Problems Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before June 15, 2009

- 5062: *Proposed by Kenneth Korbin, New York, NY.*

  Find the sides and the angles of convex cyclic quadrilateral ABCD if $\overline{AB} = \overline{BC} = \overline{CD} = \overline{AD} - 2 = \overline{AC} - 2$.

- 5063: *Proposed by Richard L. Francis, Cape Girardeau, MO.*

  Euclid's inscribed polygon is a constructible polygon inscribed in a circle whose consecutive central angle degree measures form a positive integral arithmetic sequence with a non-zero difference.

  a) Does Euclid's inscribed n-gon exist for any prime n greater than 5?

  b) Does Euclid's n-gon exist for all composite numbers n greater than 2?

- 5064: *Proposed by Michael Brozinsky, Central Islip, NY.*

  The Lemoine point of a triangle is that point inside the triangle whose distances to the three sides are proportional to those sides. Find the maximum value that the constant of proportionality, say λ, can attain.

- 5065: *Mihály Bencze, Brasov, Romania.*

  Let n be a positive integer and let $x_1 \le x_2 \le \cdots \le x_n$ be real numbers. Prove that

$$1) \qquad \sum_{i,j=1}^{n} |(i-j)(x_i - x_j)| = \frac{n}{2} \sum_{i,j=1}^{n} |x_i - x_j|.$$

$$2) \qquad \sum_{i,j=1}^{n} (i-j)^2 = \frac{n^2(n^2-1)}{6}.$$

- 5066: *Proposed by Panagiote Ligouras, Alberobello, Italy.*

  Let a, b, and c be the sides of an acute-angled triangle ABC. Let $abc = 1$. Let H be the orthocenter, and let d_a, d_b, and d_c be the distances from H to the sides BC, CA, and AB respectively. Prove or disprove that

$$3(a+b)(b+c)(c+a) \geq 32(d_a+d_b+d_c)^2.$$

- 5067: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be complex numbers such that $a+b+c=0$. Prove that

$$\max\{|a|,|b|,|c|\} \leq \frac{\sqrt{3}}{2}\sqrt{|a|^2+|b|^2+|c|^2}.$$

Solutions

- 5044: *Proposed by Kenneth Korbin, New York, NY.*

  Let N be a positive integer and let

$$\begin{cases} x = 9N^2+24N+14 \text{ and} \\ y = 9(N+1)^2+24(N+1)+14. \end{cases}$$

Express the value of y in terms of x, and express the value of x in terms of y.

Solution by Armend Sh. Shabani, Republic of Kosova.

One easily verifies that

$$y - x = 18N + 33. \qquad (1)$$

From $9N^2+24N+14-x=0$ one obtains $N_{1,2} = \dfrac{-4 \pm \sqrt{2+x}}{3}$, and since N is a positive integer we have

$$N = \frac{-4+\sqrt{2+x}}{3}. \qquad (2)$$

Substituting (2) into (1) gives:

$$y = x + 9 + 6\sqrt{2+x}. \qquad (3)$$

From $9(N+1)^2+24(N+1)+14-y=0$ one obtains $N_{1,2} = \dfrac{-7 \pm \sqrt{2+y}}{3}$, and since N is a positive integer we have

$$N = \frac{-7+\sqrt{2+y}}{3}. \qquad (4)$$

Substituting (4) into (1) gives:

$$x = y + 9 - 6\sqrt{2+y}. \qquad (5)$$

Relations (3) and (5) are the solutions to the problem.

Comments: **1. Paul M. Harms** mentioned that the equations for x in terms of y, as well as for y in terms of x, are valid for integers satisfying the x, y and N equations in the problem. The minimum x and y values occur when $N=1$ and are $x=47$ and $y=98$. **2. David Stone and John Hawkins** observed that in addition to $(47, 98)$, other integer lattice points on the curve of $y = 9 + x + 6\sqrt{2+x}$ in the first quadrant are $(4, 98), (98, 167), (167, 254), (254, 359)$, and $(23, 62)$.

Also solved by Brian D. Beasley, Clinton, SC; John Boncek, Montgomery, AL; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX; José Luis Díaz-Barrero, Barcelona, Spain; Bruno Salgueiro Fanego, Viveiro, Spain; Michael C. Faleski, University Center, MI; Michael N. Fried, Kibbutz Revivim, Israel; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; José Hernández Santiago (student UTM), Oaxaca, México; David Stone and John Hawkins (jointly), Statesboro, GA; David C.Wilson, Winston-Salem, NC, and the proposer.

