Problems

Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before July 1, 2007

- 4966: *Proposed by Kenneth Korbin, New York, NY.*

  Solve:
  $$16x + 30\sqrt{1-x^2} = 17\sqrt{1+x} + 17\sqrt{1-x}$$
  with $0 < x < 1$.

- 4967: *Proposed by Kenneth Korbin, New York, NY.*

  Given equilateral triangle ABC with an interior point P such that $\overline{\text{AP}}^2 + \overline{\text{BP}}^2 = \overline{\text{CP}}^2$, and with an exterior point Q such that $\overline{\text{AQ}}^2 + \overline{\text{BQ}}^2 = \overline{\text{CQ}}^2$, where points C, P, and Q are in a line. Find the lengths of $\overline{\text{AQ}}$ and $\overline{\text{BQ}}$ if $\overline{\text{AP}} = \sqrt{21}$ and $\overline{\text{BP}} = \sqrt{28}$.

- 4968: *Proposed by Kenneth Korbin, New York, NY.*

  Find two quadruples of positive integers (a, b, c, d) such that
  $$\frac{a+i}{a-i}\cdot\frac{b+i}{b-i}\cdot\frac{c+i}{c-i}\cdot\frac{d+i}{d-i} = \frac{a-i}{a+i}\cdot\frac{b-i}{b+i}\cdot\frac{c-i}{c+i}\cdot\frac{d-i}{d+i}$$
  with $a < b < c < d$ and $i = \sqrt{-1}$.

- 4969: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be positive numbers such that $abc = 1$. Prove that
  $$\frac{1}{a^2\left(\dfrac{1}{a}+\dfrac{1}{c}\right)} + \frac{1}{b^2\left(\dfrac{1}{b}+\dfrac{1}{a}\right)} + \frac{1}{c^2\left(\dfrac{1}{c}+\dfrac{1}{b}\right)} \geq \frac{3}{2}$$

- 4970: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $f : [0,1] \longrightarrow \mathbf{R}$ be a contintuous convex function. Prove that

$$\frac{3}{4}\int_0^{1/5} f(t)dt + \frac{1}{8}\int_0^{2/5} f(t)dt \geq \frac{4}{5}\int_0^{1/4} f(t)dt.$$

- 4971: *Proposed by Howard Sporn, Great Neck, NY and Michael Brozinsky, Central Islip, NY.*

  Let $m \geq 2$ be a positive integer and let $1 \leq x < y$. Prove:

$$x^m - (x-1)^m < y^m - (y-1)^m.$$

Solutions

- 4936: *Proposed by Kenneth Korbin, New York, NY.*

  Find all prime numbers P and all positive integers a such that $P - 4 = a^4$.

  Solution 1 by Daniel Copeland (student, Saint George's School), Spokane, WA.

$$\begin{aligned} P &= a^4 + 4 \\ &= (a^2+2)^2 - 4a^2 \\ &= (a^2 - 2a + 2)(a^2 + 2a + 2). \end{aligned}$$

  Since P is a prime, one of the factors of P must be 1. Since a is a positive integer, $a^2 - 2a + 2 = 1$ which yields the only positive solution $a = 1, P = 5$.

  Solution 2 by Timothy Bowen (student, Waynesburg College), Waynesburg, PA.

  The only solution is $P = 5$ and $a = 1$.
  Case 1: Integer a is an even integer. For $a = 2n$, note $P = a^4+4 = (2n)^4+4 = 4\cdot(4n^4+1)$. Clearly, P is a composite for all natural numbers n.

  Case 2: Integer a is an odd integer. For $a = 2n+1$, note that $P = a^4+4 = (2n+1)^4+4 = (4n^2+8n+5)(4n^2+1)$. P is prime only for $n = 0$ (corresponding to $a = 1$ and $P = 5$). Otherwise, P is a composite number for all natural numbers n.

  Solution 3 by Jahangeer Kholdi & Robert Anderson (jointly), Portsmouth, VA.

  The only prime is $P = 5$ when $a = 1$. Consider $P = a^4 + 4$. If a is an even positive integer, then clearly P is even and hence a composite integer. Moreover, if a is a positive integer ending in digits $\{1, 3, 7 \text{ or } 9\}$, then P is a positive integer ending with the digit of 5. This also implies P is divisible by 5 and hence a composite. Lastly, assume $a = 10k+5$ where $k = 0$ or $k > 0$; that is a is a positive integer ending with a digit of 5. Then $P = (10k+5)^4 + 4$. But

$$P = (10k+5)^4 + 4 = (100k^2 + 80k + 17)(100k^2 + 120k + 37).$$

  Hence, for all positive integers $a > 1$ the positive integer P is composite.

