F_{n-1} \le \alpha^3 F_{n-1} \le 2(F_n + 1) + F_{n-1}$. Using the Binet formula $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, where $\beta = (1-\sqrt{5})/2$, these latter inequalities are equivalent to $2\beta^{n-1} - 2 \le 0 \le 2\beta^{n-1} + 2$, both of which hold since $-1 < \beta < 0$. (We also used the identities $2\alpha + 1 - \alpha^3 = 0$ and $\alpha^3 - 1 - 2\beta = 2\sqrt{5}$.)

Finally, we note that U_n is smaller than the original upper bound for $n \geq 7$, since $\alpha^3 + 2 < 7$. Also, a quick check verifies that L_n is larger than the original lower bound for $n \geq 5$; this requires

$$(\alpha^3 - 1)^2 (F_{n-2}F_{n-1})^2 - 8(\alpha^3 + 1)F_{n-2}F_{n-1} + 16 \geq 0,$$

which holds if $F_{n-2}F_{n-1} \geq 4$.

Also solved by Paul M. Harms, North Newton, KS; Tom Leong, Scotrun, PA; Boris Rays, Brooklyn NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5096:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b + \sqrt[4]{ab^3}} + \frac{b}{c + \sqrt[4]{bc^3}} + \frac{c}{a + \sqrt[4]{ca^3}} \geq \frac{3}{2}.$$

Solution 1 by Ovidiu Furdui, Cluj, Romania

We have, since $\sqrt[4]{xy^3} \leq \dfrac{x + 3y}{4}$, that

$$\sum_{cyclic} \frac{a}{b + \sqrt[4]{ab^3}} \geq 4 \sum_{cyclic} \frac{a}{7b + a} = 4 \sum_{cyclic} \frac{a^2}{7ba + a^2} \geq 4 \frac{(a+b+c)^2}{\sum a^2 + 7\sum ab},$$

and hence it suffices to prove that

$$8(a+b+c)^2 \geq 3(a^2 + b^2 + c^2) + 21(ab + bc + ca).$$

However, the last inequality reduces to proving that

$$a^2 + b^2 + c^2 \geq ab + bc + ca,$$

and the problem is solved since the preceding inequality holds for all real a, b, and c.

Solution 2 by Ercole Suppa, Teramo, Italy

By the weighted AM-GM inequality we have

$$\begin{aligned}
&\frac{a}{b + \sqrt[4]{ab^3}} + \frac{b}{c + \sqrt[4]{bc^3}} + \frac{c}{a + \sqrt[4]{ca^3}} \\
\geq \quad & \frac{a}{b + \frac{1}{4}a + \frac{3}{4}b} + \frac{b}{c + \frac{1}{4}b + \frac{3}{4}c} + \frac{c}{a + \frac{1}{4}c + \frac{3}{4}a} \\
= \quad & \frac{4a}{a + 7b} + \frac{4b}{b + 7c} + \frac{4c}{c + 7a}.
\end{aligned}$$

So it suffices to prove that

$$\frac{a}{a + 7b} + \frac{b}{b + 7c} + \frac{c}{c + 7a} \geq \frac{3}{8}.$$

This inequality is equivalent to

$$\frac{7(13a^2b+13b^2c+13ac^2+35ab^2+35a^2c+35bc^2-144abc)}{8(a+7b)(b+7c)(c+7a)}\geq 0$$

which is true. Indeed according to the AM-GM inequality we obtain

$$13a^2b+13b^2c+13ac^2 \geq 13\cdot 3\cdot \sqrt[3]{a^3b^3c^3} \quad = \quad 39abc$$

$$35ab^2+35a^2c+35bc^2 \geq 35\cdot 3\cdot \sqrt[3]{a^3b^3c^3} \quad = \quad 105abc$$

and, summing these inequalities we obtain:

$$13a^2b+35ab^2+35a^2c+13b^2c+13ac^2+35bc^2 \geq 144abc.$$

This ends the proof. Clearly, equality occurs for $a=b=c$.