- **5045:** *Proposed by Kenneth Korbin, New York, NY.*

Given convex cyclic hexagon ABCDEF with sides

$$\begin{aligned} \overline{AB} &= \overline{BC} = 85 \\ \overline{CD} &= \overline{DE} = 104, \text{ and} \\ \overline{EF} &= \overline{FA} = 140. \end{aligned}$$

Find the area of $\triangle BDF$ and the perimeter of $\triangle ACE$.

Solution by Kee-Wai Lau, Hong Kong, China.

We show that the area of $\triangle BDF$ iis 15390 and the perimeter of $\triangle ACE$ is $\dfrac{123120}{221}$.

Let $\angle AFE = 2\alpha, \angle EDC = 2\beta$, and $\angle$CBA$=2\gamma$ so that

$$\angle ACE = \pi - 2\alpha,\ \angle CAE = \pi - 2\beta, \text{ and } \angle AEC = \pi - 2\gamma.$$

Since $\angle ACE + \angle CAE + \angle AEC = \pi$, so

$$\begin{aligned} \alpha + \beta + \gamma &= \pi \\ \cos\alpha + \cos\beta + \cos\gamma &= 4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} + 1 \text{ or} \\ (\cos\alpha + \cos\beta + \cos\gamma - 1)^2 &= 2(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma). \qquad (1) \end{aligned}$$

Denote the radius of the circumcircle by R. Applying the Sine Formula to $\triangle ACE$, we have

$$R = \frac{\overline{AE}}{2\sin 2\alpha} = \frac{\overline{EC}}{2\sin 2\beta} = \frac{\overline{CA}}{2\sin 2\gamma}.$$

By considering triangles $AFE,\ EDC$, and CBA respectively, we obtain

$$\overline{AE} = 280\sin\alpha,\ \overline{EC} = 208\sin\beta,\ \overline{CA} = 170\sin\gamma.$$

It follows that $\cos\alpha = \dfrac{70}{R},\ \cos\beta = \dfrac{52}{R}$, and $\cos\gamma = \dfrac{85}{2R}$. Substituting into (1) and simplifying, we obtain

$$\begin{aligned} 4R^3 - 37641R - 1237600 &= 0 \text{ or} \\ \Big(2R - 221\Big)\Big(2R^2 + 221R + 5600\Big) &= 0. \end{aligned}$$

Hence,

$$R = \frac{221}{2},\ \cos\alpha = \frac{140}{221},\ \sin\alpha = \frac{171}{221}$$
$$\cos\beta = \frac{104}{221},\ \sin\beta = \frac{195}{221}$$
$$\cos\gamma = \frac{85}{221},\ \sin\gamma = \frac{204}{221},$$

and our result for the perimeter of $\triangle ACE$.

It is easy to check that $\angle BFD = \alpha, \angle FDB = \beta, \angle DBF = \gamma$ so that
$\angle BAF = \pi - \beta, \angle DEF = \pi - \gamma$.

Applying the cosine formula to $\triangle BAF$ and $\triangle DEF$ respectively, we obtain $BF = 195$ and $DF = 204$.

It follows, as claimed, that the area of

$$\triangle BDF = \frac{1}{2}\left(\overline{BF}\right)\left(\overline{DF}\right)\sin\angle BFD = \frac{1}{2}(195)(204)\frac{171}{221} = 15390.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5046:** *Proposed by R.M. Welukar of Nashik, India and K.S. Bhanu, and M.N. Deshpande of Nagpur, India.*

Let $4n$ successive Lucas numbers $L_k, L_{k+1}, \cdots, L_{k+4n-1}$ be arranged in a $2 \times 2n$ matrix as shown below:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & 2n \end{array}$$
$$\begin{pmatrix} L_k & L_{k+3} & L_{k+4} & L_{k+7} & \cdots & L_{k+4n-1} \\ \\ L_{k+1} & L_{k+2} & L_{k+5} & L_{k+6} & \cdots & L_{k+4n-2} \end{pmatrix}$$

Show that the sum of the elements of the first and second row denoted by R_1 and R_2 respectively can be expressed as

$$R_1 = 2F_{2n}L_{2n+k}$$

$$R_2 = F_{2n}L_{2n+k+1}$$

where $\{L_n, n \geq 1\}$ denotes the Lucas sequence with $L_1 = 1, L_2 = 3$ and
$L_{i+2} = L_i + L_{i+1}$ for $i \geq 1$ and $\{F_n, n \geq 1\}$ denotes the Fibonacci sequence,
$F_1 = 1, F_2 = 1, F_{n+2} = F_n + F_{n+1}$.