  Also solved by Brian D. Beasley, Clinton, SC; Dionne Bailey, Elsie Campbell and Charles Diminnie (jointly), San Angelo, TX; Pat Costello, Richmond, KY;

Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; Vicki Schell, Pensacola, FL; R. P. Sealy, Sackville, New Brunswick, Canada; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins of Statesboro, GA jointly with Chris Caldwell of Martin, TN, and the proposer.

- **4937:** *Proposed by Kenneth Korbin, New York, NY.*

Find the smallest and the largest possible perimeter of all the triangles with integer-length sides which can be inscribed in a circle with diameter 1105.

Solution by Paul M. Harms, North Newton, KS.

Consider a radius line from the circle's center to one vertex of an inscribed triangle. Assume at this vertex one side has a length a and subtends a central angle of $2A$ and the other side making this vertex has a length b and subtends a central angle of $2B$.

Using the perpendicular bisector of chords, we have $\sin A = \dfrac{a/2}{1105/2} = \dfrac{a}{1105}$ and $\sin B = \dfrac{b}{1105}$. Also, the central angle of the third side is related to $2A+2B$ and the perpendicular bisector to the third side gives

$$\begin{aligned}
\sin(A+B) = \frac{c}{1105} &= \sin A\cos B + \sin B\cos A \\
&= \frac{a}{1105}\frac{\sqrt{1105^2-b^2}}{1105} + \frac{b}{1105}\frac{\sqrt{1105^2-a^2}}{1105} \\
\text{Thus}\quad c &= \frac{1}{1105}\left(a\sqrt{1105^2-b^2} + b\sqrt{1105^2-a^2}\right).
\end{aligned}$$

From this equation we find integers a and b which make integer square roots. Some numbers which do this are $\{47,1104\ 105,\ 1100,\ 169,\ 1092,\ \text{etc.}\ \}$. Checking the smaller numbers for the smallest perimeter we see that a triangle with side lengths $\{105,169,272\}$ gives a perimeter of 546 which seems to be the smallest perimeter.

To find the largest perimeter we look for side lengths close to the lengths of an inscribed equilateral triangle. An inscribed equilateral triangle for this circle has side length close to 957. Integers such as $884, 943, 952, 975$, and 1001 make integer square roots in the equation for c. The maximum perimeter appears to be 2870 with a triangle of side lengths $\{943,952,975\}$.

Comment: David Stone and John Hawkins of Statesboro, GA used a slightly different approach in solving this problem. Letting the side lengths be $a, b,$ and c and noting that the circumradius is 552.5 they obtained

$$\frac{1105}{2} = \frac{abc}{4\sqrt{a+b+c)(a+b-c)(a-b+c)(b+c-a)}}$$

which can be rewritten as

$$\sqrt{a+b+c)(a+b-c)(a-b+c)(b+c-a)} = \frac{abc}{(2)(5)(13)(17)}.$$

They then used that part of the law of sines that connects in any triangle ABC, side length $a, \angle A$ and the circumradius R; $\dfrac{a}{\sin A} = 2R$. This allowed them to find that $c^2 =$

$a^2+b^2 \mp \dfrac{2ab\sqrt{1105^2-c^2}}{1105}$. Noting that the factors of a,b, and c had to include the primes 2,5,13 and 17 and that 1105^2-c^2 had to be a perfect square, (and similarly for 1105^2-b^2 and 1105^2-a^2) they put EXCEL to work and proved that $\{105,\ 272,\ 169\}$ gives the smallest perimeter and that $\{952,\ 975,\ 943\}$ gives the largest. All in all they found 101 triangles with integer side lengths that can be inscribed in a circle with diameter 1105.

Also solved by the proposer.

- 4938: *Proposed by Luis Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Let a, b and c be the sides of an acute triangle ABC. Prove that

$$\csc^2\frac{A}{2}+\csc^2\frac{B}{2}+\csc^2\frac{C}{2}\geq 6\left[\prod_{cyclic}\left(1+\frac{b^2}{a^2}\right)\right]^{1/3}$$

Solution by proposers.