Solution 3 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

We start by considering the function

$$f(t)=\frac{1}{e^t+e^{\frac{3}{4}t}}$$

on $\Re$. Then for all $t\in\Re$,

$$f''(t)=\frac{16e^{2t}+23e^{\frac{7}{4}t}+9e^{\frac{3}{2}t}}{16\left(e^t+e^{\frac{3}{4}t}\right)^3}>0,$$

and hence, $f(t)$ is strictly convex on $\Re$.

If $x=\ln\left(\dfrac{b}{a}\right)$, $y=\ln\left(\dfrac{b}{a}\right)$, and $z=\ln\left(\dfrac{b}{a}\right)$, then

$$x+y+z=\ln\left(\frac{b}{a}\cdot\frac{c}{b}\cdot\frac{a}{c}\right)=\ln 1=0.$$

By Jensen's Theorem,

$$\begin{aligned}
\frac{a}{b+\sqrt[4]{ab^3}}+\frac{b}{c+\sqrt[4]{bc^3}}+\frac{c}{a+\sqrt[4]{ca^3}} \quad &= \quad \frac{1}{\left(\dfrac{b}{a}\right)+\left(\dfrac{b}{a}\right)^{3/4}}+\frac{1}{\left(\dfrac{c}{b}\right)+\left(\dfrac{c}{b}\right)^{3/4}}+\frac{1}{\left(\dfrac{a}{c}\right)+\left(\dfrac{a}{c}\right)^{3/4}} \\
&= \quad f(x)+f(y)+f(z) \\
&\geq \quad 3f\left(\frac{x+y+z}{3}\right)
\end{aligned}$$

$$= 3f(0)$$
$$= \frac{3}{2}.$$

Further, equality is attained if, and only if, $x = y = z = 0$, i.e., if, and only if, $a = b = c$.

Solution 4 by Shai Covo, Kiryat-Ono, Israel

Let us first represent b and c as $b = xa$ and $c = yxa$, where x and y are arbitrary positive real numbers. By doing so, the original inequality becomes

$$\frac{1}{x + x^{3/4}} + \frac{1}{y + y^{3/4}} + \frac{yx}{1 + (yx)^{1/4}} \geq \frac{3}{2}. \qquad (1)$$

Let us denote by $f(x, y)$ the expression on the left-hand side of this inequality. Clearly, $f(x, y)$ has a global minimum at some point $(\alpha, \beta) \in (0, \infty) \times (0, \infty)$, a priori not necessarily unique. This point is, in particular, a critical point of f; that is, $f_x(\alpha, \beta) = f_y(\alpha, \beta) = 0$, where f_x and f_y denote the partial derivatives of f with respect to x and y. Calculating derivatives, the conditions $f_x(\alpha, \beta) = 0$ and $f_y(\alpha, \beta) = 0$ imply that

$$\begin{cases} \dfrac{1 + \frac{3}{4}\alpha^{-1/4}}{\left(\alpha + \alpha^{3/4}\right)^2} = \dfrac{\beta\left[1 + \frac{3}{4}(\beta\alpha)^{1/4}\right]}{\left[1 + (\beta\alpha)^{1/4}\right]^2} \text{ and} \\ \\ \dfrac{1 + \frac{3}{4}\beta^{-1/4}}{\left(\beta + \beta^{3/4}\right)^2} = \dfrac{\alpha\left[1 + \frac{3}{4}(\beta\alpha)^{1/4}\right]}{\left[1 + (\beta\alpha)^{1/4}\right]^2} \end{cases}, \qquad (2)$$

respectively. From this it follows straight forwardly, that

$$\frac{1 + \frac{3}{4}\alpha^{-1/4}}{\alpha\left(1 + \alpha^{-1/4}\right)^2} = \frac{1 + \frac{3}{4}\beta^{-1/4}}{\beta\left(1 + \beta^{-1/4}\right)^2}.$$

Writing this equality as $\varphi(\alpha) = \varphi(\beta)$ and noting that φ is strictly decreasing, we conclude (by virtue of φ being one-to-one) that $\alpha = \beta$. Substituting this into (2) gives

$$\frac{1 + \frac{3}{4}\alpha^{-1/4}}{\left(\alpha + \alpha^{3/4}\right)^2} = \frac{\alpha\left(1 + \frac{3}{4}\alpha^{1/2}\right)}{\left(1 + \alpha^{1/2}\right)^2}.$$

Comparing the numerators and denominators of this equation shows that the right-hand side is greater than the left-hand side for $\alpha > 1$, while the opposite is true for $\alpha < 1$. We conclude that $\alpha = \beta = 1$. Thus f has a unique global minimum at $(x, y) = (1, 1)$, where $f(x, y) = 3/2$. The inequality (1), and hence the one stated in the problem, is thus proved.