Solution by Angel Plaza and Sergio Falcon, Las Palmas, Gran Canaria, Spain.

$R_1 = L_k + L_{k+3} + L_{k+4} + L_{k+7} + \cdots + L_{k+4n-2} + L_{k+4n-1}$, and since $L_i = F_{i-1} + F_{i+1}$, we have:

$$\begin{aligned} R_1 &= F_{k-1} + F_{k+1} + F_{k+2} + F_{k+4} + F_{k+3} + F_{k+5} + \cdots + F_{k+4n-2} + F_{k+4n} \\ &= F_{k-1} + \sum_{j=1}^{4n} F_{k+j} - F_{k+4n-1} \\ &= F_{k-1} - F_{k+4n-1} + \sum_{j=0}^{4n+k} F_j - \sum_{j=0}^{k} F_j \end{aligned}$$

And since $\sum_{j=0}^{m} F_j = F_{m+2} - 1$ we have:

$$R_1 = F_{k-1} - F_{k+4n-1} + F_{k+4n+2} - 1 - F_{k+2} + 1 = 2F_{k+4n} - 2F_k$$

where in the last equation it has been used that $F_{i+2} - F_i = F_{i+1} + F_i - F_{i-1} = 2F_i$.
Now, using the relation $L_n F_m = F_{n+m} - (-1)^m F_{n-m}$ (S. Vajda, Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications, Dover Press (2008)) in the form $L_{2n+k}F_{2n} = F_{4n+k} - (-1)^{2n} F_{2n+k-2n}$ it is deduced $R_1 = 2F_{2n}L_{2n+k}$.
In order to prove the fomula for R_2 note that

$$R_1 + R_2 = \sum_{j=0}^{4n-1} L_{k+j} = \sum_{j=0}^{4n+k-1} L_j - \sum_{j=0}^{k-1} L_j$$

As before, $\sum_{j=0}^{4n+k-1} L_j = F_{k+4n} + F_{k+4n+2}$, while $\sum_{j=0}^{k-1} L_j = F_k + F_{k+2}$, so

$$\begin{aligned} R_1 + R_2 &= F_{k+4n} - F_k + F_{k+4n+2} - F_{k+2} \\ &= L_{2n+k}F_{2n} + L_{2n+k+2}F_{2n} \end{aligned}$$

And therefore,

$$R_2 = F_{2n}\left(L_{2n+k+2} - L_{2n+k}\right) = F_{2n}L_{2n+k+1}$$

Also solved by Paul M. Harms, North Newton, KS; John Hawkins and David Stone (jointly), Statesboro, GA, and the proposers.)

- 5047: *Proposed by David C. Wilson, Winston-Salem, N.C.*

  Find a procedure for continuing the following pattern:

$$\begin{aligned} S(n,0) &= \sum_{k=0}^{n} \binom{n}{k} = 2^n \\ S(n,1) &= \sum_{k=0}^{n} \binom{n}{k} k = 2^{n-1} n \\ S(n,2) &= \sum_{k=0}^{n} \binom{n}{k} k^2 = 2^{n-2} n(n+1) \end{aligned}$$

$$S(n,3) = \sum_{k=0}^{n} \binom{n}{k} k^3 = 2^{n-3} n^2 (n+3)$$
$$\vdots$$

Solution by David E. Manes, Oneonta, NY.

Let $f(x) = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. For $m \geq 0$,

$S(n,m) = \left(x\dfrac{d}{dx}\right)^m (f(x))\Big|_{x=1}$, where $\left(x\dfrac{d}{dx}\right)^m$ is the procedure $x\dfrac{d}{dx}$ iterated m times and then evaluating the resulting function at $x=1$. For example,

$$\begin{aligned}
S(n,0) = f(1) &= \sum_{k=0}^{n} \binom{n}{k} = 2^n. \text{ Then} \\
x\frac{d}{dx}(f(x)) &= x\frac{d}{dx}(1+x)^n = x\frac{d}{dx}\left(\sum_{k=0}^{n} \binom{n}{k} x^k\right) \text{ implies} \\
nx(1+x)^{n-1} &= \sum_{k=0}^{n} \binom{n}{k} k \cdot x^k. \text{ If } x=1, \text{then} \\
\sum_{k=0}^{n} \binom{n}{k} k &= n \cdot 2^{n-1} = S(n,1).
\end{aligned}$$