First, we claim that $a^2 \geq 2(b^2+c^2)\sin^2(A/2)$. In fact, the preceding inequality is equivalent to $a^2 \geq (b^2+c^2)(1-\cos A)$ and

$$a^2-(b^2+c^2)(1-\cos A) = b^2+c^2-2bc\cos A-(b^2+c^2)+(b^2+c^2)\cos A$$
$$= (b-c)^2\cos A \geq 0.$$

Similar inequalities can be obtained for b and c. Multiplying them up, we have

$$a^2b^2c^2 \geq 8(a^2+b^2)(b^2+c^2)(c^2+a^2)\sin^2(A/2)\sin^2(B/2)\sin^2(C/2). \qquad (1)$$

On the other hand, from GM-HM inequality we have

$$\begin{aligned}\sin^2(A/2)\sin^2(B/2)\sin^2(C/2) &\geq \left(\frac{3}{1/\sin^2(A/2)+1/\sin^2(B/2)+1/\sin^2(C/2)}\right)^3\\ &= \left(\frac{3}{\csc^2(A/2)+\csc^2(B/2)+\csc^2(C/2)}\right)^3.\end{aligned}$$

Substituting into the statement of the problem yields

$$\begin{aligned}\left(\csc^2\frac{A}{2}+\csc^2\frac{B}{2}+\csc^2\frac{C}{2}\right)^3 &\geq 216\left(\frac{a^2+b^2}{c^2}\right)\left(\frac{b^2+c^2}{a^2}\right)\left(\frac{c^2+a^2}{b^2}\right)\\ &= 216\prod_{cyclic}\left(1+\frac{b^2}{a^2}\right).\end{aligned}$$

Notice that equality holds when $A=B=C=\pi/3$. That is, when $\triangle ABC$ is equilateral and we are done.

- 4939: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

For any positive integer n, prove that

$$\left\{4^n+\left[\sum_{k=0}^{n-1}3^{k+1/2}\binom{2n}{2k+1}\right]^2\right\}^{1/2}$$

is a whole number.

Solution by David E. Manes, Oneonta, NY.

Let $W = 4^n + \left[\sum_{k=0}^{n-1} 3^{k+1/2}\binom{2n}{2k+1}\right]^2$ and notice that it suffices to show that $\sqrt{W}$ is a whole number. Expanding $(\sqrt{3}+1)^{2n}$ and $(\sqrt{3}-1)^{2n}$ using the Binomial Theorem and subtracting the second expansion from the first, one obtains

$$\sum_{k=0}^{n-1} 3^{k+1/2}\binom{2n}{2k+1} = \frac{(\sqrt{3}+1)^{2n} - (\sqrt{3}-1)^{2n}}{2}.$$

Therefore,

$$\begin{aligned} W &= 4^n + \left[\frac{(\sqrt{3}+1)^{2n} - (\sqrt{3}-1)^{2n}}{2}\right]^2 \\ &= 4^n + \frac{(\sqrt{3}+1)^{4n} - 2^{2n+1} + (\sqrt{3}-1)^{4n}}{4} \\ &= \frac{2^{2n+2} + (\sqrt{3}+1)^{4n} - 2^{2n+1} + (\sqrt{3}-1)^{4n}}{4} \\ &= \frac{(\sqrt{3}+1)^{4n} + 2^{2n+1} + (\sqrt{3}-1)^{4n}}{4} \\ &= \left[\frac{(\sqrt{3}+1)^{2n} - (\sqrt{3}-1)^{2n}}{2}\right]^2. \end{aligned}$$

Consequently,

$$\begin{aligned} \sqrt{W} &= \frac{(\sqrt{3}+1)^{2n} - (\sqrt{3}-1)^{2n}}{2} = \sum_{k=0}^{n}\binom{2n}{2k}(\sqrt{3})^{2k} \\ &= \sum_{k=0}^{n}\binom{2n}{2k}3^k, \text{ a whole number.} \end{aligned}$$

Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul H. Harms, North Newton, KS, and the proposer.