Also solved by Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Tom Leong, Scotrun, PA; Paolo Perfetti, Mathematics Department Tor Vergata University, Rome, Italy, and the proposer.

- **5097:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Let $p \geq 2$ be a natural number. Find the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\lfloor \sqrt[p]{n} \rfloor},$$

  where $\lfloor a \rfloor$ denotes the **floor** of a. (Example $\lfloor 2.4 \rfloor = 2$).

Solution 1 by Paul M. Harms, North Newton, KS

Since the series is an alternating series it is important to check whether the number of terms with the same denominator is even or odd. It is shown below that the number of terms with the same denominator is an odd number.

Consider p=2. The series starts:

$$\frac{(-1)^1}{1} + \frac{(-1)^2}{1} + \frac{(-1)^3}{1} + \frac{(-1)^4}{1} + \ldots + \frac{(-1)^8}{2} + \frac{(-1)^9}{3} + \ldots$$

$$\begin{aligned} &= \frac{(-1)^3}{1} + \frac{(-1)^8}{2} \frac{(-1)^{15}}{3} + \ldots \\ &= -1 + \frac{1}{2} - \frac{1}{3} + \ldots . \end{aligned}$$

The terms with 1 in the denominator are from $n = 1^2$ up to (not including) $n = 2^2$, and the terms with 2 in the denominator come from $n = 2^2$ up to $n = 3^2$. The number of terms with 1 in the denominator is $2^2 - 1^2 = 3$ terms.

For $p = 2$ the number of terms with a positive integer m in the denominator is $(m+1)^2 - m^2 = 2m + 1$ terms which is an odd number of terms.

For a general positive integer p, the number of terms with a positive integer m in the denominator is $(m+1)^p - m^p$ terms. Either $(m+1)$ is even and m is odd or vice versa. An odd integer raised to a positive power is an odd integer, and an even integer raised to a positive power is an even integer. Then $(m+1)^p - m^p$ is the difference of an even integer and an odd integer which is an odd integer. Since, for every positive integer p the series starts with $\dfrac{(-1)^1}{1} = -1$ and we have an odd number of terms with denominator 1, the last term with 1 in the denominator is $\dfrac{-1}{1}$ and the other terms cancel out.

The terms with denominator 2 start and end with positive terms. They all cancel out except the last term of $\dfrac{1}{2}$.

Terms with denominator 3 start and end with negative terms. For every p we have the series

$$\frac{-1}{1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots = -\ln 2.$$

Solution 2 by The Taylor University Problem Solving Group, Upland, IN

First note that the denominators of the terms of this series will be increasing natural numbers, because $\sqrt[p]{n}$ will always be a real number greater than or equal to 1 for $n \geq 1$, meaning that its floor will be a natural number. Furthermore, for a natural number a, a^p is the smallest n for which a is the denominator, because $\lfloor \sqrt[p]{a^p} \rfloor = \lfloor a \rfloor = a$. In other words, the denominator increases by 1 each time n is a perfect *pth* power. Thus, a natural number k occurs as the denominator $(k+1)^p - k^p$ times in the series. Because multiplying a number by itself preserves parity and $k+1$ and k always have opposite parity, $(k+1)^p$ and k^p also have opposite parity, hence their difference is odd. So each denominator occurs an odd number of times. Because the numberator alternates between 1 and -1, all but the last of the terms with the same denominator will cancel each other out. This leaves an alternating harmonic series with a negative first term, which converges to $-\ln 2$.

This can be demonstrated by the fact that the alternating harmonic series with a positive first term is the Mercator series evaluated at $x = 1$, and this series is simply the opposite of that.

Incidentally, this property holds for $p = 1$ as well.

Also solved by Shai Covo, Kiryat-Ono, Israel; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Mathematics Department Tor Vergata University, Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA; Ercole Suppa, Teramo, Italy, and the proposer.