For the value of $S(n,2)$ note that if

$$\begin{aligned}
x\frac{d}{dx}\left[nx(1+x)^{n-1}\right] &= x\frac{d}{dx}\left[\sum_{k=0}^{n} \binom{n}{k} kx^k\right], \text{ then} \\
nx(nx+1)(1+x)^{n-2} &= \sum_{k=0}^{n} \binom{n}{k} k^2 x^k. \text{ If } x=1, \text{ then} \\
n(n+1)2^{n-2} &= \sum_{k=0}^{n} \binom{n}{k} k^2 = S(n,2)
\end{aligned}$$

Similarly,

$$\begin{aligned}
S(n,3) &= \sum_{k=0}^{n} \binom{n}{k} k^3 = 2^{n-3} \cdot n^2(n+3) \text{ and} \\
S(n,4) &= \sum_{k=0}^{n} \binom{n}{k} k^4 = 2^{n-4} \cdot n(n+1)(n^2+5n-2.)
\end{aligned}$$

Also solved by Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro GA, and the proposer.

- 5048: *Proposed by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy.*

  Let $a, b, c,$ be positive real numbers. Prove that

$$\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2} \ \geq \ \frac{54}{(a+b+c)^2}\frac{(abc)^3}{\sqrt{(ab)^4+(bc)^4+(ca)^4}}.$$

Solution1 by Boris Rays, Chesapeake, VA.

Rewrite the inequality into the form:

$$\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2}\cdot\Big(a{+}b{+}c\Big)^2\cdot\sqrt{(ab)^4+(bc)^4+(ca)^4} \geq 54(abc)^3 \qquad (1)$$

We will use the Arithmetic-Geometric Mean Inequality (e.g., $x+y+z \geq 3\sqrt[3]{xyz}$ and $u+v \geq 2\sqrt{uv}$) for each of the three factors on the left side of (1).

$$\begin{aligned}
\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2} &\geq \sqrt{3\sqrt[3]{c^2(a^2+b^2)^2\cdot b^2(c^2+a^2)^2\cdot a^2(b^2+c^2)^2}}\\
&\geq \sqrt{3\sqrt[3]{(abc)^2(a^2+b^2)^2(c^2+a^2)^2(b^2+c^2)^2}}\\
&\geq \sqrt{3\sqrt[3]{(abc)^2(4a^2b^2)(4c^2a^2)(4b^2c^2)}}\\
&= \sqrt{3(abc)^{2/3}\sqrt[3]{4^3a^4b^4c^4}}\\
&= \sqrt{3(abc)^{2/3}4(abc)^{4/3}}\\
&= \sqrt{3\cdot 2^2(abc)^2}\\
&= 2\sqrt{3}(abc) \qquad (2)
\end{aligned}$$

Also, since $(a+b+c) \geq 3\sqrt[3]{abc}$, we have

$$(a+b+c)^2 \geq 3^2\left(\sqrt[3]{abc}\right)^2 = 3^2(abc)^{2/3} \qquad (3)$$

$$\begin{aligned}
\sqrt{(ab)^4+(bc)^4+(ca)^4} &\geq \sqrt{3\sqrt[3]{(ab)^4(bc)^4(ca)^4}}\\
&= \sqrt{3\sqrt[3]{a^8b^8c^8}}\\
&= \sqrt{3(abc)^{8/3}}\\
&= \sqrt{3}(abc)^{4/3} \qquad (4)
\end{aligned}$$

Combining (2), (3), and (4) we obtain:

$$\begin{aligned}
\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2} \quad &\cdot \quad \left(a+b+c\right)^2 \cdot \sqrt{(ab)^4+(bc)^4+(ca)^4} \\
&\geq \quad 2\sqrt{3}(abc)\cdot 3^2(abc)^{2/3}\sqrt{3}(abc)^{4/3} \\
&= \quad 2\cdot 3^3(abc)^{1+2/3+4/3} \\
&= \quad 54(abc)^3.
\end{aligned}$$

Hence, we have shown that (1) is true, with equality holding if $a=b=c$.

Solution 2 by José Luis Díaz-Barrero, Barcelona, Spain.