- 4940: *Proposed by Michael Brozinsky, Central Islip, NY and Leo Levine, Queens, NY .*

Let $S = \{n \in N | n \geq 5\}$. Let $G(x)$ be the fractional part of x, i.e., $G(x) = x - [x]$ where $[x]$ is the greatest integer function. Characterize those elements T of S for which the function

$$f(n) = n^2\Big(G\Big(\frac{(n-2)!}{n}\Big)\Big) = n.$$

Solution by R. P. Sealy, Sackville, New Brunswick, Canada

T is the set of primes in S. One form of Wilson's Theorem states: A necessary and sufficient condition that n be prime is that $(n-1)! \equiv -1(\text{mod n})$. But $(n-1)! = (n-1)(n-2)!$ with $n-1 \equiv -1(\text{mod n})$. Therefore $(n-2)! \equiv 1(\text{mod n})$ if, and only if, n is prime. Therefore

$$f(n) = n^2\Big(G\Big(\frac{(n-2)!}{n}\Big)\Big) = n^2 \cdot \frac{1}{n} = n \text{ if, and only if, } n \geq 5 \text{ is prime.}$$

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- **4941:** *Proposed by Tom Leong, Brooklyn, NY.*

The numbers $1, 2, \cdots, 2006$ are randomly arranged around a circle.
(a) Show that we can select 1000 adjacent numbers consisting of 500 even and 500 odd numbers.
(b) Show that part (a) need not hold if the numbers were randomly arranged in a line.

Solution 1 by Paul Zorn, Northfield, MN.

Claim: Suppose we have 1003 0's and 1003 1's arranged in a circle, like a 2006-hour clock. Then there must be a stretch of length of 1000 containing 500 of each.

Proof: Call the clock positions $1, 2, \cdots, 2006$ as on an ordinary clock, and let $a(n)$ be 0 or 1, depending on what's at position n. Let $S(n) = a(n) + a(n+1) + \cdots + a(n+999)$, where addition in the arguments is mod 2006.

Note that $S(n)$ is just the number of 1's in the 1000-hour stretch starting at n, and we're done if $S(n) = 500$ for some n.

Now $S(n)$ has two key properties, both easy to show:
i) $S(n+1)$ differs from $S(n)$ by at most 1
ii) $S(1) + S(2) + S(3) + \cdots S(2006) = 1000 \cdot$(sum of *all* the 1's around the circle) $=1000(1003)$.

From **i)** and **ii)** it follows that if $S(j) > 500$ and $S(k) < 500$ for some j and k, then $S(n) = 500$ for some n *between* j and k. So suppose, toward contradiction, that (say) $S(n) > 500$ for all n. Then

$$S(1) + S(2) + S(3) + \cdots + S(2006) > 2006 \cdot 501 = 1003(1002),$$

which contradicts ii) above.

Solution 2 by Harry Sedinger, St. Bonaventure, NY.

Denote the numbers going around the circle in a given direction as $n_1, n_2, \cdots, n_{206}$ where n_i and n_{i+1} are adjacent for each i and n_{2006} and n_1 are also adjacent. Let S_i be the set of 1,000 adjacent numbers going in the same direction and starting with n_i. Let $E(S_i)$ be the number of even numbers in S_i. It is easily seen that each number occurs in exactly 1000 such sets. Thus the sum S of occurring even numbers in all such sets is 1,003 (the number of even numbers) times 1000 which is equal to 1,003,000.

a) Suppose that $E(S_i) \neq 500$ for every i. Clearly $E(S_i)$ and $E(S_{i+1})$ differ by at most one, (as do $E(S_{2006})$ and $E(S_1)$), so either $E(S_i) \leq 499$ for every i or $E(S_i) \geq 501$ for every i. In the first case $S \leq 499 \cdot 2,006 < 1003,000$, a contradiction, and in the second case $S \geq 501 \cdot 2006 > 1,003,000$, also a contradiction. Hence $E(S_i) = 500$ for some k and the number of odd numbers in S_k is also 500.

b) It is easily seen that a) does not hold if the numbers are sequenced by 499 odd, followed by 499 even, followed by 499 odd, followed by 499 even, followed by 4 odd, and followed by 4 even.

Also solved by Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Apologies Once Again

I inadvertently forgot to mention that David Stone and John Hawkins of Statesboro, GA jointly solved problems 4910 and 4911. But worse, in my comments on 4911 (Is is possible for the sums of the squares of the six trigonometric functions to equal one), I mentioned that only two of the 26 solutions that were submitted considered the problem with respect to complex arguments. (For real arguments the answer is no; but for complex arguments it is yes.) David and John's solution considered both arguments–which makes my omission of their name all the more embarrassing. So once again, mea-culpa.