The inequality claimed is equivalent to

$$\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2}\,\sqrt{(ab)^4+(bc)^4+(ca)^4} \geq \frac{54(abc)^3}{(a+b+c)^2}$$

Applying Cauchy's inequality to the vectors $\vec{u}=(c(a^2+b^2), b(c^2+a^2), a(b^2+c^2))$ and $\vec{v}=(a^2b^2, c^2a^2, b^2c^2)$ yields

$$\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2}\,\sqrt{(ab)^4+(bc)^4+(ca)^4}$$

$$\geq abc(ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2))$$

So, it will be suffice to prove that

$$(ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2))(a+b+c)^2 \geq 54a^2b^2c^2 \tag{1}$$

Taking into account GM-AM-QM inequalities, we have

$$ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2) \geq 2(a^2b^2+b^2c^2+c^2a^2) \geq 6abc\sqrt[3]{abc}$$

and

$$(a+b+c)^2 \geq 9\,\sqrt[3]{a^2b^2c^2}$$

Multiplying up the preceding inequalities (1) follows and the proof is complete

Solution 3 by Kee-Wai Lau, Hong Kong, China.

By homogeneity, we may assume without loss of generality that $abc=1$. We have

$$\begin{aligned}
&\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2} \\
&= \quad \sqrt{\left(\frac{a^2+b^2}{ab}\right)^2+\left(\frac{c^2+a^2}{ca}\right)^2+\left(\frac{b^2+c^2}{bc}\right)^2} \\
&= \quad \sqrt{\left(\frac{a^2-b^2}{ab}\right)^2+\left(\frac{c^2-a^2}{ca}\right)^2+\left(\frac{b^2-c^2}{bc}\right)^2+12}
\end{aligned}$$

$$\geq \quad 2\sqrt{3}.$$

By the arithmetic-geometric mean inequality, we have $(a+b=c)^2 \geq 9(abc)^{2/3} = 9$ and $\sqrt{(ab)^4+(bc)^4+(ca)^4} \geq \sqrt{3}(abc)^{4/3} = \sqrt{3}$. The inequality of the problem now follows immediately.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ovidiu Furdui, Campia Turzii, Cluj, Romania; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Armend Sh. Shabani, Republic of Kosova, and the proposer.

5049: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Find a function $f:\Re \to \Re$ such that

$$2f(x)+f(-x) = \begin{cases} -x^3-3, & x \leq 1, \\ 3-7x^3, & x > 1. \end{cases}$$

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX .

If $x > 1$, then

$$2f(x)+f(-x) = 3-7x^3. \qquad (1)$$

Also, since $-x < -1$, we have

$$2f(-x)+f(x) = -(-x)^3-3 = x^3-3. \qquad (2)$$

By (1) and (2), $f(x) = 3-5x^3$ and $f(-x) = -3+3x^3$ when $x > 1$. Further, $f(-x) = -3+3x^3$ when $x > 1$ implies that $f(x) = -3+3(-x)^3 = -3-3x^3$ when $x < -1$.

Finally, when $-1 \leq x \leq 1$, we get $-1 \leq -x \leq 1$ also, and hence,

$$2f(x)+f(-x) = -x^3-3, \qquad (3)$$

$$2f(-x)+f(x) = -(-x)^3-3 = x^3-3. \qquad (4)$$

As above, (3) and (4) imply that $f(x) = -x^3-1$ when $-1 \leq x \leq 1$.

Therefore, $f(x)$ must be of the form

$$f(x) = \begin{cases} -3-3x^3 & \text{if } \; x < -1, \\ -1-x^3 & \text{if } \; -1 \leq x \leq 1, \\ 3-5x^3 & \text{if } \; x > 1. \end{cases} \qquad (5)$$

With some perseverance, this can be condensed to

$$f(x) = \left|x^3+1\right| - 2\left|x^3-1\right| - 4x^3$$

for all $x \in \Re$. Since it is straightforward to check that this function satisfies the given conditions of the problem, this completes the solution.

Also solved by Brian D. Beasely, Clinton, SC; Michael Brozinsky, Central Islip, NY; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; N. J. Kuenzi, Oshkosh, WI; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; David C. Wilson, Winston-Salem, NC, and the proposer.

Late Solutions

Late solutions were received from **Pat Costello of Richmond, KY** to problem 5027; **Patrick Farrell of Andover, MA** to 5022 and 5024, and from **David C. Wilson of Winston-Salem, NC** to 5